(AST 461) LECTURE 1: Basics of Radiation Transfer

Almost all we know about the astronomical universe comes from from radiation emanating from faint sources. We need to know how to interpret this radiation.

Electromagnetic spectrum, wavelength and frequency:

$$\nu\lambda = c, \tag{1}$$

ν is frequency, λ is wavelength. The speed of light depends on the index of refraction.

Justification for Macroscopic Treatment of Radiation

Scale of system $>>$ λ of the radiation, the radiation can been considered to travel in "straight lines" =rays. $dA >> \lambda^2$

Amount of energy passing through source dA is in time dt is $FdAdt$ where F is the flux. Units are (erg/cm.s^2) in CGS.

Flux from an isotropic source

Isotropic means energy emitted in all directions. Consider 2 different spherical surfaces S_1 and S_2. (fig 1.)

$$F_1 dA_1 dt_1 = F_2 dA_2 dt_2 \tag{2}$$

but $dt_1 = dt_2$ so that

$$F_1/F_2 = dA_2/dA_1 = r_2^2/r_1^2. \tag{3}$$

This is the inverse square law.

Intensity

Flux measures energy from all rays in given area. A more detailed approach is to consider energy carried from individual rays, or sets of rays differing infinitesimally from the initial ray. (fig 2.)

$$dE = (\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}) I_\nu dA dt d\Omega d\nu, \tag{4}$$

where I_ν is the Specific Intensity and represents Energy /(time × area × sold angle × frequency).

Now suppose we have an isotropic radiation field. The differential flux at a given frequency is

$$dF_\nu = (\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}) I_\nu d\Omega = I_\nu cos\theta d\Omega. \tag{5}$$

But if I_ν is isotropic, then $F_\nu = 0$.

Constancy of Intensity in Free Space:

Specific intensity is constant along a ray in free space.

Here's why: Consider all rays passing through both dA_1 and dA_2 and use conservation of energy: (fig 3)

$$dE_1 = I_{\nu,1} dA_1 dt_1 d\Omega_1 d\nu_1 = dE_2 = I_{\nu,2} dA_2 dt_2 d\Omega_2 d\nu_2 \tag{6}$$

Note that $dt_1 = dt_2$, $d\nu_1 = d\nu_2$. Note also that $d\Omega_1$ is the solid angle subtended by dA_2 at dA_1 and $d\Omega_2$ is the solid angle subtended by dA_1 at dA_2. Thus we have

$$d\Omega_1 = dA_2/r^2 \tag{7}$$

and

$$d\Omega_2 = dA_1/r^2. \tag{8}$$

Thus

$$I_{\nu,1} = I_{\nu,2}. \tag{9}$$

The intensity is constant along free space.

Application to a Telescope (fig 3.)

Assume large distance between object and lens $r >> f$, where f is the focal length. Determine image intensity I_i given object intensity I_0. Infinitesimal amount of surface area of object dA_0 has intensity I_0. We have

$$I_0 dA_0 d\Omega_{T,0} = I_0 A_T dA_0 / r^2. \tag{10}$$

where $d\Omega_{T,0}$ is the solid angle of the telescope as measured from the dA_0 of the object. All photons from dA_0 must strike dA_1 on focal plane.

Thus

$$I_0 dA_0 d\Omega_{T,0} = I_0 dA_0 A_T / r^2 = I_i dA_i d\Omega_{T,i} = I_i dA_i A_T / f^2, \tag{11}$$

but solid angle sub-tending image from telescope equals solid angle subtending object from telescope so

$$dA_0 / r^2 = dA_i / f^2, \tag{12}$$

and as expected, $I_i = I_o$.

Do telescopes measure flux or intensity?

If resolution of telescope is crude, and we cannot resolve object, then we measure flux. If telescope can resolve object, then we measure intensity.

Why? Consider the case when the source is unresolved. Now imagine pushing the source to farther distance. As the distance increases, the number of photons falls as r^2. The flux is measured.

If instead the the source is resolved, then as the source is pushed farther away, more area of the source would be included with the solid angle, which compensates for the the increased distance and the collected number of photons remain the same.

Flux from a uniformly bright sphere: read in text.

Relationship of Intensity to Stat Mech Quantities:

Consider photon distribution function f such that $f_\alpha(\mathbf{x}, \mathbf{p}, t)d^3xd^3p$ is the number of photons in $\mathbf{x}, \mathbf{p}$ space, with spin index α.

$$\mathbf{p} = h\mathbf{k} = (h\nu/c)\hat{\mathbf{k}} \tag{13}$$

so

$$dE = \sum_{\alpha=1}^{2} h\nu f_\alpha(\mathbf{x}, \mathbf{p}, t)d^3xd^3p. \tag{14}$$

But photons traveling in direction $\hat{\mathbf{k}}$ for time dt, through an element of area dA whose normal $= \hat{\mathbf{n}}$, occupy

$$d^3x = cdt(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}})dA \tag{15}$$

and

$$d^3p = p^2 d\Omega dp = (E^2/c^2)d\Omega dE/c = (h^3\nu^2/c^3)d\Omega d\nu, \tag{16}$$

where we have used $E = pc$ and $E = h\nu$. Thus,

$$dE = (\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}) \sum_{\alpha=1}^{2} (h^4\nu^3/c^3) f_\alpha(\mathbf{x}, \mathbf{p}, t)dAdtd\Omega d\nu \tag{17}$$

so

$$I_\nu = \sum_{\alpha=1}^{2} (h^4\nu^3/c^3) f_\alpha(\mathbf{x}, \mathbf{p}, t). \tag{18}$$

In stat mech, the occupation number for each photon is

$$N_\alpha = h^3 f_\alpha, \tag{19}$$

which is dimensionless so

$$I_\nu = \sum_{\alpha=1}^{2} (h\nu^3/c^2) N_\alpha(\mathbf{x}, \mathbf{p}, t). \tag{20}$$

Later we will see for example that I_ν corresponds to B_ν, the Planck function for b-body radiation. The occupation number corresponds to $N_\alpha = (e^{h\nu_\alpha/kT - \mu_\alpha} - 1)^{-1}$ for Bose-Einstein stats.

Radiation Pressure

photon carries momentum $p_\nu = E_\nu/c$. (fig. 4)

Can compute pressure by considering incident photons reflecting off of a wall: The angle of incidence equals the angle of reflection. The change in the z component of momentum of photon between frequency ν and $\nu + d\nu$ reflected in time dt from area dA is

$$dp_{z,\nu}d\nu = [(p_{z,\nu})_f - (p_{z,\nu})_i]d\nu = (1/c)(E_\nu cos\theta - (-E_\nu cos\theta))d\nu = (2/c)E_\nu cos\theta d\nu$$
$$= \left(\frac{2}{c}\right) I_\nu d\nu dt dA cos^2\theta d\Omega, \tag{21}$$

where we used (4). Now the change in momentum per unit area per unit time is a differential force/area= differential pressure. Integrating over solid angle gives the total pressure:

$$P_{rad,\nu} = \frac{dp_\nu}{dtdA} = (2/c)\int I_\nu cos^2\theta d\Omega_{hem} = (2/c)\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} I_\nu cos^2\theta sin\theta d\theta d\phi. \tag{22}$$

Now imagine removing the reflecting surface. Thus instead of the factor of 2 in momentum, we would have photons coming in from the other side. Thus we can remove the factor of 2, and integrate over the full sphere.

$$P_{rad,\nu} = dp_\nu/dtdA = (1/c)\int I_\nu cos^2\theta d\Omega = (1/c)\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\nu cos^2\theta sin\theta d\theta d\phi. \tag{23}$$

Energy Density

Consider a cylinder of cross section dA. (fig 5)

$$dE = u_\nu(\Omega)dV d\nu d\Omega = u_\nu(\Omega)cdtd\nu dAd\Omega = I_\nu dAd\Omega dtd\nu \tag{24}$$

so the energy density is

$$u_\nu(\Omega) = (I_\nu/c). \tag{25}$$

Then

$$u_\nu = \int u_\nu(\Omega)d\Omega = (1/c)\int I_\nu d\Omega = 4\pi\bar{I}_\nu/c \tag{26}$$

For isotropic radiation field

$$\overline{I}_\nu = I_\nu \tag{27}$$

Note also that the pressure for an isotropic radiation field from (23) and (26) is

$$P = (1/c)\int I_\nu cos^2\theta d\Omega d\nu = \int (\overline{I}_\nu/c)\int cos^2\theta d\Omega d\nu = \frac{1}{3}\int u_\nu d\nu. \tag{28}$$

LECTURE 2

Equation of Radiative Transfer

Condition that I_ν is constant along rays means that

$$dI_\nu/dt = 0 = \partial_t I_\nu + c\mathbf{k}\cdot\nabla I_\nu, \tag{29}$$

where $\mathbf{k}\cdot\nabla = dI_\nu/ds$ is the ray-path derivative. This is equation is the statement that there are no sources or sinks. But real systems may have sources and sinks: emission (source) and absorption (sink).

If light travels past a system much faster than the time scale for which the system evolves then $\partial_t I_\nu \simeq 0$ and we have then

$$dI_\nu/dt = 0 = c\mathbf{k}\cdot\nabla I_\nu = dI_\nu/ds = \text{sources } - \text{ sinks}, \tag{30}$$

where ds is the path. We now construct the source and sink terms in this radiative transfer equation.

Emission coefficient (source)

Spontaneous emission coefficient j_ν is defined as the energy emitted per unit time per unit solid angle, per unit volume per unit frequency

$$dE = j_\nu dV d\Omega dt d\nu, \tag{31}$$

where j_ν has units erg/(cm$^3\cdot$steradian$\cdot$sec$\cdot$hz). The emissivity is defined as $\epsilon_\nu = 4\pi j_\nu/\rho$, where ρ is the mass density. Since $dV = dAds$, the "source" contribution to the radiative transfer equation is given by

$$dI_\nu = j_\nu ds. \tag{32}$$

Absorption coefficient (sink)

Consider the propagation of a beam through a cylinder (again) of cross section dA and length ds, populated with a number density n of absorbers (particles), with absorption cross section σ_ν. (fig 6)

The loss of intensity, or the energy absorbed out of the beam is given by

$$dE_\nu = (I_{i,\nu} - I_{f,\nu})dAd\Omega dt d\nu = I_\nu(n\sigma_\nu ds)dAd\Omega dt d\nu. \tag{33}$$

Thus the "sink" contribution to the radiative transfer equation is

$$dI_\nu = -n\sigma_\nu I_\nu ds. \tag{34}$$

The absorption coefficient α_ν is defined such that

$$dI_\nu = -\alpha_\nu I_\nu ds, \tag{35}$$

so $\alpha_\nu = n\sigma_\nu$ and α_ν is units of cm^{-1}. Sometimes it's written $\alpha_\nu = \rho\kappa_\nu$, where ρ is the mass density and κ_ν is the mass absorption or "opacity" coefficient. Note that the emission and absorption coefficients have different units. Also, note that absorbing particles are assumed to be randomly distributed and much smaller than inter-particle spacing.

The term "absorption" is used somewhat loosely: Stimulated emission is also included in the absorption coefficient because it is proportional to the incident intensity.

We have thus derived the radiative transfer equation which we will soon solve:

$$dI_\nu/ds = -\alpha_\nu I_\nu + j_\nu. \tag{36}$$

Solutions of Equation of Radiative Transfer

A primary goal is to see what the values of α_ν and j_ν are for different absorption and emission processes. Scattering complicates things since radiation from the initial solid angle is scattered into a different solid angle for which the scatter depends on the initial solid angle, so we need numerics. Consider some simple cases here.

Emission Only:

$$dI_\nu/ds = j_\nu, \tag{37}$$

where the solution is

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')ds'. \tag{38}$$

Intensity increase is equal to the integrated emission coefficient along the propagation path.

Absorbtion Only: Then $j_\nu = 0$. In this case

$$I_\nu(s) = I_\nu(s_0)Exp[-\int_{s_0}^{s} \alpha(s')ds']. \tag{39}$$

The intensity decreases exponentially along the ray path.

Optical Depth

Define

$$d\tau_\nu = \alpha_\nu ds, \tag{40}$$

or

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha(s')ds'. \tag{41}$$

The optical depth is measure of how transparent a medium is to radiation. For $\tau_\nu \geq 1$ the medium is optically thick, and for $\tau_\nu < 1$ the medium is optically thin. We rewrite the transfer equation (36) as

$$dI_\nu/d\tau_\nu = -I_\nu + S_\nu, \tag{42}$$

where we define the source function $S_\nu \equiv j_\nu/\alpha_\nu$ as the ratio of the emission to the absorption coefficient. Optical depth is a physically meaningful variable because it contains the information about transparency along the path.

Now solve (42) by integrating factor: Multiplying by e^{τ_ν}, we obtain:

$$d(I_\nu e^{\tau_\nu})/d\tau_\nu = e^{\tau_\nu} S_\nu, \tag{43}$$

so the solution is

$$I_\nu e^{\tau_\nu} = I_\nu(0) + \int_0^{\tau_\nu} S_\nu(\tau'_\nu)e^{\tau'_\nu}d\tau'_\nu \tag{44}$$

or

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\tau'_\nu)e^{-(\tau_\nu-\tau'_\nu)}d\tau'_\nu. \tag{45}$$

The right side is the sum of the initial intensity attenuated by absorption, and the emission source attenuated by absorption. For a constant source function we have

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + S_\nu(1-e^{-\tau_\nu}) = S_\nu + e^{-\tau_\nu}(I_\nu(0) - S_\nu). \tag{46}$$

At large optical depth, $I_\nu \simeq S_\nu$. Why? because the source function is the local input of radiation–little is contributed to the to I_ν from matter far from the location of interest when $\tau_\nu >> 1$.

Note also that there is a relaxation process (evidenced by (42) where I_ν tends toward the source function: If $I_\nu > S_\nu$, then $dI_\nu/d\tau_\nu < 0$. If $I_\nu < S_\nu$, then $dI_\nu/d\tau_\nu > 0$.

Mean Free Path

Interpreting the exponential in (39) (which equals $e^{-\tau_\nu}$) as the probability of a photon traveling an optical depth τ_ν, the mean optical depth traveled

$$\langle\tau_\nu\rangle = \int_0^\infty \tau_\nu e^{-\tau_\nu} d\tau_\nu = 1. \quad (47)$$

The associated mean free path (MFP) is

$$l_\nu = \langle\tau_\nu\rangle/\alpha_\nu = 1/n\sigma_\nu. \quad (48)$$

MFP is important concept for particles as well as radiation.

Thermal Radiation and Blackbody Radiation

Thermal radiation is radiation emitted from matter in thermal equilibrium.

A blackbody absorbs all incident radiation.

Blackbody radiation is radiation which itself is in thermal equilibrium and in thermal equilibrium with surrounding matter.

Consider a thermally insulated box kept at temp T. $I_\nu = I_\nu(T) = B_\nu$, the Planck function for such a black body box. Dependence only on temperature can be seen by considering two connected boxes at same T. Energy cannot flow without violating second law. Thus I_ν is same in both boxes.

I_ν is isotropic because otherwise there would be a net energy flow from one part of the box to another which is not possible when entire system is at uniform T.

For a blackbody,

$$I_\nu = B_\nu(T) = \frac{2h\nu^3/c^2}{Exp[h\nu/kT] - 1}, \quad (49)$$

the Planck function, and is only a function of the temperature and frequency.

Kirchoff's Law

Put thermally emitting material at temperature T with source function $S_\nu(T)$ in the box.

If $S_\nu > B_\nu$ then the intensity I_ν will tend to increase such that $I_\nu > B_\nu$ and if $S_\nu < B_\nu$ then the intensity will decrease to make $I_\nu < B_\nu$. But in equilibrium, the new box is still a blackbody, and so $S_\nu = B_\nu$ or

$$j_\nu = \alpha_\nu B_\nu, \quad (50)$$

which is Kirchoff's law for a thermal emitter. In general, Kirchoffs law relates emission to absorption.

Consider again the radiative transfer equation (42) with $S_\nu = B_\nu$:

$$dI_\nu/ds = -\alpha_\nu I_\nu + \alpha_\nu B_\nu(T). \tag{51}$$

For a black body (or for free space) the left side is zero, and thus $I_\nu = B_\nu$. For thermal radiation in a not necessarily optically thick system, the left side need not be zero for example if the source function temperature changes rapidly with position and I_ν cannot keep up. Thermal radiation is $S_\nu = B_\nu$ whilst blackbody radiation is $I_\nu = B_\nu$.

Blackbody Thermodynamics

The text derives some results: The Stefan-Boltzmann law for the energy density:

$$u(T) = aT^4, \tag{52}$$

where a is a constant. Recall from (26) that $u = \frac{4\pi}{c}\int B_\nu(T)d\nu$ for $I_\nu = B_\nu$. The integral of (49) over frequency is $B(T) = (ac/4\pi)T^4$, where $a = 8\pi^5 k^4/15c^2h^3$.

Now the flux from an uniformly bright, isotropically emitting sphere of brightness B_ν is given by

$$F_\nu = \int B_\nu cos\theta d\Omega = 2\pi B_\nu \int_{cos\theta_c}^{1} cos\theta d(cos\theta) = \pi B_\nu(1-cos^2\theta_c) = \pi B_\nu sin^2\theta_c = \pi B_\nu(R^2/r^2), \tag{53}$$

where $\theta_c = arcsin(R/r)$, R is radius of source, and r is distance to the source. At $r = R$, $F_\nu = \pi B_\nu$, Then

$$F = \int F_\nu d\nu = \pi B(T), \tag{54}$$

and the total flux for a blackbody

$$F = \sigma T^4, \tag{55}$$

where $\sigma = 0.25ac = 5.67 \times 10^{-5} erg\ cm^{-2}\ {}^oK^{-4} sec^{-1}$.

Properties of the Planck Law

1. Rayleigh-Jeans (RJ) limit

Taking the $h\nu << kT$ limit of the Planck law of (49) we expand the exponential and obtain

$$B_\nu^{RJ}(T) \sim 2\nu^2 kT/c^2, \tag{56}$$

which is the Rayleigh-Jeans limit. This usually applies in the radio regime. It is the classical regime. It cannot apply to all frequencies, because integrating would imply total energy divergence.

2. Wien Limit

If we take the limit $h\nu >> kT$ we have

$$B_\nu^{RJ}(T) \sim 2(h\nu^3/c^2)e^{-h\nu/kT}, \tag{57}$$

which is the Wien regime.

3. Temperature derivative

A blackbody curve with the higher temperature is always completely above a curve with a lower temperature. This follows from the temperature derivative of $B_\nu(T)$.

4. Wien Displacement Law

Peak frequency can be found by maximizing B_ν by setting $\partial B_\nu/\partial\nu = 0$, which leads to

$$\nu_{max}/T = 2.82k/h = 5.9 \times 10^{10} Hz/^oK \tag{58}$$

Peak frequency shifts linearly with the temperature. This is the Wien Displacement Law. Same can be done for wavelength to find

$$\lambda_{max}T = 0.290cm^oK. \tag{59}$$

Note that $\lambda_{max}\nu_{max} \neq c$. Can see that the Rayleigh-Jeans law is valid for $\nu << \nu_{max}$.

Other temperatures commonly used

1. Brightness Temperature

This is defined by the temperature of the blackbody having the same brightness (=intensity) at that frequency. We define it by

$$I_\nu = B_\nu(T_b). \tag{60}$$

In radio astronomy one often sees this formula in the Rayleigh-Jeans regime ($h\nu << kT$). One has there

$$T_b = (c^2/2\nu^2 k)I_\nu \tag{61}$$

and for thermal emission $S_\nu = B_\nu \sim 2\nu^2 kT/c^2$. In the RJ regime we can then set $S_\nu = B_\nu$ in (46) and divide that equation by $2\nu^2 k/c^2$ to obtain

$$T_b = T_b(0)e^{-\tau_\nu} + T(1 - e^{-\tau_\nu}). \tag{62}$$

$T_b \sim T$ for $\tau_\nu >> 1$.

Only if source is a blackbody is the brightness temperature the same at all frequencies.

For optically thin thermal emission, brightness temperature gives lower temp than that of the actual source; this is because a blackbody gives maximum intensity of thermal emitter at temperature T. (to see this use (62) with $T_b(0) = 0$).

Why is this?: A body which is not optically thick can be thought of as a system that is not fully "black" and absorbs less than a blackbody. If this system were in thermal equilibrium, then its emission must equal its absorption. But if it absorbs less than a blackbody it must also emit less then a blackbody to remain in equilibrium.

2. Color Temperature

The color temperature is the temperature obtained by fitting a blackbody shape to an observed, approximate blackbody flux.

3. Effective Temperature

This is obtained by setting the observed flux to that of blackbody form from the Stefan-Boltzmann law. That is

$$T_{eff} = (F/\sigma)^{1/4}. \tag{63}$$

LECTURE 3

Einstein Coefficients

Kirchoff's law relating emission to absorption for a thermal emitter must involve microscopic physics. Consider system with two energy states with statistical weights g_1 and g_2 respectively. Transition from 2 to 1 is by emission and from 1-2 by absorption. State 1 has energy E and state 2 has energy $E + h\nu$.

1. Define the Einstein A_{21} coefficient as the probability per unit time for spontaneous emission.

2. The absorption of a photon $\propto$ density of photons or the mean intensity at frequency ν_0. Energy difference between two levels is not sharp, but broadened, and so we also need to consider the line profile function ϕ obeying $\int_0^\infty \phi(\nu)d\nu = 1$. This is usually quite narrow. The transition probability per unit time for absorption is

$$B_{12}\bar{J} = \frac{B_{12}}{4\pi}\int\int_0^\infty I_\nu \phi(\nu)d\nu d\Omega, \tag{64}$$

where B_{12} is the Einstein B coefficient and $J_\nu = \frac{1}{4\pi}\int I_\nu d\Omega$, the intensity averaged over solid angle. Here $\bar{J} = \int_0^\infty J'_\nu \phi(\nu')d\nu' \sim J_{\nu_0}$ for narrow line profiles.

Derivation of Planck's law led Einstein to include stimulated emission. It can be thought of as negative absorption and has a coefficient B_{21} such that $B_{21}\bar{J}$ is the transition probability per time for stimulated emission, also proportional to the intensity.

Relation between Coefficients

Micro-physical relations that are independent of temperature will hold regardless of whether processes are in thermodynamic equilibrium or not, but we can use case of thermodynamic equilibrium to get at them when possible.

In thermodynamic equilibrium the number of transitions per unit time per volume into state 1 are equal to the transitions out. If n_1 and n_2 are number densities of atoms in states 1 and 2 we have

$$n_1 B_{12}\bar{J} = n_2 A_{21} + n_2 B_{21}\bar{J}. \tag{65}$$

This gives

$$\bar{J} = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1}, \tag{66}$$

where

$$n_1/n_2 = (g_1/g_2)e^{h\nu/kT} \tag{67}$$

so

$$\bar{J} = \frac{A_{21}/B_{21}}{(g_1/g_2)(B_{12}/B_{21})e^{h\nu/kT} - 1}. \tag{68}$$

But $J_\nu \sim B_\nu$ for an emitter whose matter is in thermodynamic equilibrium, we have the Einstein detailed balance relations which relate atomic properties

$$g_1 B_{12} = g_2 B_{21} \tag{69}$$

and

$$A_{21} = 2(h\nu^3/c^2)B_{21}. \tag{70}$$

These do not depend on temperature and must hold independently of thermodynamic equilibrium. If we can determine any one of the Einstein coefficients we get the other two.

Wien's law follows if you don't include stimulated emission. Recall that was the $h\nu >> kT$ regime, so $n_2 << n_1$ and the stimulated emission term is small. (Multiply top and bottom of (68) times g_2/g_1 to see this limit.)

Absorption and Emission Coefficients in Terms of Einstein Coefficients

Assume that line profile for spontaneous emission is same as that for absorption, and is $\phi(\nu)$ (which has units of $\frac{1}{\nu}$). Then matching units using the definition of the emission coefficient j_ν

$$dE = j_\nu dV d\Omega d\nu dt = (h\nu_0/4\pi)\phi(\nu) n_2 A_{21} dV d\Omega d\nu dt, \tag{71}$$

so the emission coefficient is

$$j_\nu = (h\nu_0/4\pi) n_2 A_{21} \phi(\nu). \tag{72}$$

For the absorption coefficient the energy absorbed in dV and in dt is

$$dE = h\nu_0 n_1 B_{12} \bar{J} dV dt = h\nu_0 n_1 B_{12} dV dt \left[\frac{1}{4\pi}\int I_\nu \phi(\nu) d\nu d\Omega\right], \tag{73}$$

where we take $dV = dsdA$ (the cylinder). Then using $\int_0^\infty I'_\nu \phi(\nu')d\nu' = \bar{I}_\nu$ and $d\bar{I}_\nu = \alpha_\nu \bar{I}_\nu ds$ for the absorprtion, we have

$$\alpha_{\nu,abs} = (h\nu_0/4\pi) n_1 B_{12} \phi(\nu). \tag{74}$$

Stimulated emission can be thought of as inverse absorption so the full absorpion coefficient is then

$$\alpha_\nu = (h\nu/4\pi)(n_1 B_{12} - n_2 B_{21})\phi(\nu). \tag{75}$$

Transfer Equation in terms of the Einstein Coefficients

$$dI_\nu/ds = -(h\nu/4\pi)(n_1 B_{12} - n_2 B_{21})\phi(\nu)I_\nu + (h\nu/4\pi)n_2 A_{21}\phi(\nu). \tag{76}$$

The source function is j_ν/α_ν and so

$$S_\nu = n_2 A_{21}/(n_2 B_{12} - n_2 B_{21}). \tag{77}$$

Using the Einstein relations (69) and (70) gives

$$\alpha_\nu = (h\nu/4\pi)n_1 B_{12}(1 - g_1 n_2/g_2 n_1)\phi(\nu) \tag{78}$$

and

$$S_\nu = (2h\nu^3/c^2)(g_2 n_1/g_1 n_2 - 1)^{-1}. \tag{79}$$

Masers/Lasers:

For a system in thermal equilibrium we have

$$n_1/n_2 = (g_1/g_2)e^{h\nu/kT} \tag{80}$$

which implies

$$n_1/g_1 > n_2/g_2 \tag{81}$$

But it is possible to pump atoms into the upper state to reverse this relation. Then (75) (or 78) is negative, implying negative absorption. The intensity actually increases along the ray path exponentially with the optical depth.

Many interesting maser sources in astrophysics. NGC4258 is a galaxy for which H_20 masers (22.235, 321..GHz) in the central region trace a Keplerian warped accretion disk, which has provided some of the best evidence for black holes in nature.

Also, OH/IR sources: large cool giant star or supergiant star losing mass rapidly in winds and detected only in IR or masers in OH (1.665 Ghz). Can use OH masers in these sources to estimate their distances. The maser emission line from the expanding wind shell has a red and blue component because the near side part of the wind is moving at us, and the far side is moving away (centered at the star). But variability in the stellar wind (i.e. turning on and off rather than being steady) will show up with a delay between the red and blue parts of the line due to the cross time across the source. Since radio telescopes can resolve the angular diameter, we can get the source diameter and thus the distance. More explicitly, $d = r/\theta = ct/\theta$, where d is distance, r is the “wind span” t is the measured time delay and θ is the measured angular diameter.

Scattering

So far we have ignored scattering. Scattering can be considered an emission process that depends on the amount of radiation incident upon the medium doing the scattering. (Contrast: Thermal radiation does not depend on incident radiation.)

Consider electron scattering (scattering of photons by electrons). Here we assume isotropic, coherent (elastic) scattering. More elaborate coverage of these assumptions later.

The scattering emission coefficient is found by equating the power “absorbed”(think of scattering as absorption and immediate re-emission) per unit volume and frequency to the power emitted isotropically

$$j_\nu = \sigma_\nu J_\nu. \tag{82}$$

Where here σ_ν is a scattering coefficient with units of $1/length$ like the absorption coefficient (and here σ_ν should not to be confused with the cross section or Stefan Boltzmann constant!) Note that J_ν is angle averaged version of I_ν. So by analogy to $j_\nu = \alpha_\nu S_\nu$ discussed earlier, here we have $S_\nu = J_\nu$ and radiative transfer equation has scattering coefficient in both the emission and ”absorption” terms:

$$dI_\nu/ds = -\sigma_\nu(I_\nu - J_\nu). \tag{83}$$

The solution cannot be easily extracted from (45) since I_ν is not known initially and J_ν depends on the angular integral of I_ν.

To make progress, instead think of the scattering, absorption and emission processes in probabilistic terms for a single photon. Consider a photon in an infinite homogeneous scattering region. Displacement of photon after N free paths, where the ith path is of length $\mathbf{r}_i$

$$\mathbf{R} = \sum_{a=1}^{N} \mathbf{r}_{(a)}. \tag{84}$$

where the sum is over free displacements, not vector indices. Mean square photon displacement is then given by

$$l_*^2 = \langle \mathbf{R}^2 \rangle = \langle \sum_{a=1}^{N} \mathbf{r}_{(a)} \cdot \mathbf{r}_{(a)} \rangle, \tag{85}$$

as the cross terms vanish for isotropic scattering when there are a large number of scatterings. (Book does not mention this requirement.) Statistically, each term of the sum on the right of (85) contributes the same average mean free path l squared, so we have

$$l_*^2 = N l^2, \tag{86}$$

which indicates the mean square displacement of the photon For a finite medium, we can determine the number of scatterings N. For large optical depths, N is found by setting $l^* = L$, the typical size of the medium. so that $L/l = \tau$, the optical depth to scattering and $N \simeq \tau^2$. This is a large N result only, because in deriving it we assumed that each contribution to the sum on the right of (85) contributes the same amount (and that cross terms vanish). This is true only within an error of $\pm 1/N^{1/2}$, so the error is small for large N and large for small N.

For small optical depths the probability for scattering is $1 - e^{-\tau} \sim \tau << 1$, which is equal to the expected number of scatterings in transversing the medium. The reason is that $e^{-\tau}$ is probablity for a photon NOT to scatter in trasversing an optical depth τ so $1 - e^{-\tau}$ is the probability to scatter. Think e.g. if $\tau = 1/5$ then this implies 0.2 scatterings per system size.

LECTURE 4

Scattering + Absorption

Consider thermal material with absorption coefficient α_ν describing thermal absorption and emission and a scattering coefficient σ_ν describing coherent isotropic scattering. The transfer equation is then

$$dI_\nu/ds = -\alpha_\nu(I_\nu - B_\nu) + \sigma_\nu(J_\nu - I_\nu). \tag{87}$$

The source function

$$S_\nu = (\alpha_\nu B_\nu + \sigma_\nu J_\nu)/(\alpha_\nu + \sigma_\nu), \tag{88}$$

is a weighted average of the thermal and scattering source functions. We can define $d\tau_\nu = (\sigma_\nu + \alpha_\nu)ds$ as the differential optical depth to both scattering and absorption, with extinction coefficient $(\sigma_\nu + \alpha_\nu)$.

Note limits that $S_\nu = B_\nu$ for $J_\nu = B_\nu$ (e.g. at large optical depths) and $S_\nu < B_\nu$ for regions where $J_\nu \to 0$. (Note that both J_ν and σ_ν depend on the presence of matter but not necessarily to the same functional powers).

Random Walk for Combined Scattering and Absorption

Mean free path to both scattering and absorption

$$l_\nu = 1/(\sigma_\nu + \alpha_\nu). \tag{89}$$

The probability that a free photon will be absorbed rather than scattered is

$$\epsilon_\nu = \alpha_\nu/(\alpha_\nu + \sigma_\nu), \tag{90}$$

and the probability for scatter is the single scattering albedo, and equals $1 - \epsilon_\nu$.

Consider again a homogeneous infinite medium: For large number of scatterings N per absorption, N also equals the mean number of free paths before absporption and the inverse of the probability of absorption. The RMS net displacement of the photon is then (recall (85)

$$l_*^2 = N l_\nu^2 = l_\nu^2/\epsilon_\nu = 1/(\alpha_\nu(\alpha_\nu + \sigma_\nu)). \tag{91}$$

Here l_* is the net displacement for a photon from creation to absorption. It is the "effective diffusion length." because the photon takes a stochastic path to move this net distance before being absorbed. If we think of this diffusion length as a kind of "modified mean free path" then The "effective optical depth"

$$\tau_* = R/l^* = [\tau_a(\tau_a + \tau_s)]^{1/2}, \tag{92}$$

where R is scale of system. When $\tau_* << 1$ we have a transluscent medium. Why not transparent?: Because there can still be a lot of scattering. A transparent medium would also be translucent by this definition, the former being a stronger constraint in that addition to not being absorbed, there would be little scattering. Transparent is both $\tau_s << 1$ and $\tau_a << 1$.

Note however that the analysis that led to (92) above really assumes multiple scatterings and is most applicable when $\sigma_s >> \alpha_s$ in which case $\tau_* \sim (\tau_a\tau_s)^{1/2}$.

For $\tau_* << 1$ the monochromatic luminosity from a thermal source would just be determined by the emission coefficient:

$$\tilde{L}_\nu = 4\pi j_\nu V = 4\pi\alpha_\nu B_\nu V. \tag{93}$$

For $\tau_* >> 1$ thermal equilibrium between matter and radiation is reached, and in addition, the emission comes only from a thin layer of order l_*, on surface of area A. Thus

$$L_\nu \simeq 4\pi\alpha_\nu B_\nu l_* A = 4\pi\epsilon_\nu^{1/2} B_\nu A. \tag{94}$$

This does not quite reduce to a black-body for zero scattering (i.e. $\epsilon_\nu = 1$), so numerical factor is not quite accurate. One part of the reason is that $l_*^2 = l_x^2 + l_y^2 + l_z^2 \simeq 3l_x^2$, and since the radial direction in a star for example, corresponds to only 1 of the three directions, rather than use l_* in (94) we should use $l_*/3^{1/2}$.

Note that at $\tau_* > 1$ matter and radiation approach thermal equilibrium, thus l_* is also called the "effective thermalization length."

Radiative Diffusion Equation

Source function becomes black body at large effective optical depths for homogeneous systems. Global homogeneity is not realistic though local homogeneity may be OK, as in the Sun or stars.

Consider "plane parallel" assumption: properties of an atmosphere depend only on depth. Assume planar geometry. Intensity then depends on angle from normal θ such that $ds = dz/cos\theta \equiv dz/\mu$ and

$$\mu\partial I_\nu(z,\mu)/\partial z = -(\alpha_\nu + \sigma_\nu)(I_\nu - S_\nu). \tag{95}$$

If we consider a region deep in the atmosphere, the intensity varies slowly on scale of mean free path. Intensity approximates a blackbody but with small variation. We describe this as follows: To zeroth order in z-derivative terms

$$I_\nu^{(0)} = J_\nu^{(0)} = S_\nu^{(0)} = B_\nu(T). \tag{96}$$

Then we iterate and plug the next higher term back into (95), that is

$$I_\nu^{(1)} \simeq B_\nu(T) - \frac{\mu}{\alpha_\nu + \sigma_\nu}\frac{\partial B(T)}{\partial z}. \tag{97}$$

The flux is then

$$F_\nu(z) \simeq \int I_\nu^{(1)}(z,\mu) cos\theta d\Omega = 2\pi \int_{-1}^{1} I_\nu^{(1)}(z,\mu)\mu d\mu = -\frac{4\pi}{3(\alpha_\nu + \sigma_\nu)}\frac{\partial B_\nu(T)}{\partial T}\frac{\partial T}{\partial z}, \tag{98}$$

where the first term on the right of (97) does not contribute due to isotropy. Integrating over all frequencies, we have

$$F(z) \simeq -\frac{4\pi}{3}\frac{\partial T}{\partial z}\int_0^\infty (\alpha_\nu + \sigma_\nu)^{-1}\frac{\partial B_\nu}{\partial T} d\nu \tag{99}$$

Note that

$$\int_0^\infty (\alpha_\nu + \sigma_\nu)^{-1}\frac{\partial B_\nu}{\partial T} d\nu = \frac{\int_0^\infty (\alpha_\nu + \sigma_\nu)^{-1}\frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu}\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu \equiv \frac{1}{\alpha_R}\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu, \tag{100}$$

where α_R is the Rosseland mean absorption coefficient α_R. But

$$\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu = \partial B(T)/\partial T = 4\sigma T^3/\pi, \tag{101}$$

using the Stefan-Boltzmann law. Thus

$$F(z) = \frac{-16\sigma T^3}{3\alpha_R}\frac{\partial T}{\partial z}. \tag{102}$$

Key properties: 1) Flux increases with decreasing temperature gradient. Makes sense since isotropy gets progressively violated at larger radii (nearer to the stellar surface). 2) Radiative energy transport is similar to heat conduction.

Absorption Lines vs. Emission lines, & Limb Darkening

From (88) and (90) above we have the source function

$$S_\nu = (1 - \epsilon_\nu) J_\nu + \epsilon_\nu B_\nu. \tag{103}$$

Ignore scattering for the moment, and assume thermal emission. We are back to

$$S_\nu = B_\nu. \tag{104}$$

Recall solution of radiative transfer equation in this case: we have from (46) for $I_\nu(0) = 0$

$$I_\nu = B_\nu (1 - e_\nu^{-\tau_\nu}). \tag{105}$$

For small optical depths, this goes as

$$I_\nu \simeq \tau_\nu B_\nu, \tag{106}$$

so the intensity is larger in lines where the optical depth is larger.

But for a source with $\tau \geq 1$, we do not see the whole source, only the layers which are optically thin. The temperature decreases outward in a medium whose optical depth decreases with radius. We are then looking at layers of lower temperature for lines, since the lines' optical depth is larger for a given distance of ray travel when compared to the continuum. Thus we would see absorption lines for large optical depths in a system with radially decreasing temperature because the intensity for a higher temperature blackbody exceeds that of a lower temperature blackbody at all wavelengths. It is important to emphasize that the minimum trough of an absorption line does not go to zero in intensity—just to a value lower than the continuum that would be present in the absence of the line.

For a point source emitting $I(0)$, located behind a emitting and absorbing region in local thermal equilibrium we would have

$$I_\nu = I_\nu(0) e^{-\tau_\nu} + B_\nu (1 - e^{-\tau_\nu}), \tag{107}$$

When $I_\nu(0)$ is brighter than the foreground (assuming that τ_ν here increases toward the observer) the contribution on the right from $(I_\nu(0) - B_\nu(T)) e^{-\tau_\nu}$ decreases rapidly with increasing optical depth. Then the right side is dominated by the local souce function $B_\nu(T)$. For a source with decreasing temperature toward its surface, we see into the source less deeply at the frequencies associated with line transitions so the line emission is dominated by outer regions whose local black body has a lower flux.

A related issue explains limb darkening. White light image of Sun shows edges darker than center because the edge view penetrates less deeply into Sun for given optical depth, and thus sees emission coming from regions of lower temperature than the face on view.

Eddington Approximation

Rosseland approximation employed the fact that system approached black body at large effective optical depths. Eddington only assumes isotropy, thus is good at smaller optical depths. We write a power series in $\mu = cos\theta$

$$I_\nu(\tau, \mu) = a_\nu(\tau) + b_\nu(\tau)\mu. \tag{108}$$

If we take moments of this equation (normalized by 4π) we obtain

$$J_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu d\mu = a \tag{109}$$

$$H_\nu = \frac{1}{2}\int_{-1}^{1} \mu I_\nu d\mu = b/3 \tag{110}$$

$$K_\nu = \frac{1}{2}\int_{-1}^{1} \mu^2 I_\nu d\mu = a/3. \tag{111}$$

The fact that $K_\nu = J_\nu/3$ is the Eddington approximation, and is equivalent to $p_\nu = u_\nu/3$ found in first lecture. The relation is found here for a more general system with slight anisotropy.

Writing $d\tau(z) = -(\alpha_\nu + \sigma_\nu)dz$ (note sign here; integrated toward the observer) we then have

$$\mu dI_\nu/d\tau = I_\nu - S_\nu, \tag{112}$$

where $dz = \mu ds$. Here ds is along the ray path and dz is the vertical direction through the slab. Then taking moments in μ and integrating from $-1 \leq \mu \leq 1$ we get

$$dH_\nu/d\tau = J_\nu - S_\nu. \tag{113}$$

and

$$dK_\nu/d\tau = H_\nu = \frac{1}{3}dJ_\nu/d\tau. \tag{114}$$

Combining these last two equations we obtain

$$d^2J_\nu/d\tau^2 = 3(J_\nu - S_\nu) = 3\epsilon_\nu(J_\nu - B_\nu), \tag{115}$$

where the latter equation follows from (88) and (90). If ϵ_ν does not depend on z, we can write this

$$d^2 J_\nu / d\tau_*^2 = (J_\nu - B_\nu). \tag{116}$$

using the effective optical depth $\tau_* = \tau\sqrt{3\epsilon} = \sqrt{3\tau_a(\tau_a+\tau_s)}$ keeping in mind the discussion below (94).

These are forms of the radiative transfer equation. If we have $B_\nu(\tau)$, we can solve for J_ν and get S_ν from (88) if the matter properties of the medium are known (density, cross sections). Then we have what we need to solve (112).

Text goes onto discuss two-stream approximation, for obtaining the two boundary conditions. Assume the radiation propagates at just two angles $I_\nu^\pm = I_\nu(\tau, \mu = \pm 1/3^{1/2})$ at every location (i.e. at each value of z for plane parallel approximation). Then plug into the moment equations (109) and (110) to get $J = \frac{1}{2}(I^+ + I^-)$, $H = \frac{1}{2\sqrt{3}}(I^+ - I^-)$, $K = J/3$, and use (114) to relate eliminate H. This then gives two equations for the boundary conditions

$$I_\nu^+ = J_\nu + \frac{1}{\sqrt{3}}\partial J_\nu/\partial\tau \tag{117}$$

and

$$I_\nu^- = J_\nu - \frac{1}{\sqrt{3}}\partial J_\nu/\partial\tau. \tag{118}$$

The reason for this odd choice of μ is that it reproduces exactly $K_\nu = J_\nu/3$, the Eddington approximation.

More on Moments

Consider integrating the intensity over various vector multiples of $\mathbf{k}$. We have

$$cu_\nu = \int I_\nu d\Omega \tag{119}$$

and

$$\mathbf{F}_\nu = \int \mathbf{k} I_\nu d\Omega \tag{120}$$

$$c\mathbf{P}_\nu = \int \mathbf{k}\mathbf{k} I_\nu d\Omega. \tag{121}$$

From fluid mechanics or E&M, different components of stress tensor refer to rate of momentum transfer across surfaces of different orientations. Rate of momentum transfer per photon across surface of unit normal $\mathbf{k}$ is is c$\mathbf{kp}$.

Thus the total rate of momentum transfer is given locally by the radiation stress tensor

$$\mathbf{P} = \sum_{\alpha=1}^{2} \int c\hat{\mathbf{k}}\mathbf{p} f_\alpha d^3p = \sum_{\alpha=1}^{2} \int c\hat{\mathbf{k}}\hat{\mathbf{k}}(h^4\nu^3/c^3) f_\alpha d\Omega d\nu = (1/c) \int \hat{\mathbf{k}}\hat{\mathbf{k}} I_\nu d\Omega d\nu \tag{122}$$

The total derivative of I_ν obeys

$$\partial_t I_\nu + (1/c)\mathbf{k} \cdot \nabla I_\nu = -\alpha_\nu I_\nu + j_\nu - \sigma_\nu I_\nu + \sigma_\nu \int \Phi(\hat{\mathbf{k}}, \hat{\mathbf{k}}') I_\nu(\mathbf{k}') d\Omega' \tag{123}$$

Compare this more general form to (87): In our previous discussions we have ignored the 1st term on the left and replaced the last term on the right with $\sigma_\nu \overline{J}_\nu$. If we integrate over all solid angles of photon paths, we have, for "front back symmetric scattering"

$$\partial_t u_\nu + \nabla \cdot \mathbf{F}_\nu = -c\alpha_\nu u_\nu + 4\pi j_\nu. \tag{124}$$

The next highest moment is obtained by multiplying (123) by $\mathbf{k}$ and integrating over solid angle (assuming j_ν is independent of $\mathbf{k}$) to obtain

$$\partial_t \mathbf{F}_\nu + c\nabla \cdot \mathbf{P}_\nu = -(\alpha_\nu + \sigma_\nu)\mathbf{F}_\nu. \tag{125}$$

Note that right side represents a radiation force on matter. (Sink for radiation energy/momentum appears as source for matter energy/momentum and vice versa. Removal of radiation momentum means gain in momentum of matter.) Note that to fully solve the system of equations, one needs an infinite set of equation because each equation depends on a higher moment. This is the closure problem. (Also found in turbulence and MHD).

The Eddington approximation, combined with the approximation that the systems evolve on time scales slower than the travel time associated with a mean free path (e.g. ratio is 10^{-28}) for sun, allows ignoring the time derivative terms.

Note the general form of conservation law:

$$\partial_t (quantity\ density + \nabla \cdot (quantity\ flux) = (quantity\ sources - quantity\ sinks) \tag{126}$$

LECTURE 5

The first four lectures discussed general properties of radiation transfer and the some aspects of the specific case of a blackbody. The rest of the course builds up to trying to understand the physics behind the emission, absorption, and scattering coefficients of real systems, and how this relates to the material constituents.

Basic E&M

The electromagnetic force on a particle moving with velocity $\mathbf{v}$ and charge q is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}/c), \quad (127)$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. The rate of work done is

$$\mathbf{v} \cdot \mathbf{F} = d_t(mv^2/2) = q\mathbf{v} \cdot \mathbf{E}. \quad (128)$$

For many particles the force density is

$$\mathbf{f} = \rho_c \mathbf{E} + \mathbf{j} \times \mathbf{B}/c, \quad (129)$$

where $\rho_c = \sum_a q_a n_a$ is the total charge density (sum of charge times number density over particle type) and $\mathbf{j} = \sum_a q_a n_a \mathbf{v}_a$. Rate of work per unit volume is then

$$\sum_a \mathbf{f}_a \cdot \mathbf{v}_a = \sum_a q_a n_a \mathbf{v}_a \cdot \mathbf{E} = \mathbf{j} \cdot \mathbf{E}. \quad (130)$$

The rate of work done per unit volume is the rate of change of mechanical energy density:

$$d_t u_{mech} = \mathbf{j} \cdot \mathbf{E}. \quad (131)$$

Maxwell's Equations

$$\nabla \cdot \mathbf{D} = 4\pi\rho_c \quad (132)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (133)$$

$$\nabla \times \mathbf{E} + c^{-1}\partial\mathbf{B}/\partial t = 0 \quad (134)$$

$$\nabla \times \mathbf{H} - c^{-1}\partial\mathbf{D}/\partial t = 4\pi\mathbf{j}/c. \quad (135)$$

Here $\mathbf{D} = \epsilon\mathbf{E}$. This comes from

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \tag{136}$$

where $\mathbf{P} = \sum_a n_a \langle \mathbf{p}_a \rangle$ is the electric polarization= dipole moment density averaged over spatial region. (Note dipole moment is $\int \rho_c \mathbf{x}' d^3\mathbf{x}'$). For linear response, we take $\mathbf{P} \propto \mathbf{E}$ or $\mathbf{P} = \chi\mathbf{E}$ so

$$\mathbf{D} = (1 + 4\pi\chi)\mathbf{E} \equiv \epsilon\mathbf{E}, \tag{137}$$

where ϵ is the permittivity=dielectric constant.

Similarly, for $\mathbf{B} = \mu\mathbf{H}$. Here H is the applied field and we have the magnetization $\mathbf{M} = \Sigma_a n_a \langle \mathbf{m_a} \rangle$, where the dipole moment is $\mathbf{m} = \int \mathbf{x}' \times \mathbf{j} d^3x'$. For linear response materials $\mathbf{M} = \chi_m \mathbf{H}$. This leads to $\mathbf{B} = \mu\mathbf{H}$, where $\mu = 1 + 4\pi\chi_m$ is the magnetic permeability. and χ_m is the magnetic susceptibility.

For the vacuum, $\mathbf{B} = \mathbf{H}$ and $\mathbf{D} = \mathbf{E}$

Conservation laws of E& M

The law for charge conservation is

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0 \tag{138}$$

.

Let us now derive the Poynting theorem of electromagnetic energy conservation. Use Maxwell's equation (135) and dot with $\mathbf{E}$. This gives

$$\mathbf{j} \cdot \mathbf{E} = (1/4\pi)[c(\nabla \times \mathbf{H}) \cdot \mathbf{E} - \mathbf{E} \cdot \partial_t \mathbf{D}]. \tag{139}$$

Now use a vector identity on the first term on the right:

$$\mathbf{E} \cdot \nabla \times \mathbf{H} = \mathbf{H} \cdot \nabla \times \mathbf{E} - \nabla \cdot (\mathbf{E} \times \mathbf{H}). \tag{140}$$

(prove using index tensor notation and ϵ_{ijk} tensor). Use Maxwell's, equation (134), and assume the dielectric and magnetic permeability are independent of time to get

$$(1/4\pi)\partial_t(\epsilon E^2 + B^2/\mu) = -\mathbf{j} \cdot \mathbf{E} - (c/4\pi)\nabla \cdot (\mathbf{E} \times \mathbf{H}). \tag{141}$$

This is Poynting's theorem. If we integrate we can see its true meaning:

$$(1/4\pi)\partial_t \int (\epsilon E^2 + B^2/\mu) d^3x = -\int \mathbf{j} \cdot \mathbf{E} d^3x - (c/4\pi) \int (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{A}, \tag{142}$$

where we use the divergence theorem on the last term. This shows that the change in electromagnetic energy equals that lost to work plus that flowing out the boundary. $\mathbf{S} \equiv c\mathbf{E} \times \mathbf{H}/4\pi$ is the Poynting flux. It is the energy flux vector (energy per unit time per area).

Definition of Radiation Field

For charges moving at a finite constant speed, the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ both fall as $1/r^2$. Thus at large distances $\int \mathbf{S} \cdot \mathbf{dA}$ goes as $1/r^2$.

However for the time varying fields of accelerated charges, we will find that there are components of E, B that are $\propto 1/r$, and so the Poynting flux from these terms is finite are large distances. This is the definition of the radiation field.

Plane Waves

Maxwell's equation in vacuum

$$\nabla \cdot \mathbf{E} = 0 \tag{143}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{144}$$

$$\nabla \times \mathbf{E} + c^{-1}\partial_t \mathbf{B} = 0 \tag{145}$$

$$\nabla \times \mathbf{B} - c^{-1}\partial_t \mathbf{E} = 0. \tag{146}$$

Then taking curl of (145), using (146) and the vector identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ we get

$$\nabla^2 \mathbf{E} - (1/c^2)\partial^2 \mathbf{E}/\partial t^2 = 0. \tag{147}$$

Assume

$$\mathbf{E} = \hat{\mathbf{a}}_1 E_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{148}$$

and

$$\mathbf{B} = \hat{\mathbf{a}}_2 B_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \tag{149}$$

where $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ are unit vectors and $\mathbf{k}$ is the wave vector. These are traveling wave solutions because the argument of the exponential is constant for surfaces which propagate in time along the wave direciton.

Plugging back into Maxwell's equations we have

$$i\mathbf{k} \cdot \hat{\mathbf{a}}_1 E_0 = 0, \tag{150}$$

$$i\mathbf{k}\cdot\hat{\mathbf{a}}_2 B_0 = 0 \tag{151}$$

and

$$i\mathbf{k}\times\hat{\mathbf{a}}_1 E_0 = i\omega\hat{\mathbf{a}}_2 B_0/c, \tag{152}$$

$$i\mathbf{k}\times\hat{\mathbf{a}}_2 B_0 = -i\omega\hat{\mathbf{a}}_1 E_0/c. \tag{153}$$

These imply that $\mathbf{k}\perp\hat{\mathbf{a}}_1,\hat{\mathbf{a}}_2$ and $\hat{\mathbf{a}}_2\perp\hat{\mathbf{a}}_1$. They also imply

$$E_0 = B_0\omega/kc, \quad B_0 = E_0\omega/kc, \tag{154}$$

so

$$\omega^2 = c^2k^2 \tag{155}$$

and $E_0 = B_0$. The phase velocity $v_{ph}\equiv w/k = c$. Group velocity is also c because there is no medium and the group velocity depends on the index of refraction n:

$$v_g = \frac{c}{n(\omega)+\omega(\partial n/\partial\omega)}, \tag{156}$$

but for the vacuum $n = 1$ and is independent of ω so $v_g = v_{ph} = c$.

Wave Energy Flux and Density

Note that $\mathbf{E}$ and $\mathbf{B}$ vary sinusoidally in time, but in astro, we measure time averages for measurements of interest. (The measuring time (observing time) assumed to be longer than wave period.)

We are going to calculate products of complex quantities (the electromagnetic fields) but we measure only the real part. So we are interested in the products of the real parts. Take two complex quantities $Q_1 = E_0e^{iwt}$ and $Q_2 = B_0e^{iwt}$. Then

$$Re(Q_1) = \frac{1}{2}(Q_1 + Q_1^*) = \frac{1}{2}(E_0e^{i\omega t} + E_0^*e^{-i\omega t}), \tag{157}$$

where $*$ indicates complex conjugate. Similarly,

$$Re(Q_2) = \frac{1}{2}(B_0e^{i\omega t} + B_0^*e^{-i\omega t}). \tag{158}$$

(These follow from recognizing any complex number can be written as $a + bi$ with a and b real.) Then

$$Re(Q_1)Re(Q_2) = (1/4)(E_0B_0e^{2i\omega t} + E_0B_0^* + E_0^*B_0 + E_0^*B_0^*e^{-2i\omega t}) \tag{159}$$

Take time average, then the terms with exponentials vanish when the averages are taken over sufficiently long time compared to the ω^{-1}. Thus we have

$$\langle Re(Q_1)Re(Q_2)\rangle = \frac{1}{4}\langle E_0 B_0^* + E_0^* B_0\rangle. \tag{160}$$

But only the real part of each term contributes. Thus

$$\langle Re(Q_1)Re(Q_2)\rangle = \frac{1}{4}\langle Re(E_0 B_0^* + E_0^* B_0)\rangle = \frac{1}{2}\langle Re(E_0^* B_0)\rangle = \frac{1}{2}\langle Re(E_0 B_0^*)\rangle. \tag{161}$$

Applying this to the time averaged Poynting flux for waves gives

$$\langle \mathbf{S}\rangle = \frac{c}{4\pi}\langle Re(\mathbf{E})\times Re(\mathbf{B})\rangle = \frac{c}{4\pi}\hat{\mathbf{k}}\langle |Re(\mathbf{E})||Re(\mathbf{B})|\rangle = \frac{c}{8\pi}\hat{\mathbf{k}}Re(E_0 B_0^*) = \frac{c}{8\pi}\hat{\mathbf{k}}|E_0|^2 = \frac{c}{8\pi}|B_0|^2, \tag{162}$$

where we have again eliminated $e^{\pm 2i\omega t}$ terms by averaging them away. For the electromagnetic energy density, we find

$$u_{em} = \frac{1}{16\pi}\langle Re(E_0 E_0^* + B_0 B_0^*)\rangle = \frac{1}{16\pi}|E_0|^2 + \frac{1}{16\pi}|B_0|^2 = \frac{1}{8\pi}|B_0|^2, \tag{163}$$

so $\langle u_{em}\rangle\hat{\mathbf{k}} = \langle \mathbf{S}\rangle/c$, implying energy flows at speed c for vacuum plane waves (not surprising).

LECTURE 6 Radiation Spectrum

The radiation spectrum (histogram of energy as a function of frequency) cannot be measured at an instant of time because of the "uncertainty" relation $\delta\omega\delta t \gtrsim 1$. True for waves in general. Instead we will eventually consider radiation received from a fixed location averaged over a "long" time.

Consider finite radiation pulse and consider the scalar $E(t) = \hat{\mathbf{a}}_1 \cdot \mathbf{E}$ pulling out one component of $\mathbf{E}$. We then have, using Fourier transforms,

$$\tilde{E}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t) e^{i\omega t} dt, \tag{164}$$

so that

$$E(t) = \int_{-\infty}^{\infty} \tilde{E}(\omega) e^{-i\omega t} d\omega, \tag{165}$$

also note

$$\tilde{E}(-\omega) = \tilde{E}^*(\omega), \tag{166}$$

since E is real. The $\tilde{E}$ has the power spectrum info. Write the energy/(time$\cdot$ Area) as

$$\frac{dW}{dt dA} = cE(t)^2/4\pi \tag{167}$$

from the result immediately above for the Poynting flux. For the total energy per area we have, integrating (167)

$$\frac{dW}{dA} = c \int_{-\infty}^{\infty} E^2(t) dt/4\pi. \tag{168}$$

But

$$= c \int_{-\infty}^{\infty} E(t)^2 dt/4\pi = (c/2) \int_{-\infty}^{\infty} |\tilde{E}(\omega)|^2 d\omega \tag{169}$$

by Parsevals theorem relating the Fourier transform of a product to the product of the Fourier transforms. Further, since $|\tilde{E}(\omega)|^2 = |\tilde{E}(-\omega)|^2$, we take the integral from 0 and multiply by 2 to obtain:

$$dW/dA = c \int_{0}^{\infty} |\tilde{E}(w)|^2 d\omega, \tag{170}$$

and

$$\frac{dW}{dA d\omega} = c|\tilde{E}(w)|^2. \tag{171}$$

But this is the total energy per frequency per area for the entire pulse. If pulse repeats over the time T or if it is more or less stationary for long times, we can determine the spectrum for some finite part of an "infinitely" long pulse, based on the results for finite pulse. (The finite pulse issue enters to ensure that the integrals above are finite). Spectrum is not useful if the characteristic time changes $T \leq 1/\delta\omega$, where $\delta\omega$ is the characteristic frequency resolution. We can write

$$\frac{dW}{dAd\omega T} \simeq \frac{dW}{dAd\omega dt} = c|E(w)|^2/T, \tag{172}$$

which is a power spectrum per unit area for the duration T, assuming that the signal has approximately same properties over that duration. (fig 7)

Elliptical Polarization, Stokes Parameters for Monochromatic Waves

Plane waves are linearly polarized. Waves of arbitrary polarization can be generated from a linear combination of two orthogonal linearly polarized waves. Consider the electric field

$$\mathbf{E} = (E_1\hat{\mathbf{x}} + E_2\hat{\mathbf{y}})e^{-iwt}. \tag{173}$$

We can write the complex amplitudes

$$E_1 = \varepsilon_1 e^{i\phi_1} \tag{174}$$

and

$$E_2 = \varepsilon_2 e^{i\phi_2}, \tag{175}$$

where ε_1, ϕ_1 and ε_2, ϕ_2 are real. so

$$\mathbf{E} = \varepsilon_1\hat{\mathbf{x}}e^{i(\phi_1 - wt)} + \varepsilon_2\hat{\mathbf{y}}e^{i(\phi_2 - wt)}. \tag{176}$$

The real part is thus

$$E_x = \varepsilon_1 cos(\omega t - \phi_1) \tag{177}$$

$$E_y = \varepsilon_2 cos(\omega t - \phi_2). \tag{178}$$

(fig 8)

These equations show that the electric field traces out an ellipse, as we will now see. The general equation for an ellipse can be written as

$$(X/a)^2 + (Y/b)^2 = 1. \tag{179}$$

We write

$$E_{x'} = \varepsilon_0 cos\beta coswt \tag{180}$$

and

$$E_{y'} = -\varepsilon_0 sin\beta sinwt, \tag{181}$$

for $-\pi/2 < \beta < \pi/2$. These satisfy (179) using $X = E_{x'}$, $Y = E_{y'}$, $a = \varepsilon_0 cos\beta$ and $b = \varepsilon_0 sin\beta$. When the wave is propagating toward us, the ellipse is traced out clockwise for $0 < \beta < \pi/2$ and counterclockwise for $-\pi/2 < \beta < 0$. These represent left-handed and right-handed polarization respectively (negative and positive helicity). When $\beta = \pm\pi/4$ the ellipse becomes a circle and we have circular polarization. For $\beta = 0, \pm\pi/2$ we have linear polarization.

To show that (177) and (178) also form an ellipse, perform a clockwise rotation on the x', y' components (180) and (181) by angle χ (multiply by the rotation matrix for clockwise rotation) so that we have

$$E_x = \varepsilon_0(cos\beta cos\chi cos\omega t + sin\beta sin\chi sin\omega t) \tag{182}$$

and

$$E_y = \varepsilon_0(cos\beta sin\chi cos\omega t - sin\beta cos\chi sin\omega t). \tag{183}$$

Then these match (177) and (178) if we set

$$\varepsilon_1 cos\phi_1 = \varepsilon_0 cos\beta cos\chi \tag{184}$$

$$\varepsilon_1 sin\phi_1 = \varepsilon_0 sin\beta sin\chi \tag{185}$$

$$\varepsilon_2 cos\phi_2 = \varepsilon_0 cos\beta sin\chi \tag{186}$$

$$\varepsilon_2 sin\phi_2 = -\varepsilon_0 sin\beta cos\chi. \tag{187}$$

We can solve for $\varepsilon_0, \beta, \chi$ given ε_1, ϕ_1, ε_2, ϕ_2. This is normally done with the monochromatic Stokes Parameters:

$$I = \varepsilon_0^2 = \varepsilon_1^2 + \varepsilon_2^2 = E_1^2 + E_2^2 \tag{188}$$

$$Q = \varepsilon_0^2 cos2\beta cos2\chi = \varepsilon_1^2 - \varepsilon_2^2 = E_1^2 - E_2^2 \tag{189}$$

$$U = \varepsilon_0^2 cos2\beta sin2\chi = 2\varepsilon_1\varepsilon_2 cos(\phi_1 - \phi_2) = E_1 E_2^* + E_2 E_1^* \tag{190}$$

$$V = \varepsilon_0^2 sin2\beta = 2\varepsilon_1\varepsilon_2 sin(\phi_1 - \phi_2) = -i(E_1 E_2^* - E_2 E_1^*) \tag{191}$$

To prove (188-191) start with the left side of (184-187) and construct combinations that give the right sides of (188-191): For example to prove (188) we add the squares of (184-187) and use $sin^2\phi_1 + cos^2\phi_1 = sin^2\phi_2 + cos^2\phi_2 = 1$.

To prove (189) we add the squares of (184) (185) and subtract from the sum of (186) and (187), and then use the identities $cos^2\chi - sin^2\chi = cos2\chi$ and $cos^2\beta - sin^2\beta = cos2\beta$.

To prove (190) first note that $cos(\phi_1 - \phi_2) = cos\phi_1 cos\phi_2 + sin\phi_1 sin\phi_2)$ then add the product of (184) times (186) to the product of (185) times (187) and use the identities $2sin\chi cos\chi = sin2\chi$ and $cos^2\beta - sin^2\beta = cos2\beta$.

Finally, to prove (191) first note that $sin(\phi_1 - \phi_2) = sin\phi_1 cos\phi_2 - sin\phi_2 cos\phi_1$ then subtract the product of (184) times (187) from the product of (185) times (186) and use the identities $2sin\beta cos\beta = sin2\beta$ and $cos^2\chi + sin^2\chi = 1$.

Pure elliptical polarization (polarization of monochromatic waves) is controlled by the 3 parameters $(\varepsilon_0, \chi, \beta)$ but there are 4 Stokes parameters. This is because they are related by

$$I^2 = Q^2 + U^2 + V^2. \tag{192}$$

That

$$I = \varepsilon_0^2, \tag{193}$$

highlights how I is related to the intensity of the wave. (Recall our calculation of the energy density of a plane wave above). We have

$$sin2\beta = V/I \tag{194}$$

and thus V measures the circularity of the wave polarization (see discussion below (181), and is related to the ratio of the power of the linearly independent components. If $\beta = 0$, then the wave is linearly polarized and $V = 0$. If $\beta = \pi/4$, $V/I = 1$ implying circular polarization. Finally,

$$\tan 2\chi = U/Q, \tag{195}$$

so that U or Q relate to the orientation of the ellipse that the electric field vector traces with respect to a fixed set of axes. However for circular polarization $Q = U = 0$ and $I = V$ from (194), or (192).

Stokes Parameters for Quasi Monochromatic Waves

In general, the observed emission is a superposition of wave components with different polarizations and different frequencies. The waves are generally not monochromatic. The amplitude and phase for a given frequency can change with time. We have

$$E_1(t) = \varepsilon_1(t)e^{i\phi_1(t)} \tag{196}$$

and

$$E_2(t) = \varepsilon_2(t)e^{i\phi_2(t)}. \tag{197}$$

Assume that the time variation is relatively slow, such that over a time period of $1/\omega$ the wave looks fully polarized, but such that for $\delta t >> 1/\omega$ the amplitude and phases can vary. Then $\delta\omega\delta t \sim 1$ implies $\delta\omega << \omega$ and so we call this limit a quasi-monochromatic wave of bandwidth $\delta\omega$ and coherence time δt.

Most waves are measured in the time averaged sense by a detector. In addition, measuring devices often split waves into a linear combinations of two independent electric field polarization components. Assuming any time for the wave to pass through measurement apparatus is short compared to the coherence time δt, then the splitting of waves into the two components is a linear transformation of the form

$$E_{x'} \equiv E'_1 = \lambda_{11}E_1 + \lambda_{12}E_2 \tag{198}$$

$$E_{y'} \equiv E'_2 = \lambda_{21}E_1 + \lambda_{22}E_2 \tag{199}$$

Where the λs represent the effect of the measuring apparatus. The time average sum of the squares of the x' and y' components are measured. Using the identity (161) we have for the x' component

$$\langle Re(\mathbf{E}_{x'}^*\mathbf{E}_{x'})\rangle = 2\langle[ReE_1'e^{-iwt}]^2\rangle = |\lambda_{11}^2|\langle E_1E_1^*\rangle+|\lambda_{12}^2|\langle E_2E_2^*\rangle+|\lambda_{11}\lambda_{12}^*|\langle E_1E_2^*\rangle+|\lambda_{12}\lambda_{11}^*|\langle E_2E_1^*\rangle, \tag{200}$$

where the terms with the residual e^{-iwt} have been dropped assuming the quasi-monochromatic approximation, and we only consider averages over longer times.

The λs are known features of the apparatus. The four different terms which represent averages of the radiation field can be written in terms of Stokes parameters

$$I = \langle E_1E_1^*\rangle + \langle E_2E_2^*\rangle = \langle \varepsilon_1^2 + \varepsilon_2^2\rangle \tag{201}$$

$$Q = \langle E_1E_1^*\rangle - \langle E_2E_2^*\rangle = \langle \varepsilon_1^2 - \varepsilon_2^2\rangle \tag{202}$$

$$U = \langle E_1E_2^*\rangle + \langle E_2E_1^*\rangle = \langle 2\varepsilon_1\varepsilon_2cos(\phi_1 - \phi_2)\rangle \tag{203}$$

$$V = -i(\langle E_1E_2^*\rangle - \langle E_2E_1^*\rangle) = \langle 2\varepsilon_1\varepsilon_2sin(\phi_1 - \phi_2)\rangle. \tag{204}$$

These are the generalized Stokes parameters which reduce to the monochromatic ones when the amplitudes and phases are time independent. We cannot distinguish waves with the same Stokes parameters. But here,

$$I^2 \geq Q^2 + U^2 + V^2. \tag{205}$$

Physically this makes sense when the wave is not completely elliptically polarized: If part of the radiation is unpolarized, the right side (205), which is the intensity squared of the polarized contribution, is an undersetimate of the total I^2 . The result follows from the Schwartz inequality

$$\langle E_1E_2^*\rangle\langle E_2E_1^*\rangle \leq \langle E_1E_1^*\rangle\langle E_2E_2^*\rangle, \tag{206}$$

where the equality holds when the ratio of E_1/E_2 is independent of time. (I leave as an exercise to show this from (196) and (197)). A completely unpolarized wave has no preferred orientation so $\langle\varepsilon_1^2\rangle = \langle\varepsilon_2^2\rangle$ and $Q = U = V = 0$.

LECTURE 7

Degree of Polarization

Stokes parameters are additive for waves of uncorrelated phase. That is, if we take the net electric field vector for the sum of waves, no cross terms contribute to the time averaging, that is for $E_i = \sum_n E_{i,n}$ we have

$$\langle E_1 E_2^* \rangle = \sum_n \sum_l \langle E_{1,n} E_{2,l}^* \rangle = \sum_n \langle E_{1,n} E_{2,n}^* \rangle. \tag{207}$$

Thus $I = \sum_n I_n$, $Q = \sum_n Q_n$, $U = \sum_n U_n$, and $V = \sum_n V_n$.

We can then decompose the total Stokes parameters for an arbitrary collection of quasi-monochromatic waves into contributions from the polarized part and the unpolarized part:

$$I, Q, U, V = [I - (Q^2 + U^2 + V^2)^{1/2}, 0, 0, 0] + [(Q^2 + U^2 + V^2)^{1/2}, Q, U, V]. \tag{208}$$

The former is the unpolarized part. The degree of polarization is the ratio of the polarized part of the intensity to the total. (Note that we may be missing constants to make I exactly equal to the total intensity but intensity ratios are independent of this constant).

$$\Pi = (Q^2 + U^2 + V^2)^{1/2}/I = I_{pol}/I. \tag{209}$$

As an example, consider partial linear polarization. Then $V = 0$ and we take the plane of the polarized part to be the x' plane. One can the rotate a polarizing filter which will allow the maximum intensity when the filter is aligned with the electric field. The maximum intensity is the sum of the polarized part + 1/2 the unpolarized part because the intensity for the unpolarized part is shared equally between any two orthogonal directions and the polarized part only contributes when the filter is oriented properly. Thus

$$I_{max} = I_{unpol}/2 + I_{pol}. \tag{210}$$

The minimum intensity is

$$I_{min} = I_{unpol}/2. \tag{211}$$

But $I_{unpol} = I - (Q^2 + U^2)^{1/2}$ and $I_{pol} = (Q^2 + U^2)^{1/2}$. Thus

$$\Pi = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}, \tag{212}$$

when $V = 0$, i.e. partial linear polarization.

When does the Macroscopic Transfer theory of "rays" apply?

When size of the area we are interested in, through which radiation passes, approaches the wavelength of the radiation, then the macroscopic classical theory is not applicable. That is, the validity is for $(dp_x dx)(dp_y dy) \sim p^2 dA d\Omega \gtrsim h^2$ or

$$dA d\Omega \gtrsim \lambda^2. \tag{213}$$

In addition we require

$$d\nu dt \gtrsim 1, \tag{214}$$

from the energy uncertainty relation. Thus when λ is greater than the scale of interest, classical transfer theory of rays, as we have studied, fails (e.g. on the scale of atoms).

But now that we have studied waves, we can be more precise. Define rays as the curves whose tangents point along the direction of wave propagation. Thus, rays are well defined only if the amplitude and wave direction are nearly constant over a wavelength. That is the geometric optics limit.

To see the specific quantitative relations for the validity of this limit assume that a wave (electric field vector) is represented by

$$g(\mathbf{r}, t) = a(\mathbf{r}, t) e^{i\psi(\mathbf{r},t)}. \tag{215}$$

For a wave, g satisfies

$$c^{-2}\partial_t^2 g - \nabla^2 g = 0. \tag{216}$$

For constant $\mathbf{a}$, $\mathbf{k} = \nabla\psi$ is direction of propagation and $\omega = -\partial\psi/\partial t$ is frequency. These will be important later because we will show that the approximations which lead to a plane wave approximation using these correspondences to wavevector and frequency.

If we substitute (215) into (216) we get

$$\nabla^2 a - \frac{1}{c^2}\partial_t^2 a + ia(\nabla^2 - \frac{1}{c^2}\partial_t^2)\psi + 2i(\nabla a \cdot \nabla\psi - \frac{1}{c^2}\partial_t\psi\partial_t a) - a(\nabla\psi)^2 + \frac{a}{c^2}(\partial_t\psi)^2 = 0. \tag{217}$$

Now in the limit that the following relations are satisfied

$$\frac{1}{a}|\nabla a| << |\nabla\psi|; \ \frac{1}{a}|\partial_t a| << |\partial_t\psi| \tag{218}$$

$$\frac{1}{a}|\nabla^2 a| << |\nabla\psi|^2 \tag{219}$$

$$|\nabla^2 \psi| << |\nabla \psi|^2; \; |\partial_t^2 \psi| << |\partial_t \psi|^2, \tag{220}$$

(217) reduces to

$$(\nabla \psi)^2 - \frac{1}{c^2}(\partial_t \psi)^2 = 0, \tag{221}$$

which represents the Eikonal equation describing the geometric optics limit.

The above limits represent the case for which the the amplitude is slowly varying and the phase is rapidly varying. For nearly constant a, the direction of propagation which is perpendicular to surfaces of constant phase, is

$$\mathbf{k} = \nabla \psi. \tag{222}$$

and the frequency would be given by

$$\omega = -\partial_t \psi. \tag{223}$$

Using these in (221) then reproduces the usual plane wave relation of $k^2 = w^2/c^2$. Thus Eikonal equation expresses the geometric optics limit, and that limit leads to the relation between wave number and frequency we found earlier for plane waves. This is the relationship that applies in the regime of the classical transfer theory.

Electromagnetic Potentials

E and **B** can be expressed in terms of the electromagnetic 4-vector potential $A_\mu = (\phi, \mathbf{A})$, where ϕ is the scalar potential and **A** is the vector potential. We have

$$\mathbf{E} = -\nabla \phi - \frac{1}{c}\partial_t \mathbf{A}. \tag{224}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{225}$$

Then, incorporating the bound and free charges into ρ and $\mathbf{j}$, we have

$$-\nabla \cdot \mathbf{E} = \nabla^2 \phi - \frac{1}{c^2}\partial_t^2 \phi + \frac{1}{c}\partial_t(\nabla \cdot \mathbf{A} + \frac{1}{c}\partial_t \phi) = -4\pi\rho \tag{226}$$

and

$$\nabla \times \mathbf{B} + \frac{1}{c}\partial_t(\nabla \phi + \frac{1}{c}\partial_t \mathbf{A}) = 4\pi \mathbf{j}/c, \tag{227}$$

and $\nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

Gauge Invariance

E and **B** are invariant under gauge transformations, that is under the change

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi; \quad \phi \rightarrow \phi - \frac{1}{c}\partial_t\chi. \tag{228}$$

Only **E** and **B** are measured and so the choice of gauges is merely a choice to make the equations easier to solve. Different gauges are more or less convenient in different contexts.

The Coulomb gauge is such that $\nabla \cdot \mathbf{A} = 0$. The Lorenz gauge employs $\partial_t\phi + \nabla \cdot \mathbf{A} = 0$. For the latter case we obtain for the potentials

$$\nabla^2\phi - \frac{1}{c^2}\partial_t^2\phi = -4\pi\rho \tag{229}$$

and

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\partial_t\mathbf{A} = -4\pi\mathbf{j}. \tag{230}$$

The formal solutions are

$$\phi(\mathbf{r},t) = \int \frac{[\rho]}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' = \int\int \frac{\rho(\mathbf{r}',t')\delta(t'-t+|\mathbf{r}-\mathbf{r}'(t')|/c)}{|\mathbf{r}-\mathbf{r}'|}dt'd^3\mathbf{r}' \tag{231}$$

and

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int \frac{[\mathbf{j}]}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' = \int\int \frac{\mathbf{j}(\mathbf{r}',t')\delta(t'-t+|\mathbf{r}-\mathbf{r}'(t')|/c)}{|\mathbf{r}-\mathbf{r}'|}dt'd^3\mathbf{r}', \tag{232}$$

where the brackets indicate evaluation at the retarded time. Thus these are the retarded time potentials. The retarded time means that the quantity is evaluated at a time $t_{ret} = t - |\mathbf{r}-\mathbf{r}'|/c$ (where t is the present time) due to finite speed of light travel. Thus e.g. $[\rho] = [\rho(\mathbf{r}', t_{ret})]$.

Formally, given these potentials, we can solve for **E** and **B**.

Potential of Moving Charges

Consider charge q following path $\mathbf{r} = \mathbf{r}_0(t)$, $\mathbf{u} = \dot{\mathbf{r}}_0$. Then the localized charge and current densities are

$$\rho(\mathbf{r}',t') = q\delta(\mathbf{r}'-\mathbf{r}_0(t')) \tag{233}$$

$$\mathbf{j}(\mathbf{r}',t') = q\mathbf{u}(t')\delta(\mathbf{r}'-\mathbf{r}_0(t')). \tag{234}$$

The volume integrals give the total charge q and current $q\mathbf{u}$, for a single particle.

Using the formal solutions (231) and (232), we can calculate retarded potentials for this single particle, these are

$$\phi(\mathbf{r},t) = q\int \frac{\delta(t'-t+R(t')/c)}{R(t')}dt' \tag{235}$$

and

$$\mathbf{A}(\mathbf{r},t) = q\int \mathbf{u}(\mathbf{r}')\frac{\delta(t'-t+R(t')/c)}{R(t')}dt', \tag{236}$$

where $R(t') = |\mathbf{R}(t')|$ and

$$\mathbf{R}(t') = \mathbf{r} - \mathbf{r}_0(t'). \tag{237}$$

If we now use $t_{ret} = t - R(t')/c$ or $c(t - t_{ret}) = R(t')$. We then can change variables. Let

$$t'' = t' - t_{ret} \tag{238}$$

so that

$$\begin{aligned} dt'' = dt' - dt_{ret} = dt' + dt'\tfrac{\dot{R}(t')}{c} = dt' + dt'\tfrac{1}{2cR(t')}\tfrac{dR^2(t')}{dt'} &= dt' + dt'\tfrac{1}{2cR(t')}\tfrac{d\mathbf{R}^2(t')}{dt'} \\ &= dt'(1 - \hat{\mathbf{n}}(t')\cdot\tfrac{\mathbf{u}(t')}{c}), \end{aligned} \tag{239}$$

where $\hat{\mathbf{n}} = \mathbf{R}/R$ and $\mathbf{u}(t') = \dot{\mathbf{r}}_0 = -\dot{\mathbf{R}}$. (Note that we consider t and $\mathbf{r}$ fixed in this differentiation ; we are not integrating or differentiating over t or $\mathbf{r}$, there are just the time and position at which the quantities are being measured.) Thus using (239) in (235) and (236) gives

$$\phi(\mathbf{r},t) = q\int \frac{1}{R(t')(1-\hat{\mathbf{n}}(t')\cdot\mathbf{u}(t')/c)}\delta(t'')dt'' = \frac{q}{R(t_{ret})(1-\hat{\mathbf{n}}(t_{ret})\cdot\mathbf{u}(t_{ret})/c)} \tag{240}$$

and

$$\mathbf{A}(\mathbf{r},t) = q\int \mathbf{u}(t')/c\frac{1}{R(t')(1-\hat{\mathbf{n}}(t')\cdot\mathbf{u}(t')/c)}\delta(t'')dt'' = \frac{qu(t_{ret})}{R(t_{ret})(1-\hat{\mathbf{n}}(t_{ret})\cdot\mathbf{u}(t_{ret})/c)}, \tag{241}$$

where we have integrated $\delta(t'')$ by setting $t' = t_{ret}$ using (238). Recall that integrating a function over a δ function works like this: $\int \delta(q-q_0)f(q)dq = f(q_0)$.

Notice the factor appearing on the bottom which "concentrates" the potentials along the direction of motion for strongly relativistic flows. This is related to the concept of relativistic beaming of radiation into narrow cones along the direction of motion that we will derive later. These potentials are also evaluated at the retarded time–again important for very relativistic motion. This retarded time is also the key to getting the radiation part of the electromagnetic fields, that is the part that falls off as $1/r$ rather than $1/r^2$. The potentials above are the Liénard-Wiechart potentials.

LECTURE 8

Velocity and Radiation Components of the Electromagnetic Fields

To get **E** and **B** at **r** and t note that they depend on the retarded position and time r_{ret} and t_{ret}. At the retarded time, the particle has $\mathbf{u} = \dot{\mathbf{r}}_0(t_{ret})$ and acceleration $\dot{\mathbf{u}} = \ddot{\mathbf{r}}_0(t_{ret})$. There is some messy algebra to obtain, from the potentials presented at the end of last lecture, the fields (see Jackson)

$$\mathbf{E}(\mathbf{r}, t) = q\left[\frac{(\hat{\mathbf{n}} - \vec{\beta})(1 - \beta^2)}{\kappa^3 R^2}\right] + \frac{q}{c}\left[\frac{\hat{\mathbf{n}} \times ((\hat{\mathbf{n}} - \vec{\beta}) \times \dot{\vec{\beta}})}{\kappa^3 R}\right] \tag{242}$$

and

$$\mathbf{B}(\mathbf{r}, t) = [\hat{\mathbf{n}} \times \mathbf{E}], \tag{243}$$

where $\vec{\beta} = \mathbf{u}/c$ and $\kappa = (1 - \hat{\mathbf{n}} \cdot \vec{\beta})$. Where the brackets again indicate retarded time, and so only the conditions at t_{ret} matter even though particle is currently at $\mathbf{r}, t$.(fig3.1).

The first term in the electric field is just the relativistic generalization of Coulomb's law for moving particles. It falls off as $1/R^2$. This is the "velocity field." It points in the line toward the current position of the particle.

The second term is the radiation field. It falls off as $1/R$. Alas we see why the Poynting flux associated with radiation from an accelerating particle can be finite at large distances.

The effect can be seen from the Figure below. (fig 3.2 of R&L).

Suppose the particle is originally moving at constant speed and then it stops. This information is not immediately communicated to the exterior. Only for distances less than $c(t_1 - t_0)$ is the field informed that the particle decelerated, but outside this field is that of the particle had it not decelerated (that is, as if it were at x=1). Thus there is the kink in the field if the field is to match together for the two regions.

The thickness of this the interface region is constant and equal to the light travel distance during the period of deceleration or acceleration. Now take a ring whose central axis is the line of particle motion and the plane of the ring lies perpendicular to the field lines from the interface region, such that they penetrate the ring. Since Gauss' law implies the volume integrated electric flux is constant,

$$\int \mathbf{E} \cdot d\mathbf{S} = 0 \tag{244}$$

With time, the ring keeps same thickness, but its radius increases as R if it is to keep up with the interface region field lines. To conserve flux the field must fall of as $1/R$.

Energy/(Freq· Solid angle) for single particle

From Lecture 6 we have that

$$\frac{dW}{dAdt} = \frac{c}{4\pi} E^2(t). \tag{245}$$

Then using $dA = R^2 d\Omega$ we have

$$\frac{dW}{d\Omega dt} = \frac{c}{4\pi}(R(t)E(t))^2 = \frac{c}{4\pi}(Q(t))^2 = \tag{246}$$

where $\mathbf{Q}(t) = R(t)\mathbf{E}(t)$, so that (using the radiation part of the electric field)

$$\frac{dW}{d\Omega d\omega} = c\tilde{Q}(\omega)^2 = \frac{c}{4\pi^2}\left|\int_{-\infty}^{\infty}[R\mathbf{E}(t)]e^{iwt}dt\right|^2 = \frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty}[\hat{\mathbf{n}} \times ((\hat{\mathbf{n}} - \vec{\beta}) \times \dot{\vec{\beta}})\kappa^{-3}]e^{iwt}dt\right|^2. \tag{247}$$

The brackets again mean evaluated at retarded time $t' = t - R(t')/c$. Changing time variables using $dt = \kappa dt'$ and using the Taylor expansion of $R(t')$ in $\mathbf{r}_0$ for small r_0/r ($R(t') = \sqrt{(\mathbf{r} - \mathbf{r}_0)^2} \sim (r^2 - 2\mathbf{r} \cdot \mathbf{r}_0)^{1/2} = |\mathbf{r}| - \frac{\mathbf{r} \cdot \mathbf{r}_0}{(r^2)^{1/2}} = |\mathbf{r}| - \hat{\mathbf{n}} \cdot \mathbf{r}_0$) gives

$$\frac{dW}{d\Omega d\omega} = \frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{n}} \times ((\hat{\mathbf{n}} - \vec{\beta}) \times \dot{\vec{\beta}})\kappa^{-2} e^{iw(t' - \hat{\mathbf{n}} \cdot \mathbf{r}_0(t')/c)} dt'\right|^2. \tag{248}$$

Then using the identity

$$\hat{\mathbf{n}} \times ((\hat{\mathbf{n}} - \vec{\beta}) \times \dot{\vec{\beta}})\kappa^{-2} = d_{t'}(\kappa^{-1}\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \vec{\beta}) \tag{249}$$

we can integrate (248) by parts to obtain

$$\frac{dW}{d\Omega d\omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \vec{\beta})e^{iw(t' - \hat{\mathbf{n}}\cdot\mathbf{r}_0(t')/c)}dt'\right|^2, \tag{250}$$

where a factor $|e^{i\omega|r|/c}|^2 = 1$ has been extracted from the integrand.

The integration by parts (really just the chain rule for differentiation) led to two terms, one of which was of the form $\int d_{t'}(UV)dt'$, which vanished under the assumption that $\mathbf{E}$ falls off rapidly at $t \to \pm\infty$.

Radiation from Non-relativistic systems

In the non-relativistic limit $\beta << 1$. Also the ratio of the second term on the right of (242) to the first term on the right is $E_{rad}/E_{vel} = R\dot{u}/c^2$. For a particular frequency of oscillation ν

$$E_{rad}/E_{vel} = uR\nu/c^2 = (u/c)(R/\lambda). \tag{251}$$

For the near zone, $R \leq \lambda$, the velocity field dominates, whilst for the far zone $R \geq \lambda c/u$, the radiation field dominates.

When $\beta << 1$,

$$\mathbf{E}_{rad} = (q/c)[\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \dot{\vec{\beta}})/R] \tag{252}$$

and

$$\mathbf{B}_{rad} = [\hat{\mathbf{n}} \times \mathbf{E}_{rad}]. \tag{253}$$

Then

$$E_{rad} = B_{rad} = \frac{q\dot{\beta}}{Rc} sin\Theta, \tag{254}$$

where Θ is the angle between $\dot{\vec{\beta}}$ and $\hat{\mathbf{n}}$. The Poynting flux is given by

$$\frac{dW}{dAdt} = S = \frac{c}{4\pi}E_{rad}^2 = \frac{c}{4\pi}\frac{q^2\dot{\beta}^2}{R^2c^2} sin^2\Theta. \tag{255}$$

The vector Poynting flux $\mathbf{S} \propto \mathbf{E} \times \mathbf{B}$ is directed along $\hat{\mathbf{n}}$. Using $dA = R^2 d\Omega$ we have

$$\frac{dW}{dtd\Omega} = \frac{q^2\dot{\beta}^2}{4\pi c} sin^2\Theta. \tag{256}$$

The total power emitted is then

$$dW/dt = \frac{q^2\dot{\beta}^2}{2c}\int_{-1}^{1}(1 - cos^2\Theta)d(cos\Theta)) = \frac{2q^2\dot{\beta}^2}{3c}, \tag{257}$$

which is Larmor's formula for emission from a single particle. (fig3.3)

Note that as a function of angle, there is a dipole pattern: no radiation emitted along the direction of acceleration. The maximum is emitted perpendicular to the acceleration. Also, the instantaneous direction of $\mathbf{E}_{rad}$ is determined by $\dot{\vec{\beta}}$ and $\mathbf{n}$ and 1-D acceleration means linear polarization in the $\dot{\vec{\beta}}$ and $\mathbf{n}$ plane.

Dipole Approximation

For a system of particles, differences in the retarded times can be ignored if the time scales for a system to change are $\tau >> L/c$, the light crossing time. Since this also characterizes an inverse frequency of radiation, the equivalent condition for ignoring retarded time differences is if $L << c/\omega = \lambda$ Since the time for variation also characterizes the time for change in velocity of a particle, a long time for change in velocity over a particle orbit of radius l implies $\tau \sim l/u >> L/c$. But $l \leq L$, so $u << c$ is also an expression of the same limit.

We can use the non-relativistic radiation fields in these regimes (generalizing (252) for many particles)

$$\mathbf{E}_{rad} = \sum_a \frac{q_a}{c}[\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \dot{\vec{\beta}})/R_a]. \tag{258}$$

For large distances from the source, $R_a \sim R_0$, where R_0 can be defined as the distance from the geometric centroid of the particle system to the observer. Then

$$\mathbf{E}_{rad} = [\frac{1}{c^2 R_0}\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \ddot{\mathbf{d}})], \tag{259}$$

where $\mathbf{d} = \sum_a q_a \mathbf{r}_a \simeq \int \rho_c \mathbf{r} d^3\mathbf{r}$, where the latter similarity follows for a nearly continuous charge distribution. We have

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} sin^2\Theta \tag{260}$$

and

$$P = \frac{2\ddot{\mathbf{d}}^2}{3c^3} \tag{261}$$

fig (3.5)

Using Fourier transforms

$$d(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\tilde{d}(\omega)d\omega \tag{262}$$

so

$$\ddot{d}(t) = -\int_{-\infty}^{\infty} \omega^2 e^{-i\omega t}\tilde{d}(\omega)d\omega \tag{263}$$

and

$$dW/d\omega d\Omega = \frac{1}{c^3}\omega^4|\tilde{d}(\omega)|^2 sin^2\Theta \tag{264}$$

and

$$dW/d\omega = \frac{8\pi}{3c^3}\omega^4|\tilde{d}(\omega)|^2 \tag{265}$$

It can be shown that the dipole term represents the first term in the small expansion parameter L/λ. (or $L\omega/c \sim (L/l)(u/c) << 1$), where l is the particle orbit scale and u is the particle velocity. As the system gets more and more relativistic, the lowest order expansion is not sufficient. The frequencies which show up in the radiation spectrum are not only that associated with the dipole oscillation of the particles, but higher harmonics, where the dipole produces the primary frequency ω_0 but and the quadrupole etc. produce the higher integer harmonics.

LECTURE 9

Thomson Scattering

We want to apply the dipole formula to the case for a charge radiating in response to an impinging wave. Ignore $\vec{\beta} \times \mathbf{B}$ force for non-relativistic velocities. The electric force is the main contributor and we have the equation of motion for forcing from a linear polarized wave

$$m\ddot{\mathbf{r}} = \mathbf{F} = e\vec{\epsilon}E_0 sin\omega_0 t, \tag{266}$$

or

$$\ddot{\mathbf{d}} = (e^2/m)\vec{\epsilon}E_0 sin\omega_0 t, \tag{267}$$

where e is the charge and $\vec{\epsilon}$ is the direction of E_0. The solution is an oscillating dipole

$$\mathbf{d} = e\mathbf{r} = -(e^2 E_0/m\omega_0^2) sin\omega_0 t. \tag{268}$$

fig (3.6)

Time averaged power is then

$$dP/d\Omega = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} sin^2\Theta = \frac{e^4 E_0^2}{8\pi m^2 c^3} sin^2\Theta, \tag{269}$$

where we used $\mathrm{Lim}_{T\to\infty}(1/T)\int_0^T sin^2(\omega t)dt = 1/2$. We also have upon integration over solid angle $P = e^4 E_0^2/3m^2c^3$.

Define the differential cross section $d\sigma$ for scattering an initial electromagnetic wave into solid angle $d\Omega$ from interaction with the particle. Then

$$\frac{dP}{d\Omega} = \langle S\rangle \frac{d\sigma}{d\Omega} = \frac{cE_0^2}{8\pi}\frac{d\sigma}{d\Omega}, \tag{270}$$

therefore from (269)

$$\frac{d\sigma}{d\Omega_{polarized}} = r_o^2 sin^2\Theta, \tag{271}$$

where

$$r_0^2 = \frac{e^4}{m^2 c^4}. \tag{272}$$

Note here that $\langle S \rangle$ is the incident flux.

For an electron, $r = 2.8 \times 10^{-13}$cm, and is called the classical electron radius. Integrating over solid angle gives

$$\sigma = 8\pi r_0^2/3 \tag{273}$$

which is $\sigma_T = 6.6 \times 10^{-25}\mathrm{cm}^2$, called the Thomson cross section for an electron. This remains valid when $h\nu < 0.5MeV$, the electron rest mass. Above this we are in the quantum regime, and the "Klein Nishina" cross section must be used (discussed later). Dipole approximation is also invalid for relativistic motions.

Polarization of Electron scattered radiation

The scattered radiation is polarized in the plane spanned by initial polarization vector $\vec{\epsilon}$ and $\hat{\mathbf{n}}$.

Differential cross section (as a function of angle into which radiation is scattered) for scattering of initially unpolarized radiation is found by writing the incident radiation as the sum of two orthogonal linearly polarized beams.

Take $\vec{\epsilon}_1$ to be in plane of incident and scattered beam.

Take $\vec{\epsilon}_2$ to be $\perp$ to this plane.

Take Θ as the angle between $\vec{\epsilon}_1$ and $\hat{\mathbf{n}}$. $\vec{\epsilon}_2 \perp \hat{\mathbf{n}}$, and $\mathbf{n}$ is the direction of the scattered wave.

(fig3.7)

Then using (271) the differential cross section for scattering into $d\Omega$ from an initially unpolarized beam is

$$d\sigma/d\Omega|_{unpo} = \frac{r_o^2}{2}[d\sigma(\Theta)/d\Omega + d\sigma(\pi/2)/d\Omega] = \frac{1}{2}(1 + sin^2\Theta) = \frac{1}{2}(1 + cos^2\theta), \tag{274}$$

where $\theta = \pi/2 - \Theta$. This depends only on the angle between the direction of the incident and scattered radiation. Note that integrated cross section for polarized radiation is the same as for the unpolarized case (because electron has no intrinsic direction and energy is conserved).

Recall that $\mathbf{n}$, the outgoing wave direction, is perpendicular to the polarization vector of the scattered radiation. The two differential cross sections on the right of (274) refer to the intensities of these two planes of polarization.

The ratio of the differential cross section for scattering into the plane and perpendicular to the plane of scattering is $cos^2\theta$, so we have, from (212)

$$\Pi = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{1 - cos^2\theta}{1 + cos^2\theta}. \tag{275}$$

This means that initially unpolarized radiation can be polarized upon scattering. This makes sense. If we look along incident direction, then we get no polarization since there is azimuthal symmetry around the initial wave vector axis and all directions are the same. If we look $\perp$ to the initial wave vector axis $\theta = \pi/2$, then we see 100% polarization. This is because the particle moves in the plane $\perp$ to the initial direction.

Radiation Damping Force

The radiation damping force can be assumed to be perturbation on the particles motion when $T_{rad} >> t_p$, i.e. the time over which the particles energy is changed significantly by radiation is much longer than the time for the particle to change its position (e.g. orbit time).

$$T_{rad} \sim mv^2/P_{rad} = \frac{3mc^3}{2e^2}\left(\frac{u}{\dot{u}}\right)^2. \tag{276}$$

This then requires that $t_p >> \tau = r_0/c = 10^{-23}$s, the light crossing time of a classical electron radius.

For scattering problems, an example of this holds when the distance electrons wander in an atom (typically of order 1 Angstrom) is $<<$ than the wavelength of the radiation—that is the wavelength on which the electric field varies (e.g. $>>$ 1 Angstrom.)

The radiative loss of energy, time averaged is

$$\langle dW/dt \rangle = P = (2e^2/3c^3)\langle|\ddot{\mathbf{x}}|^2\rangle, \tag{277}$$

where

$$\langle|\ddot{\mathbf{x}}|^2\rangle = \frac{1}{t_p}\int_{t-t_p/2}^{t+t_p/2} \ddot{\mathbf{x}}^2 dt, \tag{278}$$

where $t_p = 2\pi/\omega$. Since we are considering $\mathbf{x} = x_0 e^{i\omega t}$, upon integrating by parts we note that products of odd and even numbers of derivatives produce a real part that cancels out, the only term that survives is

$$\langle |\ddot{\mathbf{x}}|^2 \rangle = -\frac{1}{t_p} \int_{t-t_p/2}^{t+t_p/2} \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} dt, \tag{279}$$

We can then write

$$\langle dW/dt \rangle = -\langle F_{rad} \cdot \dot{\mathbf{x}} \rangle, \tag{280}$$

so

$$F_{rad} = \frac{2e^2}{3c^3} \dddot{\mathbf{x}}. \tag{281}$$

Radiation from Harmonically Bound Particles, Classical Line Profile

Assume that particle is bound to a center of force. Then its motion represents oscillation and its equation of motion

$$\mathbf{F} = -k\ddot{\mathbf{r}} = -m\omega_0^2 \mathbf{r}, \tag{282}$$

such that the oscillation is with frequency ω_0. A classical oscillator (Thomson atom) is not ultimately applicable, though quantum results are quoted in relation to the classical results.

The radiation reaction force for a radiating particle damps the oscillator. We assume that the damping is a perturbative effect. From (276), the validity of this classical non-relativistic regime is when $\tau/t_p << 1$ so that, τ, the light crossing time over a classical electron radius is short compared to any particle orbit period. This means the time scale to lose significant energy to radiation is long compared to particle orbit time. For oscillation along the $x - axis$:

$$-\tau \dddot{x} + \ddot{x} + \omega_0^2 x = 0. \tag{283}$$

If the 3rd derivative term is small we can assume the motion is harmonic to lowest order. Then we assume $x = cos(\omega_0 t + \phi_0)$ to this order. We then have $\tau \dddot{x} = -\omega_0^2 \dot{x}$ so

$$\ddot{x} + \omega_0^2 \tau \dot{x} + \omega_0^2 x = 0. \tag{284}$$

To solve assume $x(t) = x_0 e^{\alpha t}$ so that

$$\alpha^2 + \omega_0^2 \tau \alpha + \omega_0^2 = 0. \tag{285}$$

The solution is

$$\alpha = \pm i\omega_0 - \frac{1}{2}\omega_0^2\tau + O(\tau^2\omega_0^3), \tag{286}$$

obtained by solving quadratic and expanding for $\omega\tau << 1$. At $t = 0$ take $x(0) = x_0$ and $\dot{x}(0) \simeq 0$ [i.e. $\dot{x}(0) \sim \omega_0 x_0(\omega_0\tau) << \omega_0 x_0$]. We thus have

$$x(t) = x_0 e^{-\Gamma t/2} cos\omega_0 t = \frac{1}{2}x_0(e^{-\Gamma t/2 + i\omega_0 t} + e^{-\Gamma t/2 - i\omega_0 t}). \tag{287}$$

where $\Gamma = \omega_0^2\tau$.

If we Fourier transform we can find the power spectrum of the emitted radiation in the dipole approximation. We have

$$\tilde{x}(\omega) = \frac{1}{2\pi}\int_0^\infty x(t')e^{iwt'}dt' = \frac{x_0}{4\pi}\left[\frac{1}{\Gamma/2 - i(\omega+\omega_0)} + \frac{1}{\Gamma/2 - i(\omega-\omega_0)}\right] \sim \frac{x_0}{4\pi}\left[\frac{1}{\Gamma/2 - i(\omega-\omega_0)}\right], \tag{288}$$

where we ignore the first term on the right as we are working for the regime of ω near ω_0. The energy radiated per unit frequency is then

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\tilde{d}(\omega)|^2 = \left(\frac{8\pi\omega^4 e^2}{3c^3 16\pi^2}\right) x_0^2 \left[\frac{1}{(\Gamma/2)^2 + (\omega_0 - \omega)^2}\right]. \tag{289}$$

The power spectrum is then a Lorentz profile. (fig 3.8)

The power spectrum for a particle damped by radiation–classical line. There is a sharp max near $\omega = \omega_0$, and the width is $\delta\omega \sim \Gamma = \omega_0^2\tau << 1$. The quantity in curved parentheses can be thought of as a spring constant, characterizing the potential energy of the particle. Integrating (289) over frequency gives the total energy of $\frac{1}{2}kx_0^2$. This is the energy that is being radiated away.

Driven Harmonically Bound Particles

In this case we have a forcing from beam of radiation, so the equation of motion is

$$\ddot{x} = -\omega_0^2 x + \tau\dddot{x} + (eE_0/m)cos\omega t. \tag{290}$$

Now take x to be complex so we have

$$\ddot{x} + \omega_0^2 x - \tau \dddot{x} = (eE_0/m)e^{i\omega t}. \tag{291}$$

The solution is

$$x = x_0 e^{i\omega t} \equiv |x_0| e^{i(\omega t + \delta)}, \tag{292}$$

where

$$x_0 = \frac{eE_0/m}{w^2 - \omega_0^2 - i\omega_0^3 \tau} \tag{293}$$

and

$$\delta = tan^{-1}\left[\frac{\omega^3 \tau}{w^2 - \omega_0^2}\right]. \tag{294}$$

(We find the latter by writing x_0 in the form $x_0 = a/(b - ic) = \frac{a(b+ic)}{a^2+b^2} = qcos\delta + iqsin\delta$, with a, b, c, q, δ all real. Then $qcos\delta = \frac{ab}{b^2+c^2}$ and $qsin\delta = \frac{ac}{b^2+c^2}$ so $tan\delta = c/b$.)

The damping term from the radiation reaction provides a phase shift. For $w > w_0$, $\delta > 0$ and for $\omega < \omega_0$, $\delta < 0$. This represents an oscillating dipole with frequency ω and amplitude $|x_0|$.

The time averaged power radiated is then

$$P(\omega) = e^2|x_0|^2\omega^4/3c^3 = \frac{e^4 E_0^2}{3m^2c^2}\frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega_0^3\tau)^2}. \tag{295}$$

Dividing by the time averaged Poynting flux $\langle S \rangle = cE_0^2/8\pi$ we "project out" the cross section for scattering

$$\sigma = \sigma_T \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega^3\tau)^2}. \tag{296}$$

For $\omega >> \omega_0$ $\sigma(\omega) \to \sigma_T$. Basically this is because at high incident energies, the frequency of the particle motion is not seen by the radiation, so it appears free.

For $\omega << \omega_0$, we have $\sigma \propto \omega^4$. This is the Rayleigh-Scattering regime—bluer frequencies scattered more (thus looking at horizon means we the Sun to be redder). Inertial term in equation of motion is small, so the dipole moment of the particle is directly proportional to the incident radiation. $kx = eE$. (e.g. (290) with $\ddot{x}$ and $\dddot{x}$ terms dropped.)

For $\omega \sim \omega_0$ we have $\omega^2 - \omega_0^2 \sim 2\omega(\omega - \omega_0)$ so

$$\sigma(\omega) = \frac{\sigma_T}{\tau}\frac{\Gamma}{4(\omega-\omega_0)^2+\Gamma^2} = \frac{2\pi^2 e^2}{mc}\frac{\Gamma}{4(\omega-\omega_0)^2+\Gamma^2}, \tag{297}$$

recalling that $\Gamma = \omega_o^2\tau$. This is the Lorentz profile. This result arises because free oscillations of the unforced oscillator can be excited by a pulse of radiation $E(t) = \delta(t)$. The spectrum for this pulse is independent of frequency. (White spectrum) and so the emission is just scattering of a white spectrum. The emission is then proportional to the cross section for this scattering. $P = \langle S\rangle\sigma$

If we integrate $\sigma(\omega)$ over ω we find that

$$\int_0^\infty \sigma(\omega)d\omega = \frac{2\pi^2 e^2}{mc} \sim \sigma_T/\tau. \tag{298}$$

But this is not valid because at large ω the classical approximation breaks down (classical requires $\omega\tau << 1$, where τ is light crossing time across classical electron radius). Actually we have

$$\int_0^{\omega_{max}} \sigma(\omega)d\omega = \sigma_T\omega_{max}, \tag{299}$$

because the range of validity of the classical radiation reaction is, $\omega_{max} << 1/\tau = c/r_0$.

For quantum regime however, we write

$$\int_0^\infty \sigma(\omega)d\omega = (\sigma_T/\tau)f_{mm'} \sim \sigma_T/\tau, \tag{300}$$

where $f_{mm'}$ is the oscillator strength for $m \to m'$ transitions. (fig 3.9)

Correspondence between emission and Einstein Coeff.

Note the power radiated in the Larmor formula time averaged over 1 cycle for a harmonic oscillator is

$$P = \frac{e^2\omega_0^4}{3c^3}|x_0|^2. \tag{301}$$

But this should equal $A_{21}h\nu_{21}$ using the form of the Einstein coefficients, where ν_{21} is the resonance line frequency. We can then say that $|x_0|/2 = \langle 2|\mathbf{x}|1\rangle$ is 1/2 the displacement of the oscillation representing the expectation value between the transition from state 2 to state 1. Then we would have

$$A_{21} = \frac{8\pi e^2\omega_{21}^3}{3c^3h}|\langle 2|\mathbf{x}|1\rangle|^2, \tag{302}$$

which turns out to be not a bad approximation for specific cases in in the quantum regime (e.g. when the statistical weights of the two states are non-degenerate).

LECTURE 10

Basic Special Relativity

We review some concepts in special relativity very quickly.

First note the basic postulates:

1) laws of physics take the same form in frames in relative uniform motion.

2) speed of light is the same in free space for these frames, independent of their relative velocity.

The appropriate transformations of the coordinates x, t that preserve these relations between two frames K and K' such that K' is moving with positive velocity v on the x axis for a fixed observer in K are

$$x' = \gamma(x - vt);\; y = y'; z = z' \tag{303}$$

and

$$t' = \gamma(t - vx/c^2), \tag{304}$$

where $\gamma = 1/(1 - v^2/c^2)^{1/2}$ We talk about a 4-dimensional space-time coordinate taking place at $(t, \mathbf{x})$.

Note that a consequence of the constant speed of light is that for a pulse of light emitted at $t = 0$ where we assume the origins coincide is

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 = 0. \tag{305}$$

Length Contraction

Consider a rigid rod of length $L' = x_2' - x_1'$ carried in rest frame K'. What is the length in the unprimed frame? The length is given by $L = x_2 - x_1$ where x_2 and x_1 are the rod positions as measured at fixed time in L. We have

$$L' = x_2' - x_1' = \gamma(x_2 - x_1) = \gamma L. \tag{306}$$

Thus the length we measure at a fixed time in L is smaller than the proper length measured in L'. The length of the moving rod is contracted.

Time Dilation

Assume a clock at rest in K' measure a time interval $T_0 = t_2' - t_1'$ at fixed position $x' = 0$. This implies that

$$T = t_2 - t_1 = \gamma(t_2' - t_1') = \gamma T'. \tag{307}$$

Thus we see that more time elapses in the unprimed (=lab) frame per unit time in the source (moving) frame. Thus “moving clocks run slow.” This is the time dilation.

The point is that clocks at two different positions in K are not simulatenously synchronized as seen by K'. When $x_2' = x_1'$ in K', $x_2 \neq x_1$. At these two locations in K, clocks in K do not appear to be synchronized as measured by an observer in K' even if they are synchronized as seen by an observer in K.

Transformation of Velocities and Beaming of Radiation

Write the transformations in differential form for boost along x:

$$dx = \gamma(dx' + vdt');\;\; dy = dy';\;\; dz = dz' \tag{308}$$

and

$$dt = \gamma(dt' + vx'/c^2), \tag{309}$$

where we have also taken advantage of the symmetry of relative frames and let $v \rightarrow -v$ and $t', \mathbf{x}' \rightarrow t, \mathbf{x}$.

Then we have

$$u_x = dx/dt = \frac{\gamma(dx' + vdt')}{\gamma(dt' + vdx'/c^2)} = \frac{u_x' + v}{1 + u_x' v/c^2} \tag{310}$$

and

$$u_y = \frac{u_y'}{\gamma(1 + u_x' v/c^2)} \tag{311}$$

and

$$u_z = \frac{u_z'}{\gamma(1 + u_x' v/c^2)}. \tag{312}$$

fig 4.2

In general,

$$u_{||} = \frac{u_{||}' + v}{1 + u_{||}' v/c^2} \tag{313}$$

and

$$u_{\perp} = \frac{u_{\perp}'}{\gamma(1 + u_{\perp}' v/c^2)}. \tag{314}$$

Thus we can take the ratio of the perpendicular to the parallel velocities as measured in the lab frame (K). We have

$$tan\theta = u_\perp/u_{||} = \frac{u'_\perp}{\gamma(1+u'_\perp v/c^2)}\frac{1+u'_{||}v/c^2}{u'_{||}+v} = \frac{u'_\perp}{\gamma(u'_{||}+v)} = \frac{u'sin\theta'}{\gamma(u'cos\theta'+v)}. \quad (315)$$

For $u' = c$, corresponding to photons emitted in the source frame, we have

$$tan\theta = \frac{sin\theta'}{\gamma(cos\theta' + v/c)} \quad (316)$$

and

$$cos\theta = \frac{cos\theta' + v/c}{(1 + cos\theta' v/c)}; \; sin\theta = \frac{sin\theta'}{\gamma(1 + cos\theta' v/c)} \quad (317)$$

At $\theta' = \pi/2$, we have

$$tan\theta = \frac{1}{\gamma v/c}; \; cos\theta = v/c, \quad (318)$$

so that $sin\theta = 1/\gamma$.

This means that if a source is emitting isotropically in its rest frame (K' frame) then in the K frame, 50% of the emission is beamed into a cone of angle $1/2\gamma$.

This has many consequences in astrophysics. Both in the microphysics of radiation from moving particles, and also for relativistic bulk outflows such as Active Galactic Nuclei and Gamma-Ray Bursts. It means for example, that a for sufficiently relativistic and radiating bipolar jet, we would only see emission from the flow moving toward our line of sight.

Doppler Effect

It is particularly important in astrophysics to distinguish between local observers (those omnipresent entities that have a rod and clock at each point in space time) and actual observers (those like us fixed at a point). There is no Doppler effect for local observers.

In the rest frame of K consider the moving source to move from point 1 to point 2 and emit one radiation cycle. The period as measured in the lab frame (K) is given by the time dilation effect

$$\Delta t = \gamma \Delta t' = 2\pi\gamma/\omega', \quad (319)$$

but difference in arrival times must be considered because the source is moving relative to the observer. Consider the distance d to be positive if the source moves toward the observer. (fig 4.4)

Then accounting for the difference in arrival times gives

$$\Delta t_{obs} = \Delta t - d/c = \Delta t(1 - (v/c)cos\theta) = \gamma\Delta t'(1 - (v/c)cos\theta), \tag{320}$$

where θ is now the angle between the direction of motion and the line of sight as measured in the lab frame (K). Rearranging we then have

$$\omega_{obs} = 2\pi/\Delta t_{obs} = \frac{\omega'}{\gamma(1 - (v/c)cos\theta)} \tag{321}$$

or as the inverse

$$\omega_{obs} = \omega'\gamma(1 + (v/c)cos\theta'), \tag{322}$$

where θ' is measured in the K' frame. Note that the γ is a purely relativistic effect whereas the $(1 + (v/c)cos\theta')$ is also a non-relativistic effect.

Combined Doppler Effect and Special Relativity

Rearranging (321) and assuming the case for which $\theta \sim 0$, and $v \sim c$ we have

$$\omega' \simeq \omega_{obs}\gamma(1 - (v/c)cos\theta) = \gamma\frac{(1 - v/c)(1 + v/c)}{(1 + v/c)} \sim \omega_{obs}/2\gamma. \tag{323}$$

or

$$\Delta t_{obs} \sim \Delta t'/2\gamma. \tag{324}$$

Compare this to (319): the γ is in the opposite place! Thus there is no time dilation but rather time *contraction* for a source moving at the observer (see E. Blackman 1998, Eur. J. Phys. 19 195 for more discussion.) "A clock moving at the observer runs fast!" This highlights that for astrophysics, where we the observer are at a fixed location, that the standard time dilation and Lorentz contraction effects must be considered carefully. (See Rees 1966, Nature 211 468 for superluminal motion effects; and).

Proper time

Note that the quantity $d\tau$, the proper time, is in an invariant:

$$c^2 d\tau^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2) = c^2 d\tau'^2. \tag{325}$$

It is called the proper time because it measures the time interval between events occurring at the same spatial location. Dividing by c^2 this equation implies $dt^2(1 - v^2/c^2) = d\tau'^2$ or $dt^2 = \gamma^2 d\tau'^2$ which is the time dilation effect.

Notes on 4 vectors

The quantity $x^\mu = (ct, x, y, z)$ is a four-vector, in that it obeys the transformation properties previously described. The Greek indices indicate 4 components. Note also that

$$x_\mu = \eta_{\mu\nu} x^\nu \tag{326}$$

where summation is over repeated indices and $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$, -1 for $\mu = \nu = 0$ and 1 for $\mu = \nu > 0$. The Lorentz transformations can be written

$$x^{\mu\prime} = \Lambda^{\mu'}_{\nu} x^\nu \tag{327}$$

where $\Lambda^{\mu'}_{\nu}$ is the transformation matrix, which, for a boost along x satisfies $\Lambda^{\mu'}_{\nu} = \gamma$ for $\mu = \nu = 0, 1$; 1 for $\mu = \nu = 2, 3$; $-\beta\gamma$ for $\mu = 0, \nu = 1$, and $-\beta\gamma$ for $\mu = 1, \nu = 0$. General Lorentz transformations have more complicated matrices. We restrict ourselves to isochronous ($\Lambda^0_0 \geq 0$), and proper transformations $det\Lambda = +1$.

The norm of x^μ is given by

$$x^\mu x_\mu = \eta_{\mu\nu} x^\mu x^\nu. \tag{328}$$

Since this is a Lorentz invariant, we have

$$\Lambda^{\tau'}_{\nu} \tilde{\Lambda}^{\nu}_{\sigma'} = \delta^{\tau'}_{\sigma'} \tag{329}$$

where $\delta^{\tau'}_{\sigma'}$ is the Kronecker delta.

All 4-vectors transform in the same way as x^μ under a Lorentz transformation. Some other 4-vectors include $k^\mu = (\omega/c, \mathbf{k})$ for a electromagnetic wave, the 4-velocity $U^\mu = (\gamma c, \gamma \mathbf{v})$, the current density 4-vector $j^\mu = (\rho c, \mathbf{j})$. the momentum 4-vector $(\gamma E/c, \gamma \mathbf{p})$.

The transformation laws also hold for tensor indices. Thus two Λ matrices are required for a two index tensor. That is for example

$$A^{\mu'\nu'} = \Lambda^{\mu'}_{\sigma}\Lambda^{\nu'}_{\delta}A_{\sigma\delta}. \tag{330}$$

It turns out that the electric and magnetic field are not separately 4-vectors but comprise part of of an electromagnetic tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{331}$$

where $A_\mu = (-\phi, A_i)$ is the 4-vector potential with components (Latin indices mean only 1,2,3 components.) $F_{0i} = -E_i$, $F_{i0} = E_i$, $F_{00} = 0$, $F_{12} = B_z$, $F_{21} = -B_z$, $F_{13} = -B_y$, $F_{31} = B_y$, $F_{23} = B_x$, $F_{32} = -B_x$, and the rest zeros.

Maxwell's equations can then be written

$$\partial^\nu F_{\mu\nu} = 4\pi j_\nu / c \tag{332}$$

which includes $\nabla \cdot \mathbf{E} = 4\pi\rho$ and $\nabla \times \mathbf{B} = c^{-1}\partial_t \mathbf{E} + 4\pi \mathbf{j}/c$. and

$$\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0, \tag{333}$$

which includes $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} + c^{-1}\partial_t \mathbf{B} = 0$. Note that $F_{\mu\nu}(\mathbf{E}) = \eta_{\mu\sigma}\eta_{\nu\lambda}F^{\sigma\lambda}(\mathbf{E}) = F^{\mu\nu}(-\mathbf{E})$.

Lorentz invariants are $F^{\mu\nu}F_{\mu\nu} = 2(\mathbf{E}^2 - \mathbf{B}^2)$ and $Det(F) = (\mathbf{E} \cdot \mathbf{B})^2$.

Lorentz Transformations of the Fields

For a boost with velocity $\vec{\beta}$ we have

$$\mathbf{E}'_{||} = \mathbf{E}_{||}, \tag{334}$$

$$\mathbf{B}'_{||} = \mathbf{B}_{||} \tag{335}$$

and

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \vec{\beta} \times \mathbf{B}) \tag{336}$$

and

$$\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \vec{\beta} \times \mathbf{E}). \tag{337}$$

For a physical idea of the transformations, consider a charged capacitor with surface charge density σ' in its rest frame and plate separation d. (fig)

Place the capacitor plates perpendicular to the direction of motion (x). In the lab frame K the capacitor moves with speed v_x. Since $E'_{||} = 4\pi\sigma'$, and the area does not contract with this motion as seen in K, $E_{||} = E'_{||}$. There is no surface current either.

Now rotate the plates 90 degrees so that they are parallel to the direction of motion, parallel to the x-z plane. Now the perpendicular electric field $E_y = E'_y\gamma$ because the surface charge density as seen in K increases from the Lorentz contraction along the direction of motion (length of plate appears smaller). Now since the current per area and charge density transform as a 4-vector we have

$$\mathbf{j} = \gamma(\mathbf{j}' + v_x\hat{\mathbf{x}}\sigma'/d) \tag{338}$$

and since $\mathbf{j}' = \mathbf{0}$ we have the only component $j_x = \gamma v_x\sigma'/d$. Now the current is also given by $\nabla \times \mathbf{B}$. Since the only direction of variation is perpendicular to the plates, (the y direction in the K frame) we have

$$(\nabla \times \mathbf{B})_x = \partial_y B_z = 4\pi j_x/c, \tag{339}$$

so that

$$B_z \simeq 4\pi j_x d/c = 4\pi\gamma v_x\sigma'. \tag{340}$$

We can see that this result also follows directly from (337) which gives

$$B_z = \gamma(B'_z + \beta_x E'_y) = 4\pi\gamma v_x\sigma', \tag{341}$$

since $B'_z = 0$. This exemplifies one restricted case of the transformations above.

Fields of Uniformly Moving Charge

We want to apply the transformation relations for the fields to a charge moving with constant velocity along x. In the primed (rest) frame we have

$$E'_x = qx'/r'^3;\ B'_x = 0 \tag{342}$$

$$E'_y = qy'/r'^3;\ B'_y = 0 \tag{343}$$

$$E'_z = qz'/r'^3;\ B'_z = 0, \tag{344}$$

with $r' = (x'^2 + y'^2 + z'^2)^{1/2}$. We use the inverse transform for the relations (334-337), ($\beta \to -\beta$, and switch primed and unprimed quantities) to find

$$E_x = qx'/r'^3;\ B_x = 0 \tag{345}$$

$$E_y = q\gamma y'/r'^3;\ B_y = -q\gamma\beta z'/r'^3 \tag{346}$$

$$E_z = qz'/r'^3;\ B_z = q\gamma\beta y'/r'^3. \tag{347}$$

But these are in the primed coordinates. We can transform to unprimed coordinates by using the simple coordinate transformations to get

$$E_x = q\gamma(x - vt)/r'^3;\ B_x = 0 \tag{348}$$

$$E_y = q\gamma y/r'^3;\ B_y = -q\gamma\beta z/r'^3 \tag{349}$$

$$E_z = qz/r'^3;\ B_z = q\gamma\beta y/r'^3, \tag{350}$$

with now $r' = (\gamma^2(x - vt)^2 + y^2 + z^2)^{1/2}$.

Now let's see that these fields are exactly the same as those derived from retarded time and the Lienard-Wichart potentials. In particular, we show that the electric fields just calculated are the same as the first term on the right of (242) which is the constant velocity (including zero velocity) contribution to the electric field. Let $t_{ret} = t - R/c$, $z = 0$. fig 4.5

Then we have

$$R^2 = y^2 + (x - vt + vR/c)^2 \tag{351}$$

so solving for R gives

$$R = \gamma^2\beta(x - vt) + \gamma(y^2 + \gamma^2(x - vt)^2)^{1/2}. \tag{352}$$

The unit vector to the field point from the retarded time position is then

$$\hat{\mathbf{n}} = \frac{y\hat{\mathbf{y}} + (x - vt + vR/c)\hat{\mathbf{x}})}{R} \tag{353}$$

and

$$\kappa = 1 - \mathbf{n}\cdot\vec{\beta} = (y^2 + \gamma^2(x - vt)^2)^{1/2}/\gamma R. \tag{354}$$

Thus

$$\frac{q}{\gamma^2 R^2 \kappa^3} = \frac{\gamma R q}{(y^2 + \gamma^2(x - vt)^2)^{3/2}}. \tag{355}$$

If we combine the last 3 equations with (348) (349), and (350) we have

$$\mathbf{E} = \left[q(\hat{\mathbf{n}} - \vec{\beta})(1 - \beta^2)/\kappa^3 R^2 \right] \tag{356}$$

which is the same result we get for the velocity field of a moving charge from the Liénard-Wiechart potentials.

LECTURE 11
Field from a strongly relativistic charge
fig 4.6

For $\gamma >> 1$ at field point at $x = 0, y = b$ we have

$$E_x = -\frac{qv\gamma t}{(\gamma^2 v^2 t^2 + b^2)^{3/2}};\ B_x = 0 \tag{357}$$

$$E_y = \frac{q\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}};\ B_y = 0 \tag{358}$$

$$E_z = 0;\ B_z = \beta E_y \sim E_y. \tag{359}$$

Fields are are concentrated mainly in plane transverse to motion ($E_y direction$) in cone of angle $1/\gamma$ (found from ratio of $(E_x/E_y)_{max}$). Book calculates spectrum for this uniformly moving particle. The key point is that the spectrum has peaks at frequency given by $1/\Delta t = \gamma v/b$ which is the time period for which the electric field is significant–very short–thus a very broad peak in spectrum. $\tilde{E}^2(\omega)_{max} \sim E^2_{y,max}(\Delta t)^2 \sim (q\gamma/b^2)^2(b/\gamma v)^2 \sim q^2/(b^2v^2)$. Spectrum nearly flat for $\Delta\omega \sim 1/\Delta t \sim \gamma v/b$.
fig 4.7

Relativistic Mechanics
The four-momentum of a particle is given by

$$p^\mu = mu^\mu = mu^\mu = m(c\gamma, \gamma\mathbf{v}) = (E/c, \gamma m\mathbf{v}). \tag{360}$$

Expanding the zeroth component in the non-relativistic regime

$$p^0 c = mc^2(1-\beta^2)^{-1/2} = m(c^2 + v^2/2) + ... \tag{361}$$

which is the rest energy plus the kinetic energy. The norm is

$$p^\mu p_\mu = -m^2c^2 = -E^2/c^2 + \mathbf{p}^2 \tag{362}$$

or

$$E^2 = m^2c^4 + c^2\mathbf{p}^2 = (\gamma mc^2)^2. \tag{363}$$

For a photon we have

$$p^\mu = \frac{h}{2\pi}(\omega/c, \mathbf{k}), \tag{364}$$

and $p^\mu p_\mu = 0$.

If we take the derivative with respect to the proper time (a Lorentz invariant scalar) we have the acceleration 4-vector.

$$a^\mu = du^\mu/d\tau. \tag{365}$$

The generalization of Newton's law is

$$F^\mu = ma^\mu. \tag{366}$$

Note that

$$F^\mu u_\mu = \frac{m}{2} d(u^\mu u_\mu)/d\tau = 0. \tag{367}$$

This implies that 4-surfaces of u_μ and F^μ are orthogonal, so that as you move along F^μ as a function of its parameters, you are moving along different 4-planes in u^μ. This means that the four force is a function of u^μ so 4-force has a dependence on 4-velocity.

For the case of the electromagnetic field, the covariant generalization of the Lorentz force

$$\mathbf{F} = qE + \mathbf{v} \times \mathbf{B}/c \tag{368}$$

can be written in terms of the electromagnetic tensor as

$$F^\mu = \frac{q}{c} F^\mu_\nu u^\nu = ma^\mu. \tag{369}$$

This indeed has a 0th component which reduces to

$$dW/dt = q\mathbf{E} \cdot \mathbf{v} \tag{370}$$

and the ith components

$$dp_i/dt = qE_i + (\mathbf{v} \times \mathbf{B})_i/c. \tag{371}$$

Total Emission from Relativistic Particles

Move into "local Lorentz frame." Even if particle is accelerating, we can move into a frame for which particle moves non-relativistically for short times away from the initial time. Then we use the dipole formula, and then transform back to lab frame.

The radiation has zero momentum for "front-back symmetric" emission in this source frame. Using the lorentz transformation for the momentum and positon 4-vectors only the zeroth components then matters in the primed (co-moving) frame and we have and

$$dW = \gamma dW';\ dt = \gamma dt'. \tag{372}$$

Thus

$$dW'/dt' = dW/dt. \tag{373}$$

The non-relativistic dipole formula is

$$P = P' = \frac{2q^2}{3c^3}\mathbf{a}'^2 = \frac{2q^2}{3c^3}a_\mu a^\mu \tag{374}$$

where the last equality follows because in the instantaneous rest frame of the emitting particle, $a^{\mu'} = (0, \mathbf{a})$.

We can write this in terms of the 3-vector acceleration by quoting the result that

$$a'_{||} = \gamma^3 a_{||} \tag{375}$$

and

$$a'_\perp = \gamma^2 a_\perp \tag{376}$$

so

$$P = \frac{2q^2}{3c^3}\mathbf{a}'^2 = \frac{2q^2}{3c^3}\gamma^4(a_\perp^2 + \gamma^2 a_{||}^2). \tag{377}$$

LECTURE 12
Angular Distribution of Emitted and Received Power
fig 4.9

Now consider dW' emitted into angle $d\Omega' = sin\theta' d\theta' d\phi'$, where θ' is to the x-axis. Let $\mu = cos\theta$ and $\mu' = cos\theta'$. We do not assume forward backward symmetry here so the momentum is not zero. Then

$$dW = \gamma(dW' + vdp'_x) = \gamma(1 + \beta\mu')dW', \tag{378}$$

using the relation between photon energy and momentum. From (317) we have

$$\mu = \frac{\mu' + \beta}{1 + \beta\mu'} \tag{379}$$

so that differentiating and using $d\phi = d\phi'$ we get

$$d\Omega = \frac{d\Omega'}{\gamma^2(1 + \beta\mu')^2}. \tag{380}$$

Thus

$$dW/d\Omega = \gamma^3(1 + \beta\mu')^3(dW'/d\Omega') \tag{381}$$

To get the differential emitted power in the lab frame we divide by the time as determined from time dilation $dt = \gamma dt'$.

To get an observed differential power we must include the Doppler effect so that $dt_{obs} = \gamma(1 - \beta\mu)dt'$ from (320).

We then have

$$dP_e/d\Omega = \gamma^2(1 + \beta\mu')^3 dP'/d\Omega' = \gamma^{-4}(1 - \beta\mu)^{-3} dP'/d\Omega', \tag{382}$$

without taking into account Doppler but just time dilation and

$$dP_{obs}/d\Omega = \gamma^4(1 + \beta\mu')^4 dP'/d\Omega' = \gamma^{-4}(1 - \beta\mu)^{-4} dP'/d\Omega' \tag{383}$$

when we do take into account Doppler effect. I have used the inverse of (322) and the principle that we can interchange primed with un-primed variables and change the sign of v to obtain the inverse transformation. The factor $\gamma^{-4}(1-\beta\mu)^{-4}$ is peaked near $\theta = 0$ for small θ which is highlights the beaming effect described earlier.

(fig 4.10)

Apply to a moving particle. Recall that in the non-relativistic regime (and thus the proper frame here)

$$dP'/d\Omega' = \frac{q^2 a'^2}{4\pi c^3} sin^2\Theta' \tag{384}$$

Thus

$$dP/d\Omega = \frac{q^2}{4\pi c^3}\frac{(\gamma^2 a_{||}^2 + a_{\perp}^2)}{(1-\beta\mu)^4} sin^2\Theta', \tag{385}$$

where we have used (375) and (376).
acceleration || velocity: Then $\Theta' = \theta'$ so

$$sin^2\Theta' = sin^2\theta' = \frac{sin^2\theta}{\gamma^2(1-\beta\mu)^2}, \tag{386}$$

using the inverse of (317). Then

$$dP_{||}/d\Omega = \frac{q^2}{4\pi c^3} a_{||}^2 \frac{sin^2\theta}{(1-\beta\mu)^6}. \tag{387}$$

for acceleration $\perp$ velocity Then $Cos\Theta' = cos\phi' sin\theta'$ (where $\phi' = 0$ is aligned or anti-aligned with $\mathbf{a}_{\perp} = \mathbf{a}$) measured from acceleration so

$$sin^2\Theta' = 1 - \frac{sin^2\theta cos^2\phi}{\gamma^2(1-\beta\mu)^2}. \tag{388}$$

Then

$$dP_{\perp}/d\Omega = \frac{q^2}{4\pi c^3} a_{\perp}^2 \frac{1}{(1-\beta\mu)^4} \left(1 - \frac{sin^2\theta cos^2\phi}{\gamma^2(1-\beta\mu)^2}\right). \tag{389}$$

Extreme relativistic limit

Since

$$\beta = (1 - 1/\gamma^2)^{1/2} \sim 1 - \frac{1}{2\gamma^2} \tag{390}$$

and

$$\mu = cos\theta = 1 - \theta^2/2 \tag{391}$$

we have

$$(1 - \beta\mu) \simeq \frac{1 + \gamma^2\theta^2}{2\gamma^2}. \tag{392}$$

and (387) becomes

$$dP_{||}/d\Omega = \frac{16 q^2 a_{||}^2}{\pi c^3} \frac{\gamma^{12}\theta^2}{(1 + \gamma^2\theta^2)^6} \tag{393}$$

while (389) becomes

$$dP_{\perp}/d\Omega = \frac{4 q^2 a_{\perp}^2}{\pi c^3} \gamma^8 \frac{(1 - 2\gamma^2\theta^2 cos2\phi + \gamma^4\theta^4)}{(1 + \gamma^2\theta^2)^6}. \tag{394}$$

Notice the dependence on ϕ, the angle traced in the plane perpendicular to $\mathbf{v}$.

(fig 4.11)

The beaming as a result of the $\theta\gamma$ factors is highlighted in the figures. Notice that the maximum powers occur for $\theta \sim 1/\gamma$ in the case of parallel acceleration, and $\theta < 1/\gamma$ for perpendicular acceleration. Note also the non-azimuthal symmetry for the

Specific Intensity and Invariants

Consider group of particles moving along x in source (comoving) frame with phase space measure $d^3\mathbf{x}'d^3\mathbf{p}'$. There is slight spread in momentum but not in energy: this is because the energy of a single particle in this frame is $E' = mc^2 + mu'^2/2 + .. \sim mc^2$, which follows because $u' \sim 0$ in the comoving frame, so deviations from $u' = 0$ are quadratic in u', and we ignore second order term in the co-moving frame. This implies that phase space measure is invariant because $d^3\mathbf{x} \rightarrow d^3\mathbf{x}'/\gamma$, but $dp_x = \gamma(dp'_x + \beta dE/c)$, and since $dE = 0$ the result of invariant phase-space measure follows.

So $d^3\mathbf{x}d^3\mathbf{p}$ is an invariant and thus $dN/d^3\mathbf{x}d^3\mathbf{p} = f_\alpha(\mathbf{x}, \mathbf{p}, t)$ is also an invariant. But we related the latter to the intensity in equation (20):

$$I_\nu = \sum_{\alpha=1}^{2} (h^4\nu^3/c^3) f_\alpha(\mathbf{x}, \mathbf{p}, t). \tag{395}$$

so I_ν/ν^3 is an invariant, as is S_ν/ν^3.

Now let us find the transformation property of the absorption coefficient α_ν. Consider thickness l of material of material moving along x with β. The optical depth as seen by an observer at angle $\pi/2-\theta$ to the vertical is invariant since $e^{-\tau}$ measures the fraction of photons passing through material.

fig 4.12

Then

$$\tau_\nu = \alpha_\nu(l/sin\theta) = \nu\alpha_\nu(l/\nu sin\theta) \tag{396}$$

is invariant, but $\nu sin\theta$ is the vertical momentum component and l is perpendicular to the direction of motion, so these quantities are not affected by the motion along the $x - axis$. Thus $\nu\alpha_\nu$ must be an invariant.

Since the emission coefficient $j_\nu = S_\nu\alpha_\nu = ndP/d\Omega d\nu$, j_ν/ν^2 is also an invariant.

LECTURE 13

Bremsstrahlung =Free-Free process

Radiation from acceleration of charge in Coulomb field of another charge. Quantum treatment required, as process can produce photons comparable in energy to that of emitting particle, but do classical treatment first.
Bremsstrahlung is zero in dipole approximation for two of the same type of particles: think of diplole moment for like particles: its zero.
Electrons provide most of the radiation since acceleration is inversely proportional to mass.

Emission from Single Speed Electrons

Consider electron moving in field of nearly stationary ion (at origin).
Consider small-angle scattering regime.

Consider electron of charge $-e$ moving past ion of charge Ze with impact parameter b.

fig 5.1

Dipole moment is $\mathbf{d} = -e\mathbf{R}$ where R is the separation, and so

$$\ddot{\mathbf{d}} = -e\dot{\mathbf{v}}, \tag{397}$$

so Fourier transform gives

$$-\omega^2\tilde{\mathbf{d}}(\omega) = -\frac{e}{2\pi}\int_{-\infty}^{\infty}\dot{\mathbf{v}}e^{iwt}dt. \tag{398}$$

Consider asymptotic limits of large and small frequencies. Since $\tau \sim b/v$ is the interaction time, then the integral mainly contributes over this time.

When $\omega\tau >> 1$ exponential oscillates rapidly and integral is small.

When $\omega\tau << 1$ exponential is unity so

$$\tilde{\mathbf{d}} \sim \frac{e}{2\pi\omega^2}\Delta\mathbf{v}, \tag{399}$$

with $\Delta\mathbf{v}$ the change in velocity over the interaction time.

Using

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\tilde{d}(\omega)|^2 \tag{400}$$

we have

$$\frac{dW}{d\omega} = \frac{2e^2}{3\pi c^3}|\Delta \mathbf{v}|^2. \tag{401}$$

The acceleration is mainly perpendicular to the electron path in the small angle approximation. The change in velocity is then found by integrating the perpendicular component of acceleration:

$$\Delta v = \int a_\perp dt = (e^2 Z/m_e)\int_{-\infty}^{\infty} b/(b^2+v^2t^2)^{3/2}dt = 2e^2Z/m_e bv, \tag{402}$$

where we are in the $\omega\tau << 1$ limit, and extend the integral to infinity.

For small angle scatterings then

$$\frac{dW}{d\omega} = \frac{8e^2Z^2}{3\pi c^3 m_e^2 v^2 b^2} \tag{403}$$

For $\omega\tau << 1$ and 0 otherwise.

To get the total energy per unit frequency for a number density of ions n_i and electrons n_e with velocity $\mathbf{v}$, we note that the flux of electrons per ion is $n_e v$. Then, the number of electrons per unit time per ion that interact is $n_e v 2\pi b db$. Then multiplying by the number density of ions and integrating, we have

$$\frac{dW}{d\omega dV dt} = 2\pi n_i n_e v \int_{b_{min}}^{\infty} (dW(b)/d\omega) b db. \tag{404}$$

The main contribution to the integral, by assumption of our regime, is for $\omega\tau << 1$ or $b << v/\omega$ and so substitute (403) for the integrand

$$\frac{dW}{d\omega dV dt} = \frac{16Z^2e^6}{3c^3m_e^2v}n_i n_e \; ln(b_{max}/b_{min}) \tag{405}$$

$b_{max} = v/\omega$. There are two possibilities for b_{min}. One is classical, when $\Delta v = v$ (i.e. the limit defined by when small angle approximation becomes invalid). From (402) this can be approximated (for convenience 2 is replaced by $\frac{4}{\pi}$)

$$b_{min} \sim \frac{4Ze^2}{\pi m_e v^2}. \tag{406}$$

Second is the quantum limit $\Delta x \Delta p = h/2\pi$ or $b_{min,q} = h/(2\pi mv)$, so

$$b_{min,q} = h/mv. \tag{407}$$

When $b_{min,q} < b_{min}$, then classical treatment is ok. When $b_{min,q} > b_{min}$ then $mv^2 > Z^2 Ry \sim Z^2 13.6\text{eV}$. Here the interaction time is short, the scale is small, and need quantum treatment.

Convention is to write the intergal with either limit as a Gaunt factor $g_{ff}(v,\omega)$ so

$$\frac{dW}{d\omega dV dt} = \frac{16\pi Z^2 e^6}{3\sqrt{3}c^3 m_e^2 v} n_i n_e g_{ff}. \tag{408}$$

The Gaunt factor is of order 1 usually.

Thermal Bremsstrahlung Emission = Thermal Free-Free Emission

We can integrate single speed expression above over thermal distribution of electrons. The probability for electron to have velocity in range $d^3\mathbf{v}$ for an isotropic distribution of velocities is

$$dQ = e^{-E/kt} d^3\mathbf{v} = v^2 e^{-mv^2/2kT} dv \tag{409}$$

A subtlety is that the lower bound for the velocity must satisfy $2h\nu < mv^2$, so that photon can be produced (remember we are in non-relativistc limit so relevant energy is $mv^2/2$). Then

$$\frac{dW(T,\omega)}{dV dt d\omega} = \frac{\int_{v_{min}}^{\infty} [dW(v,\omega)/dV d\omega dt] v^2 e^{-mv^2/2kT} dv}{\int_0^{\infty} v^2 e^{-mv^2/2kT} dv} \tag{410}$$

The emission coefficient is then given by:

$$4\pi j_\nu^{ff} = \epsilon_{ff,\nu} = \frac{dW(T,\omega)}{dV dt d\omega} = 6.8 \times 10^{-38} Z^2 n_e n_i T^{-1/2} e^{-h\nu/kT} \bar{g}_{ff}(T,\nu). \tag{411}$$

Note the dependence on the number density squared for $n_e = n_i$.

The Gaunt factors tend to be of order a few for a wide range of velocity. (problem in book fig 5.3 with the Gaunt factor X-axis graph. $u = h\nu/kT$)

Note that the exponential tail is due to the cutoff in the velocity. Also, the spectrum is more or less flat until $h\nu/kT \sim 1$.

To get the total power per volume, one integrates over frequency. This ultimately gives

$$\frac{dW}{dt dV} = \left(\frac{2\pi kT}{3}\right)^{1/2} \left(\frac{32\pi e^6}{3hm^{3/2}c^3}\right) Z^2 n_e n_i g_B(T) = 10^{-27} T^{1/2} n_e n_i Z^2 g_B(T). \tag{412}$$

where $g_B(T)$ is frequency averaged Gaunt factor.

Thermal Bremsstrahlung Absorption = Thermal Free-Free Absorption

Consider electron absorbing radiation while moving in field of an ion. (Free-free absorption).

For thermal processes we have Kirchoff's law

$$j_\nu^{ff} = \alpha_\nu^{ff} B_\nu(T), \tag{413}$$

where α_ν^{ff} is the free-free abs. coefficient and j_ν^{ff} is the associated emission coefficient as above. Plugging in for the Planck function and using ϵ_ν^{ff} above we have

In the Rayleigh-Jeans regime $h\nu/kT << 1$, expand exponential and then:

$$\alpha_\nu^{ff} = 0.018 T^{-3/2} Z^2 n_e n_i \nu^{-2} \bar{g}_{ff} \tag{414}$$

In the limit $h\nu >> kT$, the Wien regime, we have instead

$$\alpha_\nu^{ff} = 3.7 \times 10^8 T^{-1/2} Z^2 n_i n_e \nu^{-3} \bar{g}_{ff}. \tag{415}$$

Recall that the Rosseland mean absorption coefficient was defined when we considered plane-parallel atmospheres in the limit of Kirchoffs law at large optical depths leading to

$$F(z) = -\frac{16\sigma T^3}{3\alpha_R}\frac{\partial T}{\partial z}. \tag{416}$$

For Bremsstrahlung,

$$\frac{1}{\alpha_R^{ff}} = \frac{\int_0^\infty \frac{1}{\alpha_\nu^{ff}}\frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu} = \frac{1}{1.7 \times 10^{-25} T^{-7/2} Z^2 n_e n_i \bar{g}_R}. \tag{417}$$

Relativistic Bremsstrahlung

One can use the method of virtual quanta to get a relativistic correction to Bremsstrahlung emission. (Electrons moving relativistically). Ultimately, we want a correction in the lab frame, in which the electron is moving. Here is the procedure

1) First move to rest frame of electron.

2) The electron sees the ion's field as pulse of electromagnetic radiation emitted by the ion (the virtual quanta). This pulse is scattered by the electron.

3) Then transform back to the rest frame of the ion to get the emission.

Roughly, consider the pulse of virtual quanta in the electron frame $dW'/dA'd\omega'$. Then project out the scattered part by multiplying by the electron scattering cross section. (σ_T for low freq $h\nu << mc^2$). This gives

$$(dW'/d\omega')_{sc} = \sigma_T dW'/dA'd\omega'. \tag{418}$$

Then note that dW' and $d\omega'$ transform the same way under a Lorentz transformation. So that

$$(dW/d\omega)_{sc} = (dW'/d\omega')_{sc}. \tag{419}$$

Then appeal to forward backward symmetry of the scattering, and write primed frame quantities such as ω' and b' in terms of quantities in the unprimed frame by a Lorentz transformation.

For a thermal distribution of electrons, one must integrate over the thermal distribution function. Then frequency integrating, one obtains

$$dW/dVdt = 4\pi \int j_\nu d\nu = 1.4 \times 10^{-27} Z^2 \bar{g}_B T^{1/2} n_i n_e (1 + 4.4 \times 10^{-10} T). \tag{420}$$

The second term on the right is the relativistic correction. Important for quite large temperatures.

LECTURE 14

Synchrotron Radiation

Radiation of particles (e.g. electrons) gyrating around a magnetic field is called cyclotron radiation when the electrons are non-relativistic and synchrotron radiation when the electrons are relativistic. The spectrum of cyclotron emission frequency is a peak at the frequency of gyration. Synchrotron frequency picks up higher harmonics and is not as simple.

Total emitted synchrotron power:

Motion of a particle in a magnetic field is governed by

$$\mathbf{a}^{\mu} = \frac{e}{mc} F^{\mu}_{\nu} u^{\nu} \tag{421}$$

with components

$$d_t(\gamma \mathbf{v}) = \frac{q\mathbf{v} \times \mathbf{B}}{mc} \tag{422}$$

and

$$d_t \gamma = \frac{q\mathbf{v} \cdot \mathbf{E}}{mc^2} = 0. \tag{423}$$

Thus γ and $|\mathbf{v}|^2$ are constants so

$$\gamma d_t \mathbf{v} = \frac{q\mathbf{v} \times \mathbf{B}}{mc} \tag{424}$$

and

$$\gamma d_t \mathbf{v}_{||} = 0 \tag{425}$$

and

$$d_t \mathbf{v}_{\perp} = \frac{q}{\gamma mc} \mathbf{v}_{\perp} \times \mathbf{B}. \tag{426}$$

fig 6.1

The magnitude of **v** is a constant, and the parallel component is unchanged, the motion represents helical motion along a magnetic field. This is circular motion in the frame where $\mathbf{v}_{||} = 0$. The gyration frequency is $\omega_B = qB/\gamma mc$. The acceleration ($a_\perp = \omega_B v_\perp$) is perpendicular to the velocity and so the power emitted is

$$P = \frac{2q^2}{3c^3}\mathbf{a}'^2 = \frac{2q^2}{3c^3}\gamma^4(a_\perp^2 + \gamma^2 a_{||}^2) = \frac{2q^2}{3c^3}\gamma^4 a_\perp^2 = \frac{2}{3}r_0^2 c\beta_\perp^2\gamma^2 B^2, \tag{427}$$

where we are now assuming that the electrons are raditating so $r_0 = e^2/m_e c^2$ is the classical elecron radius as defined earlier.

If we average over pitch angles $0 \le \psi \le \pi$ then

$$\langle\beta_\perp^2\rangle = \frac{\beta^2}{4\pi}\int sin^2\psi d\Omega = \frac{2\beta^2}{3}. \tag{428}$$

Then

$$P = \frac{4}{9}cr_0^2\beta^2\ \gamma^2 B^2 = \frac{4}{3}\sigma_T c\beta^2\gamma^2 U_m \simeq \frac{4}{3}\sigma_T c\gamma^2 U_m. \tag{429}$$

where $\sigma_T = 8\pi r_0^2/3$.

Spectrum of Synchrotron Radiation

fig 6.2

To estimate this, note that observer sees emission during the time the particle moves from point 1 to 2 below. Consider the extremely relativistic regime $\gamma >> 1$.

The emission from synchrotron radiation will be spread over a broader range of frequency than just the gyro-frequency because the time interval over which observer sees emission is much smaller than the gyro-frequency. The uncertainty relation then implies the broader spectrum.

Radius of curvature is $a = \Delta s/\Delta\theta$. We see that $\Delta\theta = 2/\gamma$ so $\Delta s = \frac{2a}{\gamma}$. The radius of curvature is also given by $asin\alpha = v/\omega_B$, which follows from

$$\gamma m\frac{\Delta v}{\Delta t} = (q/c)\mathbf{v}\times\mathbf{B} = \gamma m v^2\frac{\Delta\theta}{\Delta s}, \tag{430}$$

where $|\Delta v| = v\Delta\theta$ and $\Delta s = v\Delta t$.

Thus

$$a = \Delta s/\Delta\theta = \gamma mcv/qBsin\alpha = v/\omega_B sin\alpha, \tag{431}$$

where α is the pitch angle. Therefore

$$\Delta s = 2v/\gamma\omega_B sin\alpha. \tag{432}$$

The particle passes between points 1 and 2 in a time $t_2 - t_1$ so that

$$(t_2 - t_1) \sim \Delta s/v = \frac{2}{\gamma\omega_B sin\alpha}. \tag{433}$$

Defining t_{A1} and t_{A2} as the arrival times of radiation at the observation point from points 1 and 2, the difference $t_{A2} - t_{A1}$ is less than $t_2 - t_1$ by an amount $\Delta s/c$, which is the time for radiation to move Δs. Thus

$$(t_{A2} - t_{A1}) \sim \frac{2}{\gamma\omega_B sin\alpha}(1 - v/c). \tag{434}$$

For $v \sim c$ this becomes

$$(t_{A2} - t_{A1}) \sim \frac{1}{\gamma^3\omega_B sin\alpha}. \tag{435}$$

Thus for a finite pitch angle, observed pulse duration is γ^3 times smaller than gyro-period. Pulse frequency cutoff will be something like

$$\omega_c \sim 1/\Delta t_A = \gamma^3\omega_B sin\alpha. \tag{436}$$

fig 6.3

Now since the electric field, as reflected in (**??**) and (394) is only a function of the product $\theta\gamma$, we can write $E(t) = F(\gamma\theta)$, with t in the lab frame. Let $s = t = 0$ when pulse is along the axis of the observer. Then

$$t = \Delta t_A = (s/v)(1 - v/c). \tag{437}$$

and

$$\theta \sim (s/a) = \frac{vt}{a(1 - v/c)}, \tag{438}$$

so

$$\theta\gamma = \frac{\gamma vt}{(1 - v/c)a} = 2\gamma^3 t\omega_B sin\alpha \sim \omega_c t. \tag{439}$$

Thus

$$E(t) \propto g(\omega_c t). \tag{440}$$

The Fourier transform is then

$$\tilde{E}(\omega) \propto \int_{-\infty}^{\infty} g(\zeta) e^{i\omega\zeta/\omega_c} d\zeta, \tag{441}$$

where $\zeta = \omega_c t$.

The spectrum $dW/d\omega d\Omega \propto \tilde{E}^2$. Integrating over solid angle and dividing by the orbital period gives the time averaged power per frequency

$$dW/dtd\omega = (\omega_B/2\pi)dW/d\omega = P(\omega) = KF(\omega/\omega_c), \tag{442}$$

where K is a constant which can be fixed by integrating and comparing to the total power derived earlier. Since the total power emitted for $\beta \sim 1$ is given by

$$P_{tot} \simeq \frac{2q^4 B^2 \gamma^2 sin^2\alpha}{3m^2c^3} = K \int_0^{\infty} F(\omega/\omega_c) d\omega = \omega_c K \int_0^{\infty} F(\xi) d\xi, \tag{443}$$

where $\xi \equiv \omega/\omega_c$. Then using

$$\omega_c = \frac{3\gamma^2 qB sin\alpha}{2mc} \tag{444}$$

$$P(\omega) = KF(\omega/\omega_c) = \omega_c KF(\xi) d\xi/d\omega = \frac{\sqrt{3}}{2\pi mc^2} q^3 B sin\alpha F(\omega/\omega_c), \tag{445}$$

where the numerical integral factor is arbitrary, and comes from choice of normalization of the dimensionless integral.

Spectral Index for power law electron distribution

Often the emission spectrum can be described by a power law in frequency for some range. That is,

$$P(\omega) \propto \omega^{-s} \tag{446}$$

where s is the spectral index. Electrons also often follow a power law distribution when they have been accelerated. For relativistic electrons, consider an energy distribution

$$N(\gamma)d\gamma = C\gamma^{-p}d\gamma. \tag{447}$$

Then the total power radiated is given by

$$P_{tot}(\omega) = C\int_{\gamma_1}^{\gamma_2} P(\omega)\gamma^{-p}d\gamma \propto \int_{\gamma_1}^{\gamma_2} F(\omega/\omega_c)\gamma^{-p}d\gamma. \tag{448}$$

If we change variables of integration to $x \equiv \omega/\omega_c$ and use $\omega_c \propto \gamma^3\omega_B \propto \gamma^2$ we get

$$P_{tot}(\omega) \propto \omega^{\frac{1-p}{2}} \int_{x_1}^{x_2} F(x)x^{\frac{p-3}{2}}dx. \tag{449}$$

For wide enough frequency limits, take $x_1 \sim 0$ and $x_2 \sim \infty$. Then we have

$$P_{tot}(\omega) \propto \omega^{\frac{1-p}{2}} \int_{0}^{\infty} F(x)x^{\frac{p-3}{2}}dx. \tag{450}$$

The spectral index is then related to the particle distribution index:

$$s = \frac{p-1}{2}. \tag{451}$$

For shocks, p is often $2-4$ so spectral index is $0.5-1.5$. Relevant for radio jets in AGN (e.g. Blackman 1996).

Note that $F(x)$ is something like a Gaunt factor, but it is narrowly peaked helping to justify the assumption of wide limits of integration. fig 6.6

Summary of Results: (1) emission into $1/\gamma$ half angle, (2) emission up to critical frequency ω_c and dependence only on ω/ω_c, and (3) spectral index for power law electron distribution is $s = (p-1)/2$.

LECTURE 15

Synchrotron Radiation Polarization

Notice that for an electron, the emission is elliptically polarized as the electron moves around the field line. Looking inside the cone of maximal emission, the electron moves around counterclockwise and the emission is left circularly polarized. Seen from outside the cone of maximal emission the electron appears to move clockwise, and the emission is right circularly polarized.

For a distribution of particles that varies relatively smoothly with $Sin\alpha$, the elliptical polarization should cancel and we would get a linear polarization, largely in plane perpendicular to the magnetic field. fig 6.5

The power perpendicular and parallel to the magnetic field as projected onto the plane of the sky can then be used to calculate the polarization. For particles of a single energy

$$\Pi(\omega) = \frac{P_\perp - P_{||}}{P_\perp + P_{||}}. \tag{452}$$

The frequency integrated value is 0.75. For a power law distribution $N(\gamma) \propto \gamma^{-p}$, the polarization is

$$\Pi = \frac{P_\perp - P_{||}}{P_\perp + P_{||}} = \frac{p+1}{p+7/3}. \tag{453}$$

Note that the larger p the higher the polarization. Why?

Transformation from Cyclotron to Synchrotron Radiation

For low energies the electric field oscillates with the same frequency of gyration of the electron around the magnetic field. This produces a single line in the emission spectrum. fig 6.8 ab

For slightly higher energies, the beaming kicks in and so the E-field is stronger when the particles move toward the observer. As this effect increases for larger and larger velocities, the spectrum picks up multiple harmonics as shown in the expansion discussed earlier in the course. fig 6.10 ab

Eventually mapping out the $F(\omega/\omega_c)$ function.

The spectrum becomes continuous for a system in which there are a distribution of particle energies, or for which the field is not exactly uniform. The electric field received by observer from a distribution of particles is the random superposition of many pulses. The spectrum is the sum of the spectra from the individual pulses as long as the average distance between particles is larger than a gyro-radius.

Distinction Between Emitted and Received Power

Emitted power is not equal to received power. We need to consider the Doppler effect as the particle moves toward us. fig 6.11

Thus for period of the projected motion of $T = 2\pi/\omega_B$, we have the period measured by the received emission.

$$T_A = T(1 - v_{||}cos\alpha/c) = T(1 - \frac{v}{c}cos^2\alpha) \sim \frac{2\pi}{\omega_B}sin^2\alpha \qquad (454)$$

for $v \simeq c$.

The fundamental observed frequency is $\omega_B/sin^2\alpha$. This implies that the spacing between harmonics is larger, being $\omega_B/sin^2\alpha$.

Recall that we got the emitted power by dividing by ω_B, so now we have instead

$$P_r = \frac{P_e}{sin^2\alpha}. \tag{455}$$

Should we include this in astrophysical situations?

In fact not usually. This is because in general the gammas of the particles are large compared to the gamma of the bulk flow, and are often more or less randomly moving. When the particle moves toward the observer, the power is increased by $sin^2\alpha$ but the particle emission only arrives to the observer per period for a time $T/sin^2\alpha$. fig 6.13 fig S.8

We can see this as follows. If the time to travel the bottle half way is $t/2$ as measured by the observer with proper clocks, then the correction for the Doppler effect is

$$\frac{t}{2}(1 - v_{||}cos\alpha/c) = \frac{t}{2}(1 - vcos^2\alpha/c) \simeq (t/2)sin^2\alpha. \tag{456}$$

Suppose we arrange the bottle such that $t/2 = \pi/\omega_B$. Then the cancellation with the factor for the observed power follows: That is

$$P_{r,ave} \sim \frac{(P_e/sin^2\alpha)(2\pi sin^2\alpha/\omega_B)}{2\pi/\omega_B} = P_e \tag{457}$$

Synchrotron self-absorption

Photons emitted by synchrotron can be re-absorbed. Or such photons can stimulate more emission in some component of phase space where photons already exist. (Stimulated emission).

There is one state for each element of phase space h^3. We break up the continuous volume in to discrete elements of size h^3 and consider transitions between states.

We must sum over all upper and lower states. We can use the Einstein coefficient formalism. We then have for the absorption coefficient, assuming a tangled magnetic field and isotropic particle distribution (thus assuming isotropy)

$$\alpha_\nu = (h\nu/4\pi)\sum_{E_1}\sum_{E_2}[n(E_1)B_{12} - n(E_2)B_{21}]\phi_{21}(\nu). \tag{458}$$

The $\phi(\nu)$ is approximately a delta function restricting sum to states differing by $h\nu = E_2 - E_1$.

We want to relate the Einstein coefficients to (445), the microscopic components of emission. We write

$$P(\nu, E_2) = 2\pi P(\omega) \tag{459}$$

where ν is the frequency and E_2 is the energy of the radiating electron.

In terms of the Einstein coefficients and using Einstein relations:

$$P(\nu, E_2) = h\nu\sum_{E_1} A_{21}\phi_{21}(\nu) = (2h\nu^3/c^2)h\nu\sum_{E_1} B_{21}\phi_{21}(\nu), \tag{460}$$

where Einstein relations have been used. Note there are no statistical weights.

Thus for stimulated emission we have

$$-\frac{h\nu}{4\pi}\sum_{E_1}\sum_{E_2} n(E_2)B_{21}\phi_{21} = -\frac{c^2}{8\pi h\nu^3}\sum_{E_2} n(E_2)P(\nu, E_2) \tag{461}$$

and for true absorption

$$\frac{h\nu}{4\pi}\sum_{E_1}\sum_{E_2} n(E_1)B_{12}\phi_{21} = \frac{c^2}{8\pi h\nu^3}\sum_{E_2} n(E_2 - h\nu)P(\nu, E_2), \tag{462}$$

where we use $B_{21} = B_{12}$. Note that the argument of n corresponds to those particles which have radiated. Thus, plug into (458) to obtain

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\sum_{E_2}[n(E_2 - h\nu) - n(E_2)]P(\nu, E_2). \tag{463}$$

Now consider an isotropic electron distribution $f(p)$ such that $f(p)d^3p$ is the number density of electrons within momentum range d^3p.

The number of quantum states per volume range d^3p is $h^{-3}\xi d^3p$, where $\xi = 2$ is the statistical weight for electrons. Electron density per quantum state is then $f(p)d^3p/h^{-3}\xi d^3p = (h^3/\xi)f(p)$. Thus we can make the following exchanges: number density is then

$$n(E_2) \to (h^3/\xi)f(p_2), \tag{464}$$

and sum over quantum states is then

$$\sum_2 \to (\xi/h^3)\int d^3p_2. \tag{465}$$

We then have

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int d^3p_2[f(p_2^*) - f(p_2)]P(\nu, E_2), \tag{466}$$

where the $*$ labels the momentum for the $E_2 - h\nu$ energy. The reason for keeping the 2 is to remind us that it is the 2 which labeled the initial energy state of particles that radiate.

Note that for a thermal distribution of particles

$$f(p) = Ce^{-E(p)/kT}. \tag{467}$$

We note that

$$f(p_2^*) - f(p_2) = Ce^{-(E_2-h\nu)/kT} - Ce^{-yE_2/kT} = f(p_2)(e^{h\nu/kT} - 1) \tag{468}$$

so

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}(e^{h\nu/kT} - 1)\int d^3p_2 f(p_2)P(\nu, E_2). \tag{469}$$

The integral is just the total power per frequency for isotropic thermal emission here, which is $4\pi j_\nu$. Thus

$$\alpha_{\nu,ther} = \frac{j_\nu}{B_\nu(T)} \tag{470}$$

which is Kirchoff's law as expected.

Using energy rather than momentum for the distribution function, we have, assuming $E = pc$ for extremely relativistic particles,

$$N(E)dE = 4\pi p^2 f(p)dp \tag{471}$$

so

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int\left(\frac{N(E-h\nu)}{(E-h\nu)^2} - \frac{N(E)}{E^2}\right)P(\nu, E)E^2dE. \tag{472}$$

Now assume $h\nu << E$ as required for classical e&m. Then from the definition of the derivative (e.g. expanding for small $h\nu$)

$$\alpha_\nu = -\frac{c^2}{8\pi\nu^2}\int P(\nu,E)E^2\frac{\partial}{\partial E}\left[\frac{N(E)}{E^2}\right]dE. \tag{473}$$

Then for thermal distribution of ultra-relativistic particles

$$N(E) = CE^2 e^{-E/kT}. \tag{474}$$

This leads to

$$\alpha_{\nu,ther} = \frac{c^2}{8\pi\nu^2 kT}\int N(E)P(\nu,E)dE = \frac{j_\nu c^2}{2\nu^2 kT}, \tag{475}$$

which is Kirchoff's law for the Rayleigh-Jeans regime.

For a power-law distribution, we have

$$-E^2\frac{d}{dE}\left(\frac{N(E)}{E^2}\right) = (p+2)CE^{-(p+1)} = \frac{(p+2)N(E)}{E} \tag{476}$$

so

$$\alpha_\nu = \frac{(p+2)c^2}{8\pi\nu^2}\int dEP(\nu,E)\frac{N(E)}{E} \tag{477}$$

Recall that in (449) we ignored ignored the hidden E dependence in the bounds of of the integral. We do the same here. This allows us to avoid the frequency dependence in $P(\nu,E)$ (the emission for a single particle, peaked near ω/ω_c) since we will just leave it in the integrand as in (443). After changing variables from E to γ to ω_c/ω, the frequency dependence in (477) then becomes

$$\alpha_\nu \propto \nu^{-(p+1)/2+1/2-4/2} = \nu^{-(p+4)/2}, \tag{478}$$

where we get the 4/2 from the ν^{-2}, the $\nu^{1/2}$ from the measure, and the $\nu^{-(p+1)/2}$ from changing $N(E)/E$ to $\gamma^{-(p+1)}$ in (476).

The source function (using (450) for $P(\nu)$)

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{P(\nu)}{4\pi\alpha_\nu} \propto \nu^{-(p-1)/2+(p+4)/2} = \nu^{5/2}, \tag{479}$$

where here $P(\nu)$ is now the emission spectrum for a power-law distribution calculated earlier.

Can derive this result also like this:

$$S_\nu = j_\nu/\alpha_\nu \propto \nu^2 \bar{E} \tag{480}$$

where $\bar{E}$ is some mean particle energy. This approximate relation comes from noting that $P(\nu) = \int dEN(E)P(\nu, E)$ and then noting that $\alpha_\nu \propto \nu^{-2} \int dEN(E)P(\nu, E)/E$ and then taking ratio to get S_ν. Then note/recall $\nu_c \propto \gamma^2 \propto \bar{E}^2$. Thus each frequency in the spectrum corresponds to a ν_c for a particle of a specific energy. Thus $\nu_c = \nu$ when applied to the source function which is integrated over all particles. Thus $S_\nu \propto \nu^{5/2}$.

For optically thin synchrotron emission

$$I_\nu \propto j_\nu \propto P_\nu \propto \nu^{-(p-1)/2}, \tag{481}$$

since for optically thin plasmas, intensity is proportional to the emission function. For optically thick plasmas, intensity is proportional to the source function

$$I_\nu \propto S_\nu \propto \nu^{5/2}. \tag{482}$$

fig6.12

This optically thin regime occurs at high frequencies, and the optically thick regime occurs at low frequencies.

The absorption produces a cutoff in the frequency. Note that the electrons are non-thermally distributed so even though the system is optically thick to synchrotron emission in the $\nu^{5/2}$ regime, the system is not in thermal equilibrium, and thus shape is different from Rayleigh jeans B-Body which is proportional to ν^2.

No synchrotron masers in a vacuum

The absorption coefficient is positive for an arbitrary distribution of particle energies $N(E)$. This means that if there were stimulated emission resulting from making $N(E)$ at a certain energy E_0 larger so that emission from E_0 to $E_0 - h\nu$ was a maser ¡, then someplace else in the distribution, there would be more positive absorption that would more than compensate. This means that α_ν is positive since it is integrated over energy. Can show this is also true for separate polarization states of synchrotron emission.

Lecture 16

Compton Scattering

For low photon energies, the scattering of radiation from free charges reduces to the classical case of Thomson scattering. For that case

$$\epsilon_i = \epsilon_f, \tag{483}$$

$$d\sigma_T/d\Omega = \frac{1}{2} r_0^2 (1 + cos^2\theta) \tag{484}$$

and

$$\sigma_T = 8\pi r_0^2/3 = 6.6 \times 10^{-25}\text{cm}, \tag{485}$$

where ϵ_i and ϵ_f are the incident and scattered photon energy, $d\sigma_T/d\Omega$ is the differential cross section for scattering into Ω and σ_T is the Thomson cross section, and θ is the angle between the incident and scattered direction. This is "elastic scattering."

In reality the scattering is not elastic because the charge recoils. fig 7.1

The initial and final 4-momenta of the photon are $P_{i\gamma} = (\epsilon_i/c)(1, \mathbf{n}_i)$ and $P_{f\gamma} = (\epsilon_f/c)(1, \mathbf{n}_f)$ respectively.

For the electron $P_{ie} = (mc, 0)$ and $P_{fe} = (E/c, \mathbf{p})$ respectively. Conservation of energy and momentum can be written in terms of the energy momentum 4-vectors for the electron and photons:

$$P_{ie} + P_{i\gamma} = P_{fe} + P_{f\gamma}, \tag{486}$$

This leads to the two equations

$$mc^2 + \epsilon_i = E + \epsilon_f \tag{487}$$

and

$$\epsilon_i \mathbf{n}_i + 0 = \epsilon_f \mathbf{n}_f + \mathbf{p}c. \tag{488}$$

Rearranging we have

$$(mc^2 + \epsilon_i - \epsilon_f)^2 = E^2 = m^2c^4 + p^2c^2 \tag{489}$$

and

$$(\epsilon_i \mathbf{n}_i - \epsilon_f \mathbf{n}_f)^2 = p^2 c^2. \tag{490}$$

where I used $E^2 = m^2c^4 + p^2c^2$. Then solving to eliminate p, we obtain

$$\epsilon_f = \frac{\epsilon_i}{1 + \frac{\epsilon_i}{mc^2}(1 - cos\theta)}. \tag{491}$$

If we write $\epsilon_i = h\nu = hc/\lambda$ and similarly for ϵ_f we have,

$$\lambda_f - \lambda = \lambda_c(1 - cos\theta), \tag{492}$$

where $\lambda_c = \frac{h}{mc} = 0.024$ angstroms for electrons is the Compton wavelength. When $\lambda >> \lambda_c$ or $h_\nu << mc^2$, the scattering is elastic in the rest frame of the electron.

When quantum effects are important, ie. $\epsilon_i \gtrsim mc^2$, the cross section becomes

$$\frac{d\sigma}{d\Omega} = \frac{r_0^2}{2}\frac{\epsilon_f^2}{\epsilon_i^2}\left(\frac{\epsilon_i}{\epsilon_f} + \frac{\epsilon_f}{\epsilon_i} - sin^2\theta\right). \tag{493}$$

For $\epsilon_f \sim \epsilon_i$ Eq. this reduces to the classical expression. Note that since $\epsilon_f \leq \epsilon_i$, the quantum regime produces a lower overall cross section than the Thomson regime. Think about the role of the uncertainty principle here.

The total cross section in the non-relativistic regime ($h\nu << mc^2$) is

$$\sigma \sim \sigma_T(1 - 2x + ...), \tag{494}$$

where $x = h\nu/mc^2$, and in the relativistic regime $x >> 1$,

$$\sigma \sim \frac{3}{8}\sigma_T x^{-1}(ln2x + \frac{1}{2}). \tag{495}$$

Inverse Compton energy transfer

When the electron has more kinetic energy in the lab frame than the photon energy, there can be energy transfer from the electron to the photon during the scattering. This is inverse Compton scattering. The scattering in the electron rest frame (K') and the lab frame (K), are shown below. fig 7.2

In frame K' (rest frame of the electron) we have

$$\epsilon'_f = \frac{\epsilon'_i}{1 + \frac{\epsilon'_i}{mc^2}(1 - cos\Theta')}. \tag{496}$$

We also have from the Doppler formulas,

$$\epsilon'_i = \epsilon_i \gamma (1 - \beta cos\theta_i) \tag{497}$$

and

$$\epsilon_f = \epsilon'_f \gamma (1 + \beta cos\theta'_f). \tag{498}$$

Since electron scattering in the rest frame of the electron is front back symmetric, a typical angle for scattering is $\theta_i \sim \pi/2$ or $\theta'_f \sim \pi/2$. Thus we have roughly, from the previous 3 equations

$$\epsilon'_f = \epsilon'_i \tag{499}$$

$$\epsilon'_i \sim \epsilon_i \gamma \tag{500}$$

and

$$\epsilon_f \sim \gamma \epsilon'_f, \tag{501}$$

so that $\epsilon_f = \gamma^2 \epsilon_i$. Thus there is a gain by a factor of γ^2 in the energy from Compton scattering where γ is the Lorentz factor of the electron. We assumed that $\gamma\epsilon << mc^2$ in the rest frame of the electron. If $\epsilon'_i = \epsilon_i \gamma$ is too large, then we see from (496) that $\epsilon'_f << \epsilon'_i$ so that the processes is less efficient. Also, when $\epsilon'_i = \epsilon_i \gamma$ is too large then the cross section is reduced, which lowers the scattering probability again making the process less efficient.

Note also that γ^2 is the proportional to the energy of the electron squared.

Inverse Compton power for isotropic photon distribution (single scattering)

To get the emitted power from an isotropic distribution of photons scattering off of an isotropic distribution of electrons incurring single scattering per photon, we must average (496-498). To do this let q be density of photons of energy in range range $d\epsilon$. Let $g(p)$ be the phase space distribution function. Then

$$q d\epsilon_i = g(p) d^3 p. \tag{502}$$

We have that $\gamma d^3 p = d^3 p'$ as we derived earlier when discussing the phase space invariants in relativity. Thus $d^3 p$ transforms as energy under Lorentz transformations. Since $g(p)$ is an invariant, we have also that $qd\epsilon_i/\epsilon_i$ is an invariant that is

$$qd\epsilon_i/\epsilon_i = q' d\epsilon_i'/\epsilon_i'. \tag{503}$$

The total power scattered in the electron's rest frame is

$$\frac{dE_f'}{dt'} = c\sigma_T \int \epsilon_f'^2 \frac{q' d\epsilon'}{\epsilon_f'}. \tag{504}$$

We assume that the energy change of photon in rest frame is small compared to that in lab frame: $\gamma^2 - 1 >> \epsilon_i/mc^2$, so that so $\epsilon_f' = \epsilon_i'$. We also have

$$\frac{dE_f'}{dt'} = \frac{dE_f}{dt} \tag{505}$$

by Lorentz invariance for isotropic emission. Thus

$$\frac{dE_f}{dt} = c\sigma_T \int \epsilon_i'^2 \frac{q' d\epsilon_i'}{\epsilon_i'} = c\sigma_T \int \epsilon_i'^2 \frac{q d\epsilon_i}{\epsilon_i}, \tag{506}$$

using the $\epsilon_f' = \epsilon_i'$ assumption. Using the Doppler formula

$$\epsilon_i' = \epsilon_i \gamma (1 - \beta cos\theta) \tag{507}$$

we have

$$\frac{dE_f}{dt} = c\sigma_T \gamma^2 \int (1 - \beta cos\theta)^2 q \epsilon_i d\epsilon_i. \tag{508}$$

Thus all quantities are now written in the K frame as desired. For isotropic distribution of photons, using $\langle \cos^2\theta \rangle = 1/3$ we have

$$\langle (1 - \beta cos\theta)^2 \rangle = 1 + \frac{1}{3}\beta^2. \tag{509}$$

Thus

$$\frac{dE_f}{dt} = c\sigma_T \gamma^2 U_{ph} (1 + \beta^2/3), \tag{510}$$

where $U_{ph} = \int \epsilon_i q(\epsilon_i) d\epsilon_i$ which is the energy density of incident photons.

Now the norm of the rate of decrease of the initial photon energy is

$$\left| \frac{dE_i}{dt} \right| = |c\sigma_T \int \epsilon_i q d\epsilon_i| = \sigma_T c^2 U_{ph}, \tag{511}$$

and so the energy actually radiated by the electron and converted into radiation is the difference between the energy scattered and the energy lost by the incoming photons. That is, it is the energy out minus the energy in. We have

$$\frac{dE_{rad}}{dt} = \frac{dE_f}{dt} - \left|\frac{dE_i}{dt}\right| = c\sigma_T U_{ph}\left[\gamma^2(1+\frac{1}{3}\beta^2) - 1\right]. \tag{512}$$

Using $\gamma^2 - 1 = \gamma^2\beta^2$ we have

$$P_c = \frac{dE_{rad}}{dt} = \frac{4}{3}\sigma_T c\gamma^2\beta^2 U_{ph}. \tag{513}$$

(When the energy transfer in the electron rest frame is not ignored, there is a correction term. (Blumenthal & Gould (1970)).)

Note that the synchrotron power we calculated for a single electron was

$$\frac{4}{3}\sigma_T c\gamma^2\beta^2 U_B. \tag{514}$$

Thus the ratio of Compton to Synchrotron power is

$$P_c/P_{syn} = U_{ph}/U_B. \tag{515}$$

The result also is true for arbitrarily small γ (i.e. the non-relativistic limit, as long as $\gamma\epsilon << mc^2$. This gives an important rough and ready tool to determine which of the two emission processes are more important.

LECTURE 17

Compton power for isotropic distributions of photons and relativistic electrons (single scattering)

Let $N(\gamma)$ be the number of electrons per volume between γ and $\gamma + d\gamma$. Then

$$P_{tot} = \int P_c N(\gamma) d\gamma \text{ erg s}^{-1}\text{cm}^{-3} \tag{516}$$

Choose a power law so that $N(\gamma) = C\gamma^{-p}$ for a range of γ. Then

$$P_{tot} = \frac{4}{3}\sigma_T c U_{ph} C \frac{\gamma_{max}^{3-p} - \gamma_{min}^{3-p}}{3-p} \text{ erg s}^{-1}\text{cm}^{-3} \tag{517}$$

for $\beta \sim 1$.

We can also compute the total power from a distribution of non-relativistic electrons. For $\gamma \sim 1$ we have $\langle\beta^2\rangle = 3kT_e/mc^2$ so that

$$P_{tot} = \frac{4kT_e}{mc^2} c\sigma_T n_e U_{ph} mma \tag{518}$$

Inverse Compton Spectra (isotropic distribution of photons and electrons)

Spectrum depends on both incident distribution of photons and energy distribution of the electrons. But we first calculate for scattering of mono-energetic photons off of mono-energetic electrons assuming isotropy and single scattering. This means we just have to later average over energy distributions of photons and electrons to get overall spectrum spectrum (that averaging over distribution functions will be done in next section.).

Assume again the Thomson scattering regime $\epsilon_i' \sim \gamma\epsilon_i << mc^2$. Let us assume that the scattering in the rest frame is isotropic. We thus assume

$$\frac{d\sigma'}{d\Omega'} = \frac{\sigma_T}{4\pi} = \frac{2}{3}r_0^2. \tag{519}$$

Following the text use I_n here to mean photon number intensity rather than energy intensity. Then $I_n d\epsilon dA d\Omega dt$ is the number of photons passing through area dA in time dt within solid angle $d\Omega$ and energy $d\epsilon$. This intensity is just the specific intensity divided by the energy (or I_ν/h). The analagous applies for source and emission functions.

Consider now isotropic incident photon field as mono-energetic

$$I_n(\epsilon_i) = F_0\delta(\epsilon_i - \epsilon_0), \tag{520}$$

where F_0 is the number of photons per unit area, per time, per steradian. (The flux per solid angle). Note that δ function always has units of 1 over the argument. We scatter these photons off of a beam of electrons of number density N and energy γmc^2 moving along the x-axis.

fig 7.2

The incident intensity in the rest frame, from Lorentz transformation properties of I_n, is

$$I'_n(\epsilon'_i, \mu') = F_0 \left(\frac{\epsilon'_i}{\epsilon_0}\right)^2 \delta(\epsilon_i - \epsilon_0). \tag{521}$$

Then using the Doppler formula $\epsilon = \gamma\epsilon'(1 + \beta\mu')$ we have

$$I'_n(\epsilon'_i, \mu') = \left(\frac{\epsilon'_i}{\epsilon_0}\right)^2 F_0 \delta\left(\gamma\epsilon'_i(1 + \beta\mu') - \epsilon_0\right) = \left(\frac{\epsilon'_i}{\epsilon_0}\right)^2 \frac{F_0}{\gamma\epsilon'_i\beta} \delta\left(\frac{\gamma\epsilon'_i - \epsilon_0}{\gamma\epsilon'_i\beta} + \mu'\right). \tag{522}$$

Here μ' is the cosine of the angle between the electron travel direction (x-axis) and the incident photon.

The emission function in the rest frame for isotropic distribution of photons is

$$j'_n(\epsilon'_f) = N'\sigma_T \frac{1}{2} \int_{-1}^{1} I'_n(\epsilon'_f, \mu') d\mu' \tag{523}$$

which here is the number of emitted photons per volume per steradian per time, and $N'\sigma_T$ is the scattering coefficient. Note $N = \gamma N'$. We replaced ϵ'_i by ϵ'_f by using the rest frame Thomson scattering approximation. Then

$$j'_n(\epsilon'_f) = \frac{N'\sigma_T \epsilon'_f F_0}{2\epsilon_0^2 \gamma\beta} \tag{524}$$

for the range

$$\frac{\epsilon_0}{\gamma(1+\beta)} < \epsilon'_f < \frac{\epsilon_0}{\gamma(1-\beta)}, \tag{525}$$

and zero otherwise. This range corresponds to the range for which $-1 < \mu' < 1$ which has to be the case since μ' is a cosine of the angle between incident photon and electron velocity.

We really want $j_n(\epsilon_f, \mu_f)$, the value in the lab frame. Recalling that j_ν/ν^2 is an invariant, we thus have j_n/ϵ is an invariant. Using this we have

$$j_n(\epsilon_f, \mu_f) = j'_n(\epsilon'_f)\frac{\epsilon_f}{\epsilon'_f} = \frac{N\sigma_T\epsilon_f F_0}{2\epsilon_0^2\gamma^2\beta} \tag{526}$$

when

$$\frac{\epsilon_0}{\gamma^2(1+\beta)(1-\beta\mu_f)} < \epsilon_f < \frac{\epsilon_0}{\gamma^2(1-\beta)(1-\beta\mu_f)}, \tag{527}$$

using the Doppler shift formula for ϵ'_f, and $N = \gamma N'$. Note also that for large β and $\mu_f = 1$ the inequality range can be written

$$\epsilon_0 < \epsilon_f < 4\epsilon_0\gamma^2, \tag{528}$$

while for large β and $\mu_f = -1$ the inequality range can be written

$$\frac{\epsilon_0}{4\gamma^2} < \epsilon_f < \epsilon_0. \tag{529}$$

For an isotropic distribution of electrons we average over the angle between the electron and emitted photon:

$$j_n(\epsilon_f) = \frac{1}{2}\int_{-1}^{1} j_n(\epsilon_f, \mu_f)d\mu_f. \tag{530}$$

Note that the limits in (530) can be written in terms of μ_f, that is the non-zero region of j_n occurs when

$$\beta^{-1}\left(1 - \frac{\epsilon_0}{\epsilon_f}(1+\beta)\right) < \mu_f < \beta^{-1}\left(1 - \frac{\epsilon_0}{\epsilon_f}(1-\beta)\right). \tag{531}$$

For $\epsilon_f/\epsilon_0 < (1-\beta)/(1+\beta)$ or for $\epsilon_f/\epsilon_0 > (1+\beta)/(1-\beta)$, μ_f does not fall in range $-1 < \mu < 1$ so the integral must vanish. The finite contribution comes from within the integral limits given by

$$-1 < \mu_f < \frac{1}{\beta}\left(1 - \frac{\epsilon_0}{\epsilon_f}(1-\beta)\right) \tag{532}$$

for $(1-\beta)/(1+\beta) < \epsilon_f/\epsilon_0 < 1$ and

$$\frac{1}{\beta}\left(1 - \frac{\epsilon_0}{\epsilon_f}(1+\beta)\right) < \mu_f < 1 \tag{533}$$

for $1 < \epsilon_f/\epsilon_0 < (1+\beta)/(1-\beta)$. Since $j_n(\epsilon_f, \mu_f)$ is isotropic, the integrals just amount to multiplication of the integrand of (530) by each of the bounds and subtracting. When the dust clears, using (526), we have

$$j_n(\epsilon_f) = \frac{N\sigma_T F_0}{4\epsilon_0\gamma^2\beta^2} Q \tag{534}$$

where

$$Q = (\beta - 1) + (\beta + 1)\frac{\epsilon_f}{\epsilon_0} \tag{535}$$

for $(1-\beta)/(1+\beta) < \epsilon_f/\epsilon_0 < 1$ and

$$Q = \frac{\epsilon_f}{\epsilon_0}(\beta - 1) + (\beta + 1) \tag{536}$$

for $1 < \epsilon_f/\epsilon_0 < (1+\beta)/(1-\beta)$. (Which is $1 < \epsilon_f/\epsilon_0 < 4\gamma^2$ for large γ).

The integrals satisfy:

$$\int_0^\infty j_n(\epsilon_f) d\epsilon_f = N\sigma_T F_0 \tag{537}$$

and

$$\int_0^\infty j_n(\epsilon_f)(\epsilon_f - \epsilon_0) d\epsilon_f = \frac{4}{3}\gamma^2\beta^2 N c\sigma_T(\epsilon_0 F_0/c). \tag{538}$$

The first means that number of photons are conserved since it is the number of photons scattering per volume per solid angle per unit time. The second is just the average gain in energy per unit time measured as radiation per electron where $\epsilon_0 F_0/c$ is the average incident photon energy density. That is it is the POWER emitted per electron.

Notice that in contrast to Bremsstrahlung, this equation is only proportional to the number density of electrons rather than the number density squared.

The quantity $Q/(4\gamma^2\beta^2) = Q/[4(\gamma^2 - 1)] = \epsilon_0 j(\epsilon_f)/N\sigma_T F_0$ is plotted below. fig 7.3a

One can see the shift with larger β as the energy is gained, highlighting how the average energy of the scattered photons is increased with larger γ. At very small γ there should be a peak around the initial photon energy with no gain or loss. As the γ is increased, there will be some loss as well as gain, because the photon distribution is isotropic and catch-up interactions deplete photon energy. But with large γ the beaming means electron sees most photons as head on, and the overall energy gain is produced.

When $\beta \sim 1$, we are in the second regime of Q , Eq. (536), which actually now corresponds to (528). Define $x = \epsilon_f/4\gamma^2\epsilon_0$, and then from (534),

$$j_n(\epsilon_f) = \frac{1}{2\gamma^2\epsilon_0} N F_0 \sigma_T (1 - x) = \frac{3}{4\gamma^2\epsilon_0} N F_0 \sigma_T f(x) \tag{539}$$

where we have used $\epsilon_f/\epsilon_0 = 4\gamma^2 x$ and $4\gamma^2 x(\beta - 1) + (\beta + 1) \sim 2x + 2$ for $\beta \sim 1$ (and thus $\gamma >> 1$). The result (539) applies only in range $0 < x < 1$ because the limit below (536) can be written as $0 < x < 1$ for large γ.

We have assumed everything to be isotropic in the electron rest frame. Small corrections result when this assumption is not made. See fig 7.3b.

LECTURE 18

Spectrum for arbitrary isotropic photon distribution (not necessarily mono-energetic) off of a power law, isotopic distribution of electrons

The $q(\epsilon) = (4\pi/c)I_n(\epsilon) = F_n/c$. We will use this below. Note also that

$$j_n(\epsilon_f) = N\sigma_T \frac{1}{2}\int_{-1}^{1} I_n(\epsilon_f, \mu)d\mu = N\sigma_T I_n \tag{540}$$

for isotropic I_n.

We have for the total scattered power per volume per energy

$$\frac{dE}{dV d\epsilon_f dt} = 4\pi\epsilon_f j_n(\epsilon_f) \tag{541}$$

Then for $N(\gamma) = C\gamma^{-p}$, we have

$$\frac{dE}{dV d\epsilon_f dt} = \frac{1}{2}c\sigma_T C \int d\epsilon_i q(\epsilon_i)(\epsilon_f/\epsilon_i) \int d\gamma \gamma^{-(p+2)}(1 - \epsilon_f/4\gamma^2\epsilon_i + ...) \tag{542}$$

Change from γ to $x = \epsilon_f/4\gamma^2\epsilon_i$. That implies $d\gamma = -(2^{1/2}/4)(\epsilon_f/\epsilon_i)^{1/2}x^{-3/2}dx$ So we have:

$$\frac{dE}{dV d\epsilon_f dt} \propto \epsilon_f^{\frac{1-p}{2}} \int d\epsilon_i \ \epsilon_i^{\frac{p-1}{2}} q(\epsilon_i) \int_{x_1}^{x_2} (1 - x + ..)dx. \tag{543}$$

with $x_1 = \epsilon_f/4\gamma_1^2\epsilon_i$ and $x_2 = \epsilon_f/4\gamma_2^2\epsilon_i$. If $\gamma_2 >> \gamma_1$ and $q(\epsilon_i)$ peaks strongly at $\bar{\epsilon}_i$ then second integral is just a number that factors out (like for synch.). We then have

$$\frac{dE}{dV d\epsilon_f dt} \propto \epsilon_f^{\frac{1-p}{2}} \int d\epsilon_i \ \epsilon_i^{\frac{p-1}{2}} q(\epsilon_i). \tag{544}$$

Note this is restricted to the regime $4\gamma_1^2\bar{\epsilon}_i << \epsilon_f << 4\gamma_2^2\bar{\epsilon}_i$.

Note that the power law is the same as for synchrotron!.

For an initial photon black body distribution, we have

$$q(\epsilon_i) = \frac{8\pi\epsilon_i^2}{h^3c^3}\frac{1}{e^{\epsilon_i/kT} - 1} \tag{545}$$

and

$$\frac{dE}{dV d\epsilon_f dt} \propto (kT)^{\frac{p+5}{2}} \epsilon_f^{\frac{1-p}{2}}. \tag{546}$$

Note the strong dependence on temperature. Also note that the temperature is that of the incident radiation (e.g. like a star radiating incident photons into a corona of highly accelerated electrons).

Energy transfer for repeated scattering, Y parameter

When does scattering affect total photon energy? again assume Thomson limit in rest frame of electron. Consider limit $\gamma\epsilon_i << mc^2$

Define y- parameter such then when $y > 1$, scattering changes spectrum and energy wheras when $y < 1$, scattering does not affect energy but can still change spectrum. The appropriate definition is the typical fractional energy change per scattering times the mean number of scatterings:

$$y = \frac{\Delta\epsilon_i}{\epsilon_i} \times (\text{mean number of scatterings}). \tag{547}$$

Consider a thermal electron distribution in the non-relativistic limit. Recall that

$$\epsilon'_f \sim \epsilon'_i \left(1 - \frac{\epsilon'_i}{mc^2}(1 - Cos\Theta)\right), \tag{548}$$

when $\epsilon'_i << mc^2$. If we average over Θ (the angle between incident and scattered photons) we get

$$\frac{\Delta\epsilon'}{\epsilon'_i} = -\frac{\epsilon'_i}{mc^2}, \tag{549}$$

where this is a small term. In the lab frame, to lowest order in the two small parameters ϵ/mc^2. and kT/mc^2 we must have

$$\frac{\Delta\epsilon}{\epsilon_i} = -\frac{\epsilon_i}{mc^2} + \alpha kT/mc^2. \tag{550}$$

We will determine α by first considering the case in which the electrons and photons are in thermal equilibrium. but interact only through scattering. Ignore stimulated emission. Photons thus have a Bose-Einstein distribution in which particle number cannot change. We are in the classical regime, where we ignore the stimulated emission. (This means that $e^{\epsilon-\mu} >> 1$), This gives

$$n_\epsilon = K\epsilon^2 e^{-\epsilon/kT}, \tag{551}$$

where K includes e^μ among the other constants. so that

$$\langle\epsilon\rangle = \frac{\int \epsilon \frac{dN}{d\epsilon} d\epsilon}{\int \frac{dN}{d\epsilon} d\epsilon} = 3kT \tag{552}$$

and

$$\langle \epsilon^2 \rangle = 12(kT)^2. \tag{553}$$

In this limit, no net energy can be exchanged between photons and electrons so we have

$$\langle \Delta\epsilon \rangle = 0 = \frac{\alpha kT}{mc^2}\langle \epsilon \rangle - \frac{\langle \epsilon^2 \rangle}{mc^2} = \frac{3kT}{mc^2}(\alpha - 4)kT, \tag{554}$$

thus $\alpha = 4$. Thus for non-relativistic electrons in thermal equilibrium,

$$(\Delta\epsilon)_{NR} = \frac{\epsilon}{mc^2}(4kT - \epsilon). \tag{555}$$

This shows that when $\epsilon > 4kT$, the energy is transferred from photons to electrons. When $\epsilon < 4kT$ energy is transferred from electrons to photons, inverse Compton scattering.

Now consider the limit $\gamma >> 1$. Then ignoring the energy transfer in the electron rest frame we recall that

$$\epsilon'_i = \epsilon_i \gamma(1 - \beta cos\theta_i) \tag{556}$$

and

$$\epsilon_f = \epsilon'_f \gamma(1 + \beta cos\theta'_f). \tag{557}$$

These give

$$\epsilon_f = \epsilon_i \gamma^2(1 - \beta cos\theta_i)(1 + \beta cos\theta'_f). \tag{558}$$

The average over angles gives

$$\epsilon_f - \epsilon_i = \epsilon_i(\gamma^2(1 - \beta^2\langle(cos\theta_i)(cos\theta'_f)\rangle) - 1). \tag{559}$$

But since the average is over both angles, only when they are equal does the average contribute and only when the cosines are equal and opposite is energy gained. The first integral then gives a delta function and the second then is an average over $-(cos\theta_i)^2$. Thus we have, using $\gamma^2 - 1 = \gamma^2\beta^2$

$$\epsilon_f - \epsilon_i = \epsilon_i(\gamma^2(1 + \beta^2/3) - 1) = . \tag{560}$$

This can be written approximately, for $\gamma > 1$.

$$(\Delta\epsilon)_R \sim \frac{4}{3}\gamma^2\epsilon_i \tag{561}$$

Then for thermal distribution of relativistic electrons $E \propto \gamma$ and thus using (553)

$$\langle \gamma^2 \rangle = \frac{\langle E^2 \rangle}{(mc^2)^2} = 12 \left(\frac{kT}{mc^2} \right)^2 . \tag{562}$$

Thus

$$(\Delta\epsilon)_R \sim \frac{4}{3}\gamma^2 \epsilon = 16\epsilon \left(\frac{kT}{mc^2} \right)^2 . \tag{563}$$

That is the first factor for the y parameter.

For the second factor (recall chap 1 of book) we recall that the mean number of scatterings is just the size of the medium over the mean free path all squared: $N_s = (R/l)^2 = \tau^2$ for $\tau >> 1$, and $N_s \sim 1 - e^{-\tau} \sim \tau$ for $\tau << 1$. Thus we take

$$N_s = Max(\tau_{es}^2, \tau_{es}) \tag{564}$$

and $\tau_{es} = \rho\kappa_{es}R$, where κ_{es} is the electron scattering opacity and is $0.4\mathrm{cm}^2/\mathrm{g}$ for ionized hydrogen.

Thus for the non-relativistic regime we have for the y parameter using (555) and (564) we have

$$y = \left(\frac{4kT}{mc^2} - \epsilon \right) Max(\tau_{es}^2, \tau_{es}) \sim \left(\frac{4kT}{mc^2} \right) Max(\tau_{es}^2, \tau_{es}), \tag{565}$$

where the latter similarity follows for $\epsilon << 4kT$.

For the relativistic regime we have using (563) and (564)

$$y = 16 \left(\frac{kT}{mc^2} \right)^2 Max(\tau_{es}^2, \tau_{es}). \tag{566}$$

In the non-relativistic regime, it can be shown that the gain in energy of the photon as is scatters off non-relativistic electrons satisfies $\epsilon_f/\epsilon_i = e^y$. (problem 7.1).

In general, the Compton y parameter is frequency dependent. Then $\tau_{es} = \tau_{es}(\nu)$. Then since absorption takes away photons before they can scatter indefinitely, we determine τ_{es} as the value such that $\tau_*(\nu) = 1$, where the latter is the effective optical depth. Define $l_*(\nu)$ as the effective path the photon travels before absorption. When $l_* < R$ we have

$$\tau_{es}(\nu) = \rho\kappa_{es} l_*(\nu), \tag{567}$$

Since

$$l_* = lN^{1/2} \sim l\left(\frac{\alpha_\nu + \sigma_\nu}{\alpha_\nu}\right)^{1/2} \sim [\alpha_\nu(\alpha_\nu + \sigma_\nu)]^{-1/2} \tag{568}$$

since $l = (\alpha_\nu + \sigma_\nu)^{-1}$. If we then note that $\kappa_a(\nu) = \rho^{-1}\alpha_\nu$ and $\kappa_{es}(\nu) = \rho^{-1}\sigma_\nu$ we have

$$\tau_{es}(\nu) = \rho\kappa_{es}l_*(\nu) = \rho\kappa_{es}[\alpha_\nu(\nu)(\alpha_\nu + \sigma_\nu)]^{-1/2} = \left(\frac{\kappa_{es}^2}{\kappa_a^2 + \kappa_{es}\kappa_a}\right)^{1/2}. \tag{569}$$

Note that κ_a and κ_{es} are the absorption and scattering opacities.

LECTURE 19

Inverse Compton Power for repeated scatterings: relativistic electrons small optical depth

Interesting result: we will find that a photon power law spectrum can arise even when electrons are not a power law, if there are repeated scatterings.

Define A as the amplification per scattering

$$A = \epsilon_f/\epsilon = \frac{4}{3}\langle\gamma^2\rangle = 16(kT/mc^2)^2, \quad (570)$$

for a thermal distribution of electrons. Consider initial photon energy distribution with mean photon energy ϵ_i satisfying

$$\epsilon_i << mc^2\langle\gamma^2\rangle^{-1/2}, \quad (571)$$

and number intensity $I_n(\epsilon_i)$. Then after k scatterings, the energy of a mean initial photon is

$$\epsilon_k = \epsilon_i A^k. \quad (572)$$

If the medium has small scattering optical depth then the probability of a photon making k scatterings is $p_k(\tau_{es}) \sim \tau_{es}^k = (1-e^{-\tau_{es}})^k$. This is because probability of a photon propagating through optical depth τ_{es} before single scattering is $e^{-\tau_{es}}$.

The intensity is also proportional to τ_{es}^k quantity since the width of the probability distribution (histogram) is of order the peak frequency. Thus

$$I_n(\epsilon_k) \sim I_n(\epsilon_i)\tau_{es}^k \sim I(\epsilon_i)\left(\frac{\epsilon_k}{\epsilon_i}\right)^{-s} \quad (573)$$

where $s = -ln(\tau_{es})/ln(A)$. To prove, take ln_τ of (573) and get back (572).

This holds only when

$$\epsilon_k \leq \langle\gamma^2\rangle^{1/2}mc^2, \quad (574)$$

to ensure last scattering is still in Thomson type limit in rest frame. The final photons are those which emerge therefore with energy of order kT for thermal distribution of relativistic electrons. The total Compton power is

$$P \propto \int^{A^{1/2}mc^2} I_n(\epsilon_k)d\epsilon_k = I_n(\epsilon_i)\epsilon_i\left[\int_1^{A^{1/2}mc^2/\epsilon_i} x^{-s}dx\right], \quad (575)$$

where the upper bound is $\sim kT$ as required for the upper limit on the allowed energy range.

You can see that the initial energy is amplified by the factor in brackets, which is important for $s < 1$. When does this happen? When

$$-ln\tau_{es} < lnA \tag{576}$$

that is, when $A\tau_{es} \geq 1$.

This is equivalent to the condition that $y_R > 1$ for small τ_{es}, since A is given by (570). Compare to (566). Thus energy amplifciation important in this regime.

Repeated Scatterings by non-relativistic electrons

In general we need to solve an explicit equation for evolution of the photon distribution function due to repeated non-relativistic inverse Compton scatterings (Kompaneets equation). For frequencies with $y << 1$ and $y >> 1$ we can get away with approximations. We we consider these regimes here.

We are interested in thermal media with both emission and absorption take place by bremsstrahlung, and we want to see the effect of Compton scattering in such a medium.

Note that bremsstrahlung absorption takes place more strongly at lower frequencies.

There are three key frequencies of interest. First define the frequency ν_0 such that the scattering and the absorption opacities are equal. We recall that $\alpha_\nu = \rho\kappa_{\nu,a}$, $\sigma(\nu) = \rho\kappa_{es}(\nu)$, and $\kappa_{es} = \sigma_T/m_p = 0.4\text{cm}^2\text{g}$, we have

$$\kappa_{es} = \kappa_{ff}(\nu) \tag{577}$$

$$\alpha_\nu^{ff} = 3.7 \times 10^8 T^{-1/2} Z^2 n_i n_e \nu^{-3}(1 - e^{-h\nu/kT})\bar{g}_{ff}. \tag{578}$$

That is then

$$\frac{x_0^3}{1 - e^{-x_0}} \sim 4 \times 10^{25} T^{-7/2} \rho \bar{g}_{ff}(x_0), \tag{579}$$

or

$$x_0 = 6 \times 10^{12} T^{-7/4} \rho^{1/2} [\bar{g}_{ff}(x_0)]^{1/2}, \tag{580}$$

for $x_0 << 1$. where $\bar{g}_{ff}(x) = 3\pi^{-1/2} ln(2.25/x)$. For $x = h\nu/kT < x_0$ scattering is not important. We assume $x_0 << 1$ below.

The second frequency ν_t is that for which the medium becomes "effectively thin." Using

$$(L/l_*)^2 = 1 = \rho^2 \kappa_a(\nu)(\kappa_a(\nu) + \kappa_{es})L^2. \tag{581}$$

Then dividing by $\tau_{es}^2 = \rho^2 L^2 \kappa_{es}^2$, and ignoring the term $(\kappa_a/\kappa_{es})^2$ we have

$$\kappa_{es} = \kappa_a(\nu)\tau_{es}^2. \tag{582}$$

This gives the condition

$$\frac{x_t^3}{1-e^{-x_t}} \sim 4 \times 10^{25} T^{-7/2} \rho \bar{g}_{ff}(x_t) \tau_{es}^2. \quad (583)$$

The τ_{es} is the total optical depth to electron scattering. For values $x > x_t$ the absorption is unimportant. For $x_0 < x < x_t$ both scattering and absorption are important.

Third, define the frequency ν_{coh} where inverse Compton can be important. This is defined such that $y(\nu_{coh}) = 1$. That is for $\nu > \nu_{coh}$, inverse Compton is important. But this frequency is only of interest if the y parameter for the full medium is greater than unity, implying that scattering is important for some photons at least.

In this case we then have

$$1 = \frac{4kT}{mc^2} \tau_{es}^2(\nu) \quad (584)$$

and using

$$\tau_{es} = \left(\frac{\kappa_{es}/\kappa_a(\nu)}{1 + \kappa_a(\nu)/\kappa_{es}} \right)^{1/2} \quad (585)$$

gives

$$\kappa_{es} = \frac{mc^2}{4kT} \kappa_{ff}(\nu_{coh}). \quad (586)$$

Then again using $\kappa_{ff} = \alpha_{ff}/\rho$ we have

$$x_{coh} = 2.4 \times 10^{17} \rho^{1/2} T^{-9/4} [\bar{g}_{ff}(x_{coh})]^{1/2}. \quad (587)$$

Note that x_{coh} must be less than x_t because for $x > x_t$ the medium is effectively thin. Inverse Compton cannot be significantly important when the medium is effectively thin.

Now let us use these frequencies to get info about various regimes.

Modified blackbody spectral regime $y << 1$

For $y << 1$ there is only coherent (elastic) scattering. You derived in problem 1.10 that the formula for the intensity in a scattering and absorbing medium is

$$I_\nu = \frac{2B_\nu}{1 + \sqrt{(1 + \kappa_{es}/\kappa_{ff})}}. \quad (588)$$

For $x << x_0$, this reduces to blackbody spectrum.

For $x >> x_0$ this reduces to modified blackbody spectrum, that is

$$I_\nu^{mb} = 2B_\nu \left(\frac{\kappa_{ff}}{\kappa_{es}}\right)^{1/2} = 8.4 \times 10^{-4} T^{5/4} \rho^{1/2} \bar{g}_{ff}^{1/2} x^{3/2} e^{-x/2} (e^x - 1)^{-1/2}. \quad (589)$$

Note that for $x_0 << x << 1$, $I_\nu^{mb} \propto \nu$ not $I_\nu^{rj} \propto \nu^2$ as in Rayleigh Jeans. (Refer to (580)). Less steep than RJ regime.

The total flux is

$$F^{mb} = \sigma T^4 (\kappa_R/\kappa_{es})^{1/2} \sim 2.3 \times 10^7 T^{9/4} \rho^{1/2} \text{erg/s cm}^{-2}, \quad (590)$$

here using the Rosseland mean absorption coefficient, (defined as inverse of expectation value of $\frac{1}{\alpha_{ff}}$, weighted by $\partial B_\nu/\partial T$.) where $\alpha_R = 1.7 \times 10^{-25} T^{-7/2} Z^2 n_e n_i \bar{g}_R$ The flux is reduced from the bbody flux. This is because if you think of absorption and emission in equilibrium as "trapping" radiation density, then the radiation density is higher where there is more trapping.

** For finite media, we have to consider x_t. When $x_t < x_0$, emission is blackbody for $x < x_t$ and optically thin bremsstrahlung for $x > x_t$ where scattering is not important.

For $x_0 < x_t < 1$, the emission is (588) for $x < x_t$, and is optically thin brems. for $x > x_t$.

(all this is because scattering is more important than abs for $x > x_0$, but system is effectively thin for $x > x_t$.

For $x_t > 1$ (588) is good for all x.

Wien Spectra $y >> 1$

Inverse Compton may be important depending on whether $x_{coh} << 1$ or $x_{coh} >> 1$. In the latter case IC can be neglected since the majority of photons $x < x_{coh}$ undergo coherent scattering (previous discussion applies!)

Here consider $x_{coh} << 1$. In the limit $x_0 << 1$, then we have

$$x_{coh} = (mc^2/4kT)^{1/2} x_0. \quad (591)$$

For $x < x_{coh}$ Eq. (588) is OK. For $x > x_{coh}$ a saturated inverse Compton spectra will result. This is a Wien spectrum, maximal scattering produces a kind of thermal equilibrium. We have

$$I_\nu^W = \frac{2h\nu^3}{c^2} n_\gamma = \frac{2h\nu^3}{c^2} e^{-f} e^{-h\nu/kT}, \quad (592)$$

where e^{-f} is the rate at which photons are input into the system. Total flux is then

$$F^W = \pi \int I_\nu^W d\nu = \frac{12\pi e^{-f} k^3 T^4}{c^2 h^3}. \tag{593}$$

The mean photon energy is $3kT$.

The intensity is shown below fig 7.4

As you might expect, the flux is given by

$$F^W \sim RA\epsilon_{ff}, \tag{594}$$

which is the bremsstrahlung initial photon energy density per unit time, times an amplification factor A times a thickness over which the emission is coming from. If $x_t << 1$ the medium is effectively thin, so thickness is size of medium. For $x_t >> 1$, photons of $x > x_{coh}$ amplify to kT. So we replace R by $\bar{R}$ where $\bar{R}$ satisfies $\tau_*(\bar{R}, x = 1) = 1$.

LECTURE 20

Plasma Effects

Debye Length:

Consider plasma composed of charge Ze and electrons of charge e. For charge neutrality we have

$$n_e Z_i = n_i, \tag{595}$$

summation implied. The ions attract the electrons. Focus on a single ion and let us see the influence of the ambient plasma on shielding the potential of this ion. The Coulomb law takes the form

$$\nabla^2 \phi = -4\pi \rho_c \tag{596}$$

where

$$\rho_c = Z_i e \delta(\mathbf{r}) - e n_e e^{e\phi/kT} + e n_i Z_i e^{-eZ_i\phi/kT}, \tag{597}$$

assuming $T_i = T_e$ and n_i and n_e are the average mean densities, and assuming Boltzmann spatial distributions. Let us assume for the moment that the exponentials are small (justified later). The first terms in the expansion cancel under the assumption of lowest order charge neutrality. Then we have

$$\nabla^2 \phi = -4\pi Z_i e \delta(\mathbf{r}) + 4\pi \frac{e^2}{kT}(n_e + n_i Z_i^2)\phi = -4\pi Z_i e \delta(\mathbf{r}) + 4\pi\phi/l_d^2, \tag{598}$$

where $l_d = (kT/e^2(n_e + n_i Z_i^2))^{1/2}$ is the Debye length and is $\sim (kT/2e^2 n_e)^{1/2}$ for $Z_i = 1$. Spherical symmetry means that the only derivative components are r components, thus the laplacian becomes

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\phi}{dr} = -4\pi Z_i e \delta(\mathbf{r}) + 4\pi\phi/l_d^2. \tag{599}$$

The appropriate solution which satisfies $\phi = eZ_i/r$ at $r = 0$ and 0 at $r = \infty$ is

$$\phi = \frac{Z_i e}{r} e^{-r/l_D}. \tag{600}$$

What this means is that the Coulomb potential is reduced at large r. The statistical attraction of electrons and repulsion of ions shield the primary charge.

Ions and electrons contribute equally to the shielding, because it depends on the ratio of particle speed to mass through the temperature, and not solely on the particle mass. The message is that for $r > l_d$, the free charge potential must be modified.

Conditions for the above analysis require that there be enough particles inside a volume $\sim l_d^3$ for a statistical treatment, that is $n_d l_d^3 >> 1$. (The inverse of this quantity is called the plasma parameter.) Note that the mean distance between electrons is $n_e^{-1/3}$. Thus the mean electrostatic energy/kT

$$e^2 n_e^{1/3}/kT \sim n_e^{-2/3} l_d^{-2} << 1, \tag{601}$$

in the statistical regime, thus justifying the earlier expansion.

Plasma Frequency

The electron contribution to the inverse Debye length is

$$l_{d,e}^{-2} \sim 4\pi n_e e^2/kT = 4\pi n_e e^2/m_e v_e^2 = \omega_{p,e}^2/v_e^2, \tag{602}$$

where v_e is the electron thermal velocity and $\omega_{p,e} = 4\pi n_e^2 e^2/m_e$ is the electron plasma frequency.

The meaning is as follows. Imagine displacing the electrons of a plasma relative to the ions by an amount x. The resulting electric field is like that of a two plate capacitor with surface charge density $\pm e n_e x$, so that

$$E = 4\pi e n_e x. \tag{603}$$

This comes from integrating $\nabla \cdot \mathbf{E} = 4\pi\rho_c$. The equation of motion for the electron between the plates is then

$$\ddot{x} = -eE/m_e = -e^2 x 4\pi n_e/m_e = -\omega_{pe}^2 x/m_e. \tag{604}$$

Thus the electron plasma frequency is the natural oscillator frequency for an electron from the charge separation. Ions oscillate at a much slower rate and so one can consider them to be fixed. Numerically $\omega_{pe} = 5.6 \times 10^4 n_e^{1/2}$. Larger frequency forcing or disturbances will set up faster oscillations. Otherwise fast oscillations are set up to counteract the disturbances. No charge separation occurs for disturbances on scales larger than the Debye length. For such disturbances one is in the neutral fluid MHD regime.

Now lets come at this from a slightly different angle. Consider the linearized equation of motion of an electron fluid (take, $n, \mathbf{E}, \mathbf{B}, \mathbf{v}$ equal to zeroth order part + perturbation and consider only linear surviving terms). We get

$$m_e n_0 \partial_t \mathbf{v}_1 = -e n_0 \mathbf{E}_1 \tag{605}$$

From perturbed Maxwells equations we get

$$\nabla \times \mathbf{B}_1 = -(4\pi/c) n_o e \mathbf{v}_1 + (1/c)\partial_t \mathbf{E}_1 \tag{606}$$

and

$$\nabla \times \mathbf{E}_1 = -(1/c)\partial_t \mathbf{B}_1. \tag{607}$$

Note that no n_1 appears in these equations. Lets assume $e^{i\omega t}$ solutions for the time dependence. Then for (605) we have

$$\mathbf{v}_1 = -(ie/\omega m_e)\mathbf{E}_1. \tag{608}$$

Plugging this into (606) gives

$$\nabla \times \mathbf{B}_1 = -(i/c)\epsilon\omega \mathbf{E}_1, \tag{609}$$

where $\epsilon = 1 - \omega_p^2/\omega^2$. Note that in standard E+M, this represents the dielectric constant since by replacing of the $-iw$ in (609) with ∂_t we would recover

$$\nabla \times \mathbf{B}_1 = \partial_t \epsilon \mathbf{E}_1. \tag{610}$$

More on this in next section.

No E-M wave propagation at $\omega < \omega_{p,e}$

Assuming wave propagation solutions to Maxwell's equations of the form $e^{i\mathbf{k}\cdot\mathbf{x}-\omega t}$. Maxwell's equations for the plasma become

$$i\mathbf{k}\cdot\mathbf{E}= 4\pi\rho_c; \quad i\mathbf{k}\cdot\mathbf{B} = 0. \tag{611}$$

and

$$i\mathbf{k}\times\mathbf{E} = i\omega/c\mathbf{B}; \quad i\mathbf{k}\times\mathbf{B} = 4\pi\mathbf{j}/c - i\mathbf{E}\omega/c. \tag{612}$$

We again ignore the magnetic force (non-relativistic) and then for the equation of motion we have

$$\mathbf{v} = e\mathbf{E}/i\omega m_e. \tag{613}$$

The current density then satisfies

$$\mathbf{j} = \frac{ine^2}{\omega m_e}\mathbf{E} = \sigma\mathbf{E}. \tag{614}$$

Charge conservation gives

$$-i\omega\rho_c + i\mathbf{k}\cdot\mathbf{j} = 0, \tag{615}$$

so that

$$\rho_c = \mathbf{k}\cdot\mathbf{j}/\omega = \sigma\mathbf{k}\cdot\mathbf{E}/\omega. \tag{616}$$

If we again define the dielectric constant

$$\epsilon = 1 - 4\pi\sigma/i\omega = 1 - \omega_{p,e}^2/\omega^2, \tag{617}$$

we can then get "modified" vacuum equations:

$$i\mathbf{k} \cdot \epsilon\mathbf{E} = 0; \quad i\mathbf{k} \cdot \mathbf{B} = 0 \tag{618}$$

and

$$i\mathbf{k} \times \mathbf{E} = i\omega\mathbf{B}/c; \quad i\mathbf{k} \times \mathbf{B} = -i\omega\epsilon\mathbf{E}/c. \tag{619}$$

These are the source free Maxwell's equations. We solved them earlier, now we have a revised relation:

$$c^2k^2 = \epsilon\omega^2 = \omega^2[1 - (\omega_{p,e}/\omega)^2]. \tag{620}$$

Thus we see that

$$kc = (\omega^2 - \omega_{p,e}^2)^{1/2}. \tag{621}$$

Thus for $\omega < \omega_{p,e}$ the wave number is imaginary. This means that such waves do not propagate. Plugging back into $e^{i\mathbf{k}\cdot\mathbf{x} - \omega t}$ means that a wave propagating in the direction $\mathbf{x}$ will decay on length scale $\sim 2\pi c/\omega_p$, and instead reflects! This allows probing the density of the ionosphere (90-150km above surface).

Send a pulse of radiation upward. Where n is larger than critical, $\omega_p > \omega$ and the radiation is reflected. Electron density can be determined as a function of height. (MHz range) when one uses also the time measured for the signal to go up and come back down.

Dispersion Measure

For $\omega > \omega_p$, the E-M radiation propagates with phase velocity

$$v_{ph} = \omega/k = c/n_r \geq c \tag{622}$$

where $n_r = \epsilon^{1/2} = (1 - \omega_p^2/\omega^2)^{1/2}$ is the index of refraction. The group velocity travels at $v \leq c$

$$v_g = \partial\omega/\partial k = c(1 - \omega_p^2/\omega^2)^{1/2}. \tag{623}$$

When electron density varies, and thus the index of refraction, radiation travels along curved path. From Snell's law, the curved trajectories satisfy

$$d(n_r\hat{\mathbf{k}})/dl = \nabla n_r, \tag{624}$$

where l is the trajectory path length and $\hat{\mathbf{k}}$ is the ray direction.

One application of the spread in group velocities for waves of different frequencies is for pulsars. Pulsars have pulses with a range of frequencies per pulse. Interaction with ISM means that different frequency ranges will travel at different group velocities and thus reach observer at different times. If pulsar is a distance d away, then the time elapsed before the pulse reaches earth is

$$t_p = \int_0^d ds/v_g, \tag{625}$$

where ds is line of sight distance differential. Interstellar space has kHz plasma frequencies (Radio waves are in GHz). Thus assume $\omega >> \omega_p$. Then we have

$$v_g^{-1} = \frac{1}{c}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2} \sim \frac{1}{c}\left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right). \tag{626}$$

Thus

$$t_p \sim \frac{d}{c} + \frac{1}{2c\omega}\int_0^d \omega_p^2 ds. \tag{627}$$

The second term is the plasma correction. To get a convenient measured quantity take the frequency derivative to obtain

$$dt_p/d\omega = -\frac{4\pi e^2}{cm\omega^3}D, \tag{628}$$

where

$$D = \int_0^d n_e ds \tag{629}$$

is the dispersion measure. If we assume that the density is constant across interstellar space between the observer and the pulsar, we get the pulsar distance.

Faraday Rotation

Consider a fixed magnetic field. This means the plasma is anisotropic since the plasma properties will be different perpendicular to and parallel to the magnetic field. We also can expect to find the cyclotron frequency entering the problem for non-relativistic plasmas. Ignoring the particle pressure is called the cold plasma approximation, which we will do.

The dielectric constant is now a tensor, and also waves have different polarizations so in general the coefficient multiplying the $e^{(i\mathbf{k}\cdot\mathbf{x}-\omega t)}$ in the waves just discussed is not a constant.

Lets assume that the fixed field is stronger than the fields associated with the propagating wave. Then

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - \frac{e}{c}\mathbf{v}\times\mathbf{B} \tag{630}$$

Consider a circularly polarized wave

$$\mathbf{E}(t) = Ee^{-i\omega t}(\hat{\epsilon}_1 \mp i\hat{\epsilon}_2). \tag{631}$$

Assume that the waves propagate along the uniform field, which we take to be in the z-direction, $\hat{\epsilon}_3$. Using $d/dt = -i\omega$ we then have

$$\mathbf{v}(t) = \frac{e}{im(\omega \pm \omega_B)}\mathbf{E}(t). \tag{632}$$

Comparing to (613) and (617), we have a new dielectric constant

$$\epsilon_{R,L} = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_B)}, \tag{633}$$

where R, L corresponds to $-, +$. These travel with different velocities since the dielectric constant enters the wave velocity. Thus the plane of a linearly polarized wave, which is the superposition of two circularly polarized waves rotates as the wave propagates. That is Faraday rotation. fig 8.1

For a constant wavenumber, the phase angle of rotation of the electric field vector is $\mathbf{k}\cdot\mathbf{d}$ where d is the distance the wave travels. (This dot product is $= 2\pi d/\lambda$ where λ is the wavelength.) This can be computed more generally, for a space-varying wavenumber, from

$$\phi_{R,L} = \int_0^d k_{R,L} ds, \tag{634}$$

where

$$k_{R,L} = \frac{\omega}{c}\sqrt{\epsilon}_{R,L}. \tag{635}$$

The plane polarized wave gets rotated by an angle $\Delta\theta = 0.5(\phi_R - \phi_L)$. If we assume that $\omega >> \omega_p$ and $\omega >> \omega_B$ we have

$$k_{R,L} = \frac{\omega}{c}\left[1 - \frac{\omega_p^2}{2\omega^2}(1 \mp \omega_B/\omega)\right]. \tag{636}$$

Thus

$$\Delta\theta = \frac{1}{2}\int_0^d (k_R - k_L) ds = \frac{1}{2}\int_0^d (c\omega^2)^{-1}\omega_B \omega_p^2 ds. \tag{637}$$

Then substituting for ω_p and ω_B we have

$$\Delta\theta = \frac{2\pi e^3}{m^2 c^2 \omega^2}\int_0^d nB_{||} ds = 1.2 \times 10^4 \lambda^2 \int_0^d nB_{||} ds. \tag{638}$$

By measuring the relative rotation at different frequencies we can get info about the magnetic field.

Combining the pulsar dispersion measure and the Faraday rotation, we can get an estimate of the mean magnetic field. Suppose for example that we have a signal from a pulsar measured to have a time delay of $dt_p/d\omega = 10^{-5}\text{s}^2$ and a Faraday rotation that varies with frequency as $d\Delta\theta/d\omega = 2 \times 10^{-4}\text{sec}$. Let us assume the measurements are made around frequency of 100MHz and that we don't know the distance of the pulsar. From this we can estimate the mean magnetic field defined as

$$\langle B_{||} \rangle = \frac{\int nB_{||} ds}{\int nds}. \tag{639}$$

We simply take the derivative of (638) and divide by (628) to get

$$\frac{d\Delta\theta/d\omega}{dt_p/d\omega} \sim 2 \times 10^7 B_{||}\text{s}^{-1}. \tag{640}$$

Substituting $dt_p/d\omega = 10^{-5}\text{s}^2$ and $d\Delta\theta/d\omega = 2 \times 10^{-4}\text{sec}$, we get $B_{||} \sim 10^{-6}$ G. Which is typical for the ISM. Note that this is independent of the frequency of emission.

LECTURE 21

Effects of plasma on high energy radiation

The radiation from all of the processes that we have discussed is subject to the influence of the plasma, e.g. little radiation will propagate below the plasma frequency. Faraday polarization will also set up a depolarization screen.

There are some effects on the high energy processes themselves which change the radiative process fundamentally. Need to then consider the induced motions and subsequent emission from particles in the ambient medium responding to the radiation. We consider a dielectric constant of the plasma which is independent of frequency, which is a cheat, but will capture basic ideas.

Maxwell's equations in the presence of a constant dielectric become

$$\nabla \cdot \mathbf{E} = 4\pi\rho/\epsilon;\ \nabla \cdot \mathbf{B} = 0 \tag{641}$$

$$\nabla \times \mathbf{E} = \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t};\ \nabla \times \mathbf{B} = \frac{4\pi \mathbf{j}}{c} + \frac{\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{642}$$

These equations result from Maxwell's equations with the substitution

$$\mathbf{E} \to \mathbf{E}\sqrt{\epsilon};\ \mathbf{B} \to \mathbf{B}, \tag{643}$$

$$\mathbf{A} \to \mathbf{A};\ \phi \to \sqrt{\epsilon}\phi \tag{644}$$

and

$$e \to \frac{e}{\sqrt{\epsilon}};\ c \to \frac{c}{\sqrt{\epsilon}}. \tag{645}$$

Recall that these are solved by the Liénard-Wiechart potentials

$$\phi\sqrt{\epsilon} = \frac{-e/\sqrt{\epsilon}}{\kappa(t_{ret})R(t_{ret})} \tag{646}$$

where

$$\tilde{\kappa} = 1 - \mathbf{n} \cdot \vec{\beta}\sqrt{\epsilon} \tag{647}$$

and

$$\mathbf{A} = -\frac{e\mathbf{u}(t_{ret})}{c\tilde{\kappa}(t_{ret})R(t_{ret})}, \tag{648}$$

so that

$$\sqrt{\epsilon}\mathbf{E}_{rad}(\mathbf{r}, t) = \frac{-e}{c}\left[\frac{\mathbf{n}}{\tilde{\kappa}^3 R} \times \{(\mathbf{n} - \tilde{\beta}) \times \dot{\tilde{\beta}}\}\right] \tag{649}$$

and

$$\mathbf{B}_{rad} = [\mathbf{n} \times \sqrt{\epsilon}\mathbf{E}_{rad}] \tag{650}$$

Cherenkov effect

Charge moving uniformly in a vacuum cannot radiate: No acceleration, no force, no energy change. Also true for a charge moving uniformly in dielectric medium if the phase velocity exceeds particle velocity. This is because the modified L-W potentials still then has only the term proportional to $1/R^2$.

However, if medium has index of refraction larger than 1, then the particle speed can be larger than the phase velocity. Then $\tilde{\kappa} = 1-\beta n_r cos\theta$ can vanish when $cos\theta = 1/n_r\beta$. ($n_r \equiv \sqrt{\epsilon}$). The potentials can become quite large at places where this holds, and now there can be effective radiation from the $1/R^2$ term (consider Poynting flux argument, not necessarily zero at large distances in this case).

Note also that for $v > c/n_r$, the potentials can be determined by two retarded positions. fig 8.2

Note the cone, for this regime. The resulting radiation is confined within the cone of angle θ. The faster the particle the narrower the cone. Radiation propagates at c/n_r at the angle θ. Note that

$$c/n_r < v < c \tag{651}$$

for Cherenkov.

Note that the Cherenkov effect operates for weakly ionized plasmas, because for highly ionized plasmas, the conductivity is large and the conductivity dominates in determining the index of refraction. Sometimes when people refer to a "plasma" they mean a "fully ionized plasma" so be careful. Actually, for that case, we have also been ignoring the temperature contribution to the conductivity and dynamics (=cold plasma approximation).

LECTURE 22

Razin effect

For the index of refraction $n_r < 1$ as in a cold plasma, there is no Cherenkov radiation.

In this case however, beaming can be reduced and thus synchrotron radiation made less potent. Since the beaming is the result of the factor $\tilde{\kappa} = 1 - \beta n_r cos\theta$, for $n_r < 1$ beaming is suppressed since no matter how large β becomes, $\tilde{\kappa}$ is still significantly larger than zero.

The angle for the beaming in a vacuum was shown to be $\theta \sim 1/\gamma \sim (1-\beta^2)^{1/2}$. But for a plasma we have instead

$$\theta = (1 - n_r^2\beta^2)^{1/2}. \tag{652}$$

If $n_r \sim 1$ then this reduces to the approximate vacuum case. If $n_r << 1$ but $\beta \sim 1$, then

$$\theta = (1 - n_r^2)^{1/2} = \omega_p/\omega, \tag{653}$$

for a cold plasma. Thus at low frequencies, the medium will quench the wave propagation and the plasma effects dominate the beaming effects. At high frequencies, θ decreases until it gets down to $1/\gamma$ and then the vacuum results hold. Thus the plasma medium effects are unimportant when

$$\omega >> \gamma\omega_p. \tag{654}$$

The suppression of the beaming effect has a large effect on synchrotron emission. Below the frequency $\gamma\omega_p$ the spectrum will be largely cut off. This effect dominates the usual plasma cutoff since it enters for $\omega < \gamma\omega_p$, so large γ means the effect is strong.

Basically, the decrease in beaming occurs because the electron can no longer "catch up" to the wave that it had just emitted to reinforce it with another emission wave. The relativistic beaming enhancement due to this effect is what would boost synchrotron emission effect over other emission processes at a given frequency. (note that the enhancement is the same effect as the Cherenkov only in the latter the particle actually catches up completely and the particle is moving at a constant speed). Thus the Razin effect is a relative effect. Since the particle speed v cannot approach ω/k for sufficiently small n_r (since $\omega/k >> c$) the synchrotron is choked.

Selected Topics in Atomic and Molecular Physics

Note that macroscopic theory of radiative transfer requires that the scale of the system be much larger than the wavelength of the emission. This is not true on the atomic scale. Macroscopic theory OK on larger scales but we have to compute the absoprtion and emission coefficients from the proper electromagnetic or quantum processes.

Quantum calculations are needed when the De Broglie wavelengths of particles are larger than the dimension L which determines the conditions of a given problem, in this case the atomic scale. Note that if classical theory applied, the electron orbiting the atom would radiate away its energy and spiral in indefinitely.

Some special topics to address.

Distribution of energy levels and Ionization

Boltzmann Population of levels:

Relative populations of levels is not easy since it depends on detailed processes of how a level becomes populated or depopulated. But in thermal equilibrium things are simple and determined by temperature T. In any atom collection, the number in a given level is proportional to $ge^{-E/kT}$, with g as a statistical weight describing the level degeneracy.

Let E_i be the energy of the "ith" level with respect to the ground state. Let N_i be the number density of the ith level and $N = \sum_i N_i$ is the total number density, we have Boltzmann's law:

$$N_i = \frac{N}{U} g_i e^{-\beta E_i}, \tag{655}$$

where $\beta = 1/kT$, U is the partition function, and is simply

$$U = \sum_i g_i e^{-\beta E_i}. \tag{656}$$

At small temperatures the only the lowest term matters, the g_0 term. This gives

$$U = g_0. \tag{657}$$

Note that for finite temperature the sum diverges. Thus we have to think about how to cutoff the sum. In fact for a gas, the distance between particles is finite and outer electrons can be stripped by interactions between atoms. The ionization potential is thus reduced. A characteristic cutoff results from assuming the Bohr orbit corresponding to $n = n_{max}$ is that of the inter-atomic spacing.

$$n_{max}^2 a_0 Z^{-1} \sim N^{-1/3}. \tag{658}$$

or

$$n_{max} \sim \left(\frac{Z}{a_0}\right)^{1/2} N^{-1/6}. \tag{659}$$

The Bohr radius $a_0 \sim .5 \times 10^{-8}$cm marks a characteristic scale such that $e^2/a_0 \sim 27$ ev is a characteristic atomic binding energy.

For hydrogen at 10^{12}cm^{-3}, $n_{max} = 100$. However note that there is also Debye screening and such which will complicate this calculation.

In general, the cutoff is not well understood, and people spend lots of time writing thesis on this subject.

For many cases the value of the cutoff is not critical. Up to 10^4K the U is usually equal to g_0. Ok for Solar interior, ISM, AGN accretion disks.

Saha Equation

Above was for distribution of levels in a density of a single atom type in thermal equilibrium. Now we investigate distribution of among various stages of ionization of single atomic type. The Saha equation describes this. Consider the tow states being the neutral atom and the first stage of ionization.

Consider more general Boltzmann law:

$$\frac{dN_0^+(v)}{N_0} = \frac{g}{g_0} exp^{\frac{-(\chi_I + 1/2 m_e v^2)}{kT}}. \tag{660}$$

Here $dN_0^+(v)$ is the differential number of ions in the ground state with free electron in velocity range $v + dv$, χ_I is the ionization potential, N_0 is the number of atoms in the ground state and the statistical weight g. Here $g = g_0^+ g_e$, the product of statistical weights of the electron and the ground state ion. The statistical weight g_e is given by

$$g_e = \frac{2 dx_1 dx_2 dx_3 dp_1 dp_2 dp_3}{h^3}, \tag{661}$$

where the 2 represents the two spin states and the rest of the factors define the phase space. In particular $N_e dx_1 dx_2 dx_3 = 1$, when we apply to volume containing 1 electron. For the momentum space

$$dp_1 dp_2 dp_3 = 4\pi m_e^3 v^2 dv, \tag{662}$$

for non-relativistic electrons. Then

$$\frac{dN_0^+(v)}{N_0} = \frac{8\pi m_e^3}{h^3} \frac{g_0^+}{N_e g_0} exp\left[\frac{(\chi_I + \frac{1}{2} m_e v^2)}{kT}\right] v^2 dv. \tag{663}$$

To find the total number density of the ions we integrate over v to get

$$\frac{N_0^+ N_e}{N_0} = \frac{8\pi m_e^3}{h^3} \frac{g_0^+}{g_0} e^{-\chi_I/kT} \left(\frac{2kT}{m_e}\right)^{3/2} \int_0^\infty e^{-x^2} x^2 dx, \tag{664}$$

where $x \equiv (m_e/2kT)^{1/2} v$. Using the Gaussian integral result that $\int_0^\infty e^{-x^2} x^2 dx = \pi/4$, we have

$$\frac{N_0^+ N_e}{N_0} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \frac{2g_0^+}{g_0} e^{-\chi_I/kT}. \tag{665}$$

Now if we want the number of atoms or ions in any state not only the ground state, we use

$$\frac{N_0}{N} = \frac{g_0}{U(T)} \tag{666}$$

and

$$\frac{N_0^+}{N^+} = \frac{g_0^+}{U^+(T)}. \tag{667}$$

so that $N^+/N = U^+(T)/U(T)$. Thus

$$\frac{N^+ N_e}{N} = 2\frac{U^+(T)}{U(T)} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT}, \tag{668}$$

where N is number density of all neutral atoms and N^+ is that for ions. This is the Saha equation.

A similar relationship holds for successive stages of ionization, that is

$$\frac{N_{j+1} N_e}{N_j} = 2\frac{U_{j+1}(T)}{U_j(T)} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_{j,j+1}/kT}. \tag{669}$$

We can rewrite all this in terms of pressure rather than number density. The ideal gas law is

$$P = nkT, \tag{670}$$

so that

$$\frac{P_{j+1} P_e}{P_j} = \frac{2U_{j+1}(T)}{U_j(T)} \left(\frac{2\pi m_e}{h^2}\right)^{3/2} (kT)^{5/2} e^{-\chi_{j,j+1}/kT}. \tag{671}$$

When you have a mixture of various different species, one must have an equation that gives the conservation of nuclei

$$\sum N_j^{(i)} = N^{(i)} \tag{672}$$

where $N_j^{(i)}$ is the jth ionization state of the ith species, so that $N^{(i)}$ is the total number density of the ith species. In addition we would need charge conservation

$$N_e = \sum_i \sum_j Z_j N_j^{(i)}. \tag{673}$$

In general, the problem must be done numerically.

Also note that trace elements e.g. of alkali metals with low ionization potentials can make important difference in the equilibrium state because they may strongly affect the electron number density. Particularly the alkali metals, like Na, K, Rb, Cs. (Left most column on peroidic table.).

LECTURE 23

Line Broadening

Transitions between atomic states are broadened by a number of effects. Recall we used the line profile function $\phi(\nu)$ in our analyses of the Einstein coefficients.

Doppler Broadening:

This results from the thermal motion of the atom. Thus the emission and absorption frequency in the frame of the atom represents a different frequency for the observer.

The change in frequency is

$$\nu - \nu_0 = \nu_0 v_z / c \tag{674}$$

where ν_0 is the rest frame frequency. The number of atoms having velocities in the range $v_z, v_z + dv_z$ is

$$e^{(-m_a v_z^2/2kT)} dv_z. \tag{675}$$

From the above, we get

$$v_z = \frac{c(\nu - \nu_0)}{\nu_0} \tag{676}$$

and

$$dv_z = c\frac{d\nu}{\nu_0}. \tag{677}$$

Thus the strength of emission in frequency range $\nu, \nu + d\nu$ is

$$e^{\left[\frac{-m_a c^2 (\nu-\nu_0)^2}{2\nu_0^2 kT}\right]} d\nu, \tag{678}$$

so that the profile function becomes

$$\phi(\nu) = \frac{1}{\Delta\nu_d \pi^{1/2}} e^{-(\nu-\nu_0)^2/(\Delta\nu_d)^2}, \tag{679}$$

where the Doppler width is

$$\Delta\nu_d = \frac{\nu_0}{c}(2kT/m_a)^{1/2}. \tag{680}$$

The constant $1/(\Delta\nu_d)$ is determined by $\int \phi(\nu) d\nu = 1$ assuming that $\Delta\nu_d << \nu_0$.

Let us see what this does to absorption cross sections. Recall that $B_{12}\bar{J}_\nu$ is a transition probability per unit time per frequency (for absorption). This means that B_{12} has units of $(time)(steradian)(mass)^{-1}$. In addition, we have

$$\alpha_\nu = \frac{h\nu_{ul}}{4\pi} n_1 B_{12} \phi(\nu) \tag{681}$$

which comes from relating the absorption coefficient to the Einstein coefficient. Note also that

$$\alpha_\nu = \alpha(\omega) = n_1 \sigma(\omega), \tag{682}$$

which relates the cross section to absorption coefficient. Now recall that in the classical limit we had that the absorption cross section for electron scattering, based on radiation driven, harmonically bound particles, was

$$\sigma(\omega) = \frac{\sigma_T}{\tau} \frac{\Gamma}{4(\omega-\omega_0)^2 + \Gamma^2} = \frac{2\pi^2 e^2}{mc} \frac{\Gamma}{4(\omega-\omega_0)^2 + \Gamma^2}, \tag{683}$$

where $\Gamma = \omega_0^2 \tau$. τ is the time for radiation to cross a classical electron radius, and ω_0 is the frequency of the free oscillator. Integrating this, we have

$$\int \sigma(\omega) d\omega = \frac{2\pi^2 e^2}{mc} \tag{684}$$

or

$$\int \sigma(\nu) d\nu = \frac{\pi e^2}{mc} = B_{lu,clas} h\nu_{lu}/4\pi. \tag{685}$$

or

$$B_{lu,clas} = \frac{4\pi^2 e^2}{h\nu_{ul}}. \tag{686}$$

The quantum correction to this classical result is given by the absorption oscillator strength f_{lu} and is usually less than but of order 1. It is derived in the text. Then

$$B_{lu,clas} = \frac{4\pi^2 e^2}{h\nu_{ul}} f_{lu}. \tag{687}$$

Getting back to the cross section, we note that from (681) that the line center cross section for each atom, corrected for Doppler broadening, is

$$\sigma_{\nu_{lu}} = \frac{h\nu_{ul}}{4\pi} B_{12}\phi(\nu) = \frac{1}{\Delta\nu_D \pi^{1/2}} \frac{h\nu_{lu}}{4\pi} B_{12} = \pi^{1/2} r_0 c f_{12} \frac{1}{\Delta\nu_D} = \left(\frac{10^4}{\nu_{lu}}\right)\left(\frac{A}{T}\right)^{1/2} f_{12} \mathrm{cm}^2. \tag{688}$$

Note that the cross section for a given frequency is reduced for large temperatures. This is because the absorption is "spread out" over a range of frequencies.

There can be turbulent broadening in addition to the Doppler broadening. When the scale of the turbulence is small in comparison to the mean free path (micro-turbulence) the motions can be accounted for with an effective Doppler width:

$$\Delta\nu_D = \frac{\nu_{ul}}{c}\left(\frac{2kT}{m_a} + \xi^2\right)^{1/2} \tag{689}$$

assuming that the micro-turbulence is Gaussian. If the turbulence is on a scale larger than a mean free path, than we have "hydrodynamic" or "magnetohydrodynamic" turbulence. Then, one would take whatever profile one has with the line (including collisional effects as described below) and integrate over the hydrodynamic turbulent distribution. One can also have systematic effects on the shape of the line profiles.

Natural Broadening

Atomic level has a certain width due to the uncertainty relation. The spread in energy $\Delta E \Delta t \sim h/2\pi$. The spontaneous decay of state n into occurs at rate

$$\Gamma = \sum_{n'} A_{nn'} \tag{690}$$

the sum is over the lower energy states. The energy for the nth state the decays as $e^{-\Gamma t}$. The emitted spectrum is determined y the decaying sinusoid type of electric field (fig 2.3), or radiation from a harmonically bound, undriven particles. This was given by the Lorentz profile

$$\phi(\nu) = \frac{\Gamma/4\pi}{(\nu-\nu_0)^2 + (\Gamma/4\pi)^2} \tag{691}$$

also called the natural profile. Actually, the above applies to cases when the bottom state is the ground state, and the difference in energies really correspond to a broadening of the upper state. If both upper and lower states are broadened, then $\Gamma = \Gamma_u + \Gamma_d$, which means that the total width is determined by the sum of the widths of the upper and lower states.

Collisional Broadening

If the emitted radiation suffers collisions, then the phase of the emission can be lost. fig 10.3

By assuming that collisions induce a time dependent random phase in the electric field, and calculating the emitted power as a function of frequency by Fourier transforming the square of $E(t)$, then further assuming a Poisson distribution for number of collisions with given mean, one can show that the collision effect also gives Lorentz profile.

To see this, consider the electric field and let $\phi_c(t)$ be random phase resulting from collisions. We have

$$E(t) = Ae^{i\omega_o t - \Gamma t/2 + i\phi_c(t)}, \tag{692}$$

(with A as a constant, Γ is the spontaneous decay rate and ω_0 is the fundamental frequency). The averaged power spectrum is given by

$$\langle|\tilde{E}(\omega)|\rangle^2 = \langle|\int E(t)e^{i\omega t}dt|^2\rangle. \tag{693}$$

Using $E(t)$ above we have

$$|\tilde{E}(\omega)|^2 = |A|^2 \int_0^\infty \int_0^\infty dt_1 dt_2 e^{i(\omega-\omega_0)(t_1-t_2)-\Gamma(t_1+t_2)/2} e^{i[\phi_c(t_1)-\phi_c(t_2)]}. \tag{694}$$

Only the phase function is random. Thus we get

$$\langle|\tilde{E}(\omega)|^2\rangle = |A|^2 \int_0^\infty \int_0^\infty dt_1 dt_2 K(t_1,t_2)\langle e^{i[\phi_c(t_1)-\phi_c(t_2)]}\rangle \tag{695}$$

where $K(t_1,t_2) = e^{i(\omega-\omega_0)(t_1-t_2)-\Gamma(t_1+t_2)/2}$. Now write

$$\phi_c(t_1) - \phi_c(t_2) = \Delta\phi_c(\Delta t) \tag{696}$$

where $\Delta\phi_c(\Delta t)$ is the change of phase in time interval $\Delta t = t_2 - t_2$. The randomness of the collisions means that if 1 or more collisions occur in time interval $(t_1 - t_2)$, then the average vanishes (average of a rapidly oscillating function). If no collisions occur, then the quantity averages to 1.

Given that the mean rate of collisions is ν_{col}, we have that for a Poisson distribution of collisions, the probability for no collisions to occur in Δt is

$$P(x=0) = |e^{-\lambda}\lambda^x/x!|_{x=0} = e^{-|t_2-t_1|\nu_{col}}, \tag{697}$$

where $\lambda = \nu_{col}\Delta t$ is the probability of collisions per unit time times the length of time, and x is the number of collisions we are seeking the probability for.

Thus

$$\langle e^{i\Delta\phi(\Delta t)}\rangle = e^{-|\Delta t|\nu_{col}}, \tag{698}$$

and we have

$$\langle|\tilde{E}(\omega)|^2\rangle = |A|^2 \int_0^\infty \int_0^\infty dt_1 dt_2 K(t_1, t_2) e^{-|\Delta t|\nu_{col}}, \tag{699}$$

Integration gives a combined collisional/natural Lorentz profile

$$\phi(\nu) = \frac{\tilde{\Gamma}/4\pi}{(\nu - \nu_0)^2 + (\tilde{\Gamma}/4\pi)^2}, \tag{700}$$

with $\tilde{\Gamma} = \Gamma + 2\nu_{col}$, where ν_{col} is mean number of collisions per unit time.

Combined Doppler Broadening and Lorentz Profiles

One can combine the Lorentz profile, plus the Doppler effect. We can write the profile as an average of the Lorentz profile weighted by the velocity states.

$$\phi(\nu) = \frac{\tilde{\Gamma}}{4\pi^2} = \int_{-\infty}^{\infty} \frac{(m/2\pi kT)^{1/2} exp(-mv_z^2/2kT)}{(\nu - \nu_0 - \nu v_z/c)^2 + (\tilde{\Gamma}/4\pi)^2} dv_z \tag{701}$$

where v_z is the line of sight component of the velocity. This can be simplified with the Voigt function

$$H(a, u) \equiv \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{a^2 + (u - y)^2} \tag{702}$$

then

$$\phi(\nu) = (\Delta\nu_D)^{-1} \pi^{-1/2} H(a, u) \tag{703}$$

with $a = \tilde{\Gamma}/4\pi\Delta\nu_D$ and $u = (\nu - \nu_0)/\Delta\nu_D$. For small values of a, the central part of the line is dominated by the Doppler profile and the regions away from $\nu = \nu_0$ the "wings" are dominated by the Lorentz profile.