I. Fluids and Plasmas: The Big Picture

Both fluid dynamics and plasma dynamics are important for astrophysics. Let us discuss why, and the relation between the two.

- Examples of Fluids: a river, car exhaust, airs these fluids are composed of neutral particles. The bulk dynamics are modeled by equations that treat these systems as continuous media without having to worry about dynamics of individual particles. We will be more precise about this later.

- If we heat a fluid to high enough temperature, the neutral particles ionize. Even if the net charge of the system is zero the fluid particles themselves are charged. Depending on the amount of ionization we have a "partially ionized plasma" or "fully ionized plasma"

- Fluid Books often focus on bulk properties of flows without considering individual particles. This is appropriate when the inverse of the collision frequency $= (w_c^{-1})$ is short compared to the time scale of evolution t_{sys} of the multi-particle system under study.

- Similarly for a highly ionized plasma: when $(\text{collision frequency})^{-1} = w_c^{-1} / t_{bulk} << 1$ the plasma can be treated as a fluid. However, the charged particles of the plasma can carry currents and thus sustain magnetic fields. Magnetohydrodynamics represents the "simplest" generalization of fluid mechanics to include charged particles and electromagnetic fields.

- when the collision frequency is sufficiently small, the dynamics of individual particles become increasingly more important in modeling the system. Magnetohydrodynamics is thus a special limit of plasma physics which is the subject of the dynamics of charged particle systems when the collision frequencies are not necessarily large. kinetic Theory is used for the latter.

- Most astrophysical objects are made of plasma: gas with a significant ionized fraction.
- >90% of the luminous material in the universe can be classified as plasma
- sometimes the neutral fluid equations can be used even for plasmas in astrophysics. This depends on the problem being considered. For other problems MHD and/or kinetic theory is required.

In short:

plasma physics

↓ ← dynamics of individual particles unimportant

MHD

↓ ← E & M unimportant

Fluid Mechanics

[MHD & fluid mechanics are special cases of plasma physics]

Dynamical Theories

- dynamical theory implies time evolution theory
- Mechanics, E&M, QM examples
- common features: ① way of expressing state of system ② Eqns for time evolution

Mechanics: $\vec{x}_i(t)$, $\vec{p}_i(t)$: Newtons laws

E&M : $\vec{E}(\vec{x},t)$, $\vec{B}(x,t)$: Maxwells Eqn

QM : $\psi(x,t)$: Time dependent Schrodinger Eqn

- fluids & plasma need ① & ② as well

- These requirements can be expressed geometrically using concept of "phase space": the space such that each of the variables needed to define the state of the system corresponds to 1 dimension. For state functions that are continuous the phase space is infinite dimensional. The state of a system at any time corresponds to 1 point in the phase space, and equations describe a trajectory through this space.

Levels of Dynamical theory

Neutral Fluids

Level	Description of State	Dynamical Eqns
0: N Quantum particles	$\psi(\vec{x}_1 \dots \vec{x}_N)$	Schrodinger Eqns
1: N Classical particles	$(\vec{x}_1 \dots \vec{x}_N, \vec{p}_1 \dots \vec{p}_N)$	Newton's Laws
2: Distribution function	$f(\vec{x}, \vec{p}, t)$	Boltzmann Eqn
3: Continuum Model	$\rho(\vec{x},t), T(\vec{x},t), \vec{v}(\vec{x},t)$	Hydrodynamic Eqns

plasmas

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0: N Quantum particles	$\psi(\vec{x}_1, \dots \vec{x}_N)$	Schrodinger Eqn
1: N Classical Particles	$(\vec{x}_1, \dots \vec{x}_N, \vec{p}_1 \dots \vec{p}_N)$	Newtons laws
2: Distribution Function	$f(\vec{x}, \vec{p}, t)$	Vlasov equation
2.5 Two-fluid Model (ions + electrons)	$\begin{cases} \rho_e(x,t) & T_e(\vec{x},t), \vec{V}_e(\vec{x},t) \\ \rho_i(x,t) & T_i(\vec{x},t), \vec{V}_i(\vec{x},t), \end{cases} \vec{E}(x,t), \vec{B}(x,t)$	two-fluid plasma equations
3: One fluid model	$\rho(\vec{x},t)$ $T(\vec{x},t)$, $V(\vec{x},t)$, $\vec{B}(\vec{x},t)$	MHD

what determines when we can use a particular level?

More rigor later but a physical description follows $\Big\} \rightarrow$

Explanation of the Levels of Dynamical Theories

All microscopic systems obey Quantum mechanics. However we can treat a collection of N particles classically when the characteristic distance between particles is large enough so that there is little interference between their wavepackets. condition for classical treatment is that

$$n^{-1/3} \gg \lambda_d = \frac{h}{p} = \frac{h}{\sqrt{m k_b T}} \qquad (1)$$

where n is particle density and λ_d is the De Broglie wavelength. The quantity $n^{-1/3}$ is just the typical interparticle spacing and momentum $p = m V_{th} = m\sqrt{\frac{k_b T}{m}}$ for a thermal gas. Eqn (1) implies that each individual wave packet is isolated and expectation values can be treated classically. (Ehrenfest's theorem). This explains moving from level 0 to 1. on page 5.

If N is large then it is too impractical to solve equations for all individual particles. Then one moves to Level 2 and uses $f(\vec{x}, \vec{u}, t)$ distribution function, which is the particle number density in ~~phase~~ space $(\vec{x}, \vec{u})$ at time t.

A dynamical theory then requires an equation for f. (Vlasov or Boltzmann equation)

Level 3 treats the fluids as continua.

Since a gas is a flowing fluid and ~~~~ we know that a gas whose center of mass is at rest can be described by 2 variables (e.g. ρ, T) a moving gas requires 3 $(\rho(\vec{x},t), T(x,t), \vec{v}(\vec{x},t))$.

For a plasma we must also have an equation for $\vec{B}(x,t)$ since magnetic fields can be embedded. Since astrophysical plasmas are usually good conductors, on large enough scales $\vec{E} = -\vec{v} \times B$ as the plasma shorts out "microscopic" electric fields from currents. This is the MHD regime. On smaller scales, there are charge separations, and $\vec{E}_{\text{microscopic}}$ must be considered. This two-fluid regime is between MHD & Vlasov theory and is thus Level 2.5.

Comment on turbulence:

when fluid or plasma systems are subject to violent disturbances they can become turbulent: ie, incur motions which appear to be chaotic and seem unpredictable. we will see how even though system in principle has dynamical equations, in practice they cannot easily be solved for turbulent flows.

Liouville's Theorem

consider a dynamical system whose state is prescribed by position & momentum coords. $(q_s, p_s, s = 1, \ldots N)$ and satisfies Hamilton's eqn of motion

$$\dot{p}_s = -\frac{\partial H}{\partial q_s} \qquad (2)$$

$$\dot{q}_s = \frac{\partial H}{\partial p_s} \qquad (3)$$

(a classical system of N particles satisfies this system of eqns)

→

Define <u>ensemble</u>: set of many replicas of identical systems except being at different states at some given time.

each member of the ensemble is represented by a point in the phase space.

e.g. snapshot of a collection of particles

$\vec{q}_s(t), \vec{P}_s(t) \longrightarrow$ a point in $6N+1$ dimensional phase space for N particles $(1 \leq s \leq N)$

e.g. fluid

$\underbrace{\vec{v}(\vec{x},t)}_{\text{velocity}}, \underbrace{T(\vec{x},t)}_{\text{temp}}, \underbrace{\rho(\vec{x},t)}_{\text{mass density}} \rightarrow$ a point in infinite dimensional phase space

$6N+1$

we can define <u>ensemble density</u> ρ_{ens} as the density of ensemble points at a given location in phase space: e.g. $\rho_{ens}(\vec{q}_s, \vec{P}_s, t)$ for our system of particles

→ consider one member of the ensemble $\vec{q}_s, \vec{P}_s$ and its trajectory $\vec{q}_s(t), \vec{P}_s(t)$.

if we measure density as a function of time varying on this trajectory Liouville's theorem is

$$\frac{D\rho_{ens}}{Dt} = 0 \tag{4}$$

where $\frac{D}{Dt}$ is time derivative along the trajectory. To prove →

proof of Liouvilles thm

If $(q_s p_s)$ and $(q_s+\delta q_s,\ p_s+\delta p_s)$ denote the system at times t and $t+\delta t$ then

$$\frac{D\rho_{ens}}{Dt} = \lim_{\delta t \to 0} \frac{\rho_{ens}(q_s+\delta q_s,\ p_s+\delta p_s,\ t+\delta t) - \rho_{ens}(q_s p_s, t)}{\delta t} \tag{5}$$

expansion in Taylor series:

sums over particles

$$\rho_{ens}(q_s+\delta q_s,\ p_s+\delta p_s,\ t+\delta t) = \rho_{ens}(q_s, p_s, t) + \sum_{s=1}^{N} \delta q_s \frac{\partial \rho_{ens}}{\partial q_s} + \sum_{s=1}^{N} \delta p_s \frac{\partial \rho_{ens}}{\partial p_s} + \delta t \frac{\partial \rho_{en}}{\partial t}$$

plugging into (5) gives:

$$\frac{D\rho_{ens}}{Dt} = \frac{\partial \rho_{ens}}{\partial t} + \sum_s \dot{q}_s \frac{\partial \rho_{ens}}{\partial q_s} + \sum_s \dot{p}_s \frac{\partial \rho_{ens}}{\partial p_s} \tag{6}$$

now let us derive another result that we will use in conjunction with (6) to show why right side vanishes.:

The <u>continuity equation</u> applies to any mass conserving system & states that

$$\underbrace{\frac{\partial}{\partial t}\int \rho \, dV}_{\text{time derivative of mass}} = - \underbrace{\int \rho \vec{v} \cdot d\vec{S}}_{\text{outward mass flux}} \tag{7}$$

$\longrightarrow$

$\therefore$ using Gauss' theorem

$$\int \rho \vec{v} \cdot d\vec{s} = \int \nabla \cdot (\rho \vec{v})\, dV$$ so

$\Rightarrow$ (7) $\rightarrow$

$$\int \left[\partial_t \rho + \nabla \cdot (\rho \vec{v})\right] dV = 0$$

since it must be true for any volume:

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \qquad (8)$$

this applies for a density in regular 3-space OR for an ensemble density in a volume of phase space. Thus

$$\partial_t \rho_{ens} + \underbrace{\sum_s \frac{\partial}{\partial q_s}(\rho_{ens} \dot{q}_s) + \sum_s \frac{\partial}{\partial p_s}(\rho_{ens} \dot{p}_s)}_{\text{generalized divergence using all coordinates in phase space.}} = 0 \qquad (9)$$

plugging into (6) gives

$$\Rightarrow \frac{\partial \rho_{ens}}{\partial t} + \sum_s \dot{q}_s \frac{\partial \rho_{ens}}{\partial q_s} + \sum_s \dot{p}_s \frac{\partial \rho_{ens}}{\partial p_s} + \rho_{ens} \sum_s \underbrace{\left(\frac{\partial \dot{q}_s}{\partial q_s} + \frac{\partial \dot{p}}{\partial p_s}\right)}_{0 \text{ by Eqn's (2) \& (3)}} = 0$$

$$\rightarrow \boxed{\frac{\partial \rho_{ens}}{\partial t} + \sum_s \dot{q}_s \frac{\partial \rho_{ens}}{\partial q_s} + \sum_s \dot{p}_s \frac{\partial \rho_{ens}}{\partial p_s} = 0} \qquad (10)$$

as desired. QED

Collisionless Boltzmann Equation

Consider N classical particles, all of same type

- $6N$ position & velocity coords.
  call this ~~phase~~ space "Γ-space", and a point in Γ represents a state of the system
- Define "μ-space" as the 6 dimensional position & velocity space
  Each particle is represented by a point in μ space at some time
  the system state can be determined by N points in μ space. Note correspondence:
  [1 point in Γ-space $\leftrightarrow$ N points in μ space]
  Both describe state of system
- A trajectory in Γ-space gets mapped to N trajectories in μ-space

Define distribution function in μ space

$$f(\vec{x},\vec{u},t) = \lim_{\delta V \to 0^+} \frac{\delta N}{\delta V}, \quad V \text{ is } \mu \text{ space Volume}$$

$\delta V \to 0^+$ means take δV to small volume compared to system size, but still containing many particles

$f(\vec{x},\vec{u},t)$ is density of points in μ-space

$\rightarrow$

We can derive a similar equation to Liouville's thm for $f(\vec{x}, \vec{u}, t)$ if point trajectories in μ space satisfy

$$\dot{\vec{u}} = -\nabla H \quad ; \quad \nabla = \hat{e}_x \partial_x + \hat{e}_y \partial_y + \hat{e}_z \partial_z \qquad (11$$

$$\dot{\vec{x}} = \nabla_u H \quad ; \quad \nabla_u = \hat{e}_x \frac{\partial}{\partial u_x} + \hat{e}_y \frac{\partial}{\partial u_y} + \hat{e}_z \frac{\partial}{\partial u_z} \qquad (1\text{:}$$

(since we needed such relations in the proof)

→ in Γ space, Hamiltonian H is fn of $6N+1$ variables ($6N$ coords + time)

→ in μ-space H is function of 7 variables
6 coords + time

→ when the N particles are non-interacting

$$H(\vec{u}, \vec{x}, t) = \tfrac{1}{2}u^2 + \phi(\vec{x})$$

if particles interact, there are problems:
Suppose particle has coords $(\vec{u}, \vec{x})$ and interacts with nearby particle at $(\vec{u}', \vec{x}')$. This interaction can be described by potential $\phi(\vec{x}, \vec{x}')$
(and this incorporating the $6N+1$ dimensions of Γ-space is possible by considering a different x' for each particle interacting with the original)

But $\phi(\vec{x}, \vec{x}')$ cannot be written as only a function of $\vec{x}$, so <u>cannot</u> be incorporated into an H that satisfies (11) & (12)

→

$\Rightarrow$ Hamiltonian dynamics of N particles is always possible in Γ-space (phase space) but only possible in μ space when particles are not interacting, i.e. <u>collisionless</u>

For a collisionless system

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \dot{\vec{x}} \cdot \nabla f + \dot{\vec{u}} \cdot \nabla_u f = 0$$

we can write this as

$$\frac{\partial f}{\partial t} + \dot{x}_i \partial_i f + \dot{u}_i \partial_{u_i} f = 0 \qquad (13)$$

(repeated indices are summed)

Collisionless Boltzmann equation

when interactions are present, (13) must be modified