Let $n(\vec{x},t)$ be the density of markers.
If the dispersion of markers is diffusive,
then $n(\vec{x},t)$ should satisfy

$$\frac{\partial n}{\partial t} = D\nabla^2 n \qquad (187)$$

where D is the diffusion coefficient.

We now prove that $D = D_T$:

If markers are introduced near the origin
such that evolution is basically spherically
symmetric, then

$$\overline{X^2(t)} = \frac{\int_0^\infty r^2 n(r,t)\, 4\pi r^2 dr}{\int_0^\infty n(r,t)\, 4\pi r^2 dr} \qquad (188)$$

Then using (187)

$$\frac{\partial}{\partial t}\int_0^\infty r^2 n\, 4\pi r^2 dr = D\int_0^\infty \underbrace{\frac{\partial}{\partial r}\left(r^2 \frac{\partial n}{\partial r}\right)}_{= r^2 \nabla^2 n} 4\pi r^2 dr \qquad (189)$$

integrating right side by
parts twice:

$$\partial_t \int_0^\infty r^2 n\, 4\pi r^2 dr = 6D \overbrace{\int n\, 4\pi r^2 dr}^{\text{mass}}$$

or

$$\int_0^\infty r^2 n\, 4\pi r^2 dr \underset{\uparrow}{=} 6Dt \int n\, 4\pi r^2 dr \qquad (190)$$

(for conserved mass)

thus (188) then implies

$$\overline{X^2(t)} = 6Dt \quad \text{since} \quad \overline{X^2}(t) = \frac{\int r^2 n 4\pi r^2 dr}{\int n 4\pi r^2 dr} \qquad (191)$$

just as in (186) so $D = D_T$ for $t = T$

and thus

$$D_T \simeq \frac{\overline{X^2(t)}}{6T} = \frac{1}{3}\overline{v^2}\int R(\tau)\,d\tau$$

$$\simeq \frac{1}{3}\overline{v^2}\,\tau_{corr}$$

$$\simeq \frac{1}{3}|v||\ell_{corr}| \qquad (192)$$

Note that this coefficient does not depend on e.g. temperature or density unlike the molecular transport coefficients. This is because in the molecular case for example, molecules carrying more thermal energy move faster. For turbulent fluid transport however, the temperature does not have a direct influence on the turbulent velocity when the turbulent velocities are mechanically driven externally (e.g. the rate of sugar transport by turbulence in coffee does not depend on coffee temperature when externally stirred).

→

However, if the quantity transported can backreact on the turbulence, the turbulent diffusion coefficient can change.

<u>Mean Field Equations:</u>

How does turbulent viscosity affect Navier stokes equations?

For an incompressible flow, the N-V eqn is

$$\frac{\partial}{\partial t}(\rho V_i) = \rho F_i + \frac{\partial}{\partial x_j}\left(-p\delta_{ij} - \rho V_i V_j + \mu \frac{\partial V_i}{\partial x_j}\right) \quad (193)$$

Now to apply to turbulent flow, follow Reynolds (1895) and break velocity, pressure & force into mean and fluctuating parts: $V_i = \bar{V}_i + V_i'$; $p = \bar{p} + p'$; $F_i = \bar{F}_i + F_i'$. Substituting into (193) and taking average gives:

$$\frac{\partial}{\partial t}(\rho \bar{V}_i) = \rho \bar{F}_i + \frac{\partial}{\partial x_i}\left(-\bar{p}\delta_{ij} - \rho \bar{V}_i \bar{V}_j - \rho \overline{V_i' V_j'} + \mu \frac{\partial \bar{V}_i}{\partial x_j}\right) \quad (194)$$

(where we used $\rho\,\overline{V_i V_j} = \rho\left(\bar{V}_i\bar{V}_j + \overline{V_i' V_j'} + \underbrace{\cancel{\overline{V_i'\bar{V}_j}}}_{0} + \underbrace{\cancel{\overline{V_i'\bar{V}_j}}}_{0}\right)$)

(194) is the Reynolds Equation.

Note its similarity to (193) except for replacement of $\vec{v}$ by $\bar{\vec{v}}$ and the extra term $\overline{v_i' v_j'}$. This is an important term however, and is called the Reynolds stress. Note however that it leads to complications:

One possibility for dealing with it is to write

$$\partial_t (\overline{v_i' v_j'}) = v_i' \frac{\partial v_j'}{\partial t} + \frac{\partial v_i'}{\partial t} v_j' \qquad (194)$$

Then substituting for $\frac{\partial v_j'}{\partial t}$ and $\frac{\partial v_i'}{\partial t}$ by substracting (194) from (193). However, this will then produce triple correlations: $\overline{v_i' v_j' v_k'}$ → trying to deal with this triple in the same way as the double leads to 4th order correlations etc. This is the Closure problem. Since one effect of turbulence is to provide enhanced transport (as discussed in terms of D_T)

The simplest and most naive closure is to write the Reynolds stress as

$$\overline{v_i' v_j'} = -D_T \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) \qquad (194a)$$

this means that we have, for (194)

$$\frac{\partial}{\partial t}(\rho \overline{V}_i) = \rho \overline{F_i} + \frac{\partial}{\partial x_j}\left(-\overline{p}\delta_{ij} - \rho \overline{V}_i\overline{V}_j + D_T\left(\frac{\partial \overline{V}_i}{\partial x_j} + \frac{\partial \overline{V}_j}{\partial x_i}\right) + \mu \frac{\partial \overline{V}_i}{\partial x_j}\right) \quad (195)$$

Since μ is the microphysical viscosity we can see that D_T acts in a similar way and is typically <u>much larger</u> than μ since $\mu \simeq \frac{1}{3} V_{th} l_{mfp}$ and $D_T = \frac{1}{3} V_T l_T$ and $l_T \gg l_{mfp}$ for wide range of V_{th}.

note also that the diffuse nature of transport is reflected by a diffusion coefficient multiplying two spatial derivatives. For D_T independent of space, its term becomes in (195)

$$D_T \nabla^2 \overline{V}_i + D_T \frac{\partial \nabla \cdot \overline{V}}{\partial x_i} \quad \text{and}$$

for incompressible $\overline{V}$, the second term vanishes so the closure (194a) is a <u>turbulent diffusion</u> closure to the Navier-Stokes eqn. It works well in many cases.