Radial velocity as diffusion velocity

thin disks

$$\Omega \simeq \Omega_K(R) = \left(\frac{GM}{R^3}\right)^{1/2} \Rightarrow V_\phi = R\,\Omega_K$$

also have V_R, radial drift velocity, must be second order quantity. $\left(\frac{H^2}{R^2}\right)$ we'll [see] this more explicitly.

Write conservation equations:

annulus of disk lying between R & $R + \Delta R$ has mass $2\pi R \Delta R \Sigma$, and $\measuredangle$ momentum $2\pi R \Delta R \Sigma R^2 \Omega$.

Rate of change of mass is (for small changes)

$$\frac{\partial}{\partial t}(2\pi R \Delta R \Sigma) \simeq V_R(R,t)\, 2\pi R\, \Sigma(R,t) - V_R(R+\Delta R, t)\, 2\pi (R+\Delta R, t)\, \Sigma(R+\Delta R, t)$$

$$\simeq -2\pi \Delta R \frac{\partial}{\partial R}(R \Sigma V_R)$$

or

$$R\frac{\partial \Sigma}{\partial t} + \frac{\partial (R\Sigma V_R)}{\partial R} = 0 \quad \text{(mass continuity)} \quad (14.2)$$

(14.2a) $\longrightarrow$

$$R^2\Sigma R^3\Omega \frac{\partial \Sigma}{\partial t} + R^2\Omega \frac{\partial}{\partial R}(R\Sigma V_R) = 0$$

$$\frac{\partial}{\partial t}(R^3\Omega\Sigma) = -R^2\Omega \frac{\partial}{\partial R}(R\Sigma V_R) \text{ for } \frac{\partial \Omega}{\partial t} = 0$$

for ∡ momentum same idea:

$$\frac{\partial}{\partial t}\left(2\pi R\,\Delta R\,\Sigma\, R^2\Omega\right) = V_R(R,t)\,2\pi R\,\Sigma(R,t)\,R^2\Omega(R,t)$$

$$- V_R(R+\Delta R,t)\,2\pi R\,\Sigma(R+\Delta R,t)\,(R+\Delta R)^2\,\Omega(R+\Delta R,t)$$

$$+ \frac{\partial G}{\partial R}\Delta R$$

↖ from before, the torque

$$\simeq -2\pi\Delta R\frac{\partial}{\partial R}\left(\Sigma V_R R^3\Omega\right) + \frac{\partial G}{\partial R}\Delta R$$

or

$$\therefore\ \partial_t\left(\Sigma R^3\Omega\right) + \frac{\partial}{\partial R}\left(\Sigma V_R R^3\Omega\right) = \frac{1}{2\pi}\frac{\partial G}{\partial R} \quad (143)$$

∡ mom cons.

$$G(R,t) = 2\pi\, \nu\,\Sigma R^3\frac{\partial\Omega}{\partial R} \quad \text{from } (101) \qquad (143a)$$

Using (142, 142a) to eliminate 1st term of (143) (using $\frac{\partial R}{\partial t} = \frac{\partial\Omega}{\partial t} = 0$) and $\underline{\frac{\partial\Omega}{\partial t} = 0}$ (fixed grav potential)

$$\Rightarrow\ \Sigma R V_R\frac{\partial\left(R^2\Omega\right)}{\partial R} = \frac{1}{2\pi}\frac{\partial G}{\partial R} \quad (144)$$

use (142) & (144) to get V_R:

$$R\frac{\partial \Sigma}{\partial t} = -\frac{\partial}{\partial R}(R\Sigma V_R)$$

$$= -\frac{\partial}{\partial R}\left[\frac{1}{2\pi \frac{\partial}{\partial R}(\Omega R^2)} \frac{\partial G}{\partial R}\right]$$

from 143a

$$\frac{\partial}{\partial R} 2\pi R \Sigma \nu R^3 \frac{d\Omega}{dR}$$

now since $\Omega \propto R^{-3/2}$ keplerian

$\Rightarrow$

$$\boxed{\frac{\partial \Sigma}{\partial t} = \frac{1}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(\nu_T \Sigma R^{1/2})\right]} \qquad (145)$$

surface density equation

Given soln of (145)

V_R follows from (142)

$$\boxed{u_R = -\frac{1}{\Sigma R^{1/2}}\frac{\partial}{\partial R}\left[\nu_T \Sigma R^{1/2}\right]} \qquad (146a)$$

For (artificially) constant ν_T : (145) implies

$$\Rightarrow \frac{\partial}{\partial t}(R^{1/2}\Sigma) = \frac{\nu_T}{R}\left(R^{1/2}\frac{\partial}{\partial R}\right)^2 (R^{1/2}\Sigma)$$

Comment on 2

let $s = 2R^{1/2}$ $\Rightarrow$ $\frac{\partial}{\partial R} = \frac{\partial s}{\partial R}\frac{\partial}{\partial s} = R^{-1/2}\frac{\partial}{\partial s}$

$$\Rightarrow \frac{\partial}{\partial t}(R^{1/2}\Sigma) = \frac{4\nu_T}{s^2}\frac{\partial^2}{\partial s^2}(\Sigma R^{1/2})$$

$$\frac{\partial}{\partial t}(s\Sigma) = \frac{4\nu_T}{s^2}\frac{\partial^2}{\partial s^2}(\Sigma s) \rightarrow$$

$$\frac{\partial}{\partial t}\left(\Sigma R^{1/2}\right) = \frac{4\nu_T}{s^2}\frac{\partial^2}{\partial s^2}\left(\Sigma R^{1/2}\right)$$

let $Q = R^{1/2}\Sigma = C(t)e^{ik^{1/2}s}$ just as an example

$$\frac{\partial}{\partial t}\underbrace{\left(R^{1/2}\Sigma\right)}_{Q} = -\frac{4\nu_T k}{s^2}\underbrace{\left(R^{1/2}\Sigma\right)}_{Q}$$

$\Rightarrow$

$$R^{1/2}\Sigma = Q = Q_0\, e^{-\frac{4\nu_T}{s^2}kt}, \quad Q_0 = R^{1/2}\Sigma(0)$$

$\Rightarrow$ diffusion ; effect

of constant viscosity is to diffuse mass density. (true for any separable $R^{1/2}\Sigma = f(t)g(s)$)

$$t_{visc} \simeq \frac{s^2}{4\nu_T k} \simeq \frac{R}{\nu_T k}$$

$$\simeq \frac{R^2}{\nu_T} \text{ for } k \sim \frac{1}{R}$$

effect of viscosity is to spread structure of of radius R on time scale t_{visc}.

$\longrightarrow$

From (44b)

$$v_R = -\frac{1}{\Sigma R^{1/2}} \frac{\partial}{\partial R}\left(\nu_T \Sigma R^{1/2}\right)$$

$\simeq -\frac{\nu_T}{R}$ when $\Sigma \propto R^q$ $q > -\frac{1}{2}$ (for $\nu_T = \text{const}$) and

$\simeq +\frac{\nu_T}{R}$ when $q \leq -1/2$;

v_R is in general a diffusion velocity

Shakura-Sunyaev (1973) approximation

$$\frac{v_R}{R} = \alpha_{ss} C_s \frac{H}{R} \quad \text{for} \quad \underbrace{\nu_T = \alpha_{ss} C_s H}$$

Note also that $C_s \ll V_\phi$ for $\frac{H}{R} \ll 1$

this comes from hydrostatic equilibrium :

$\longrightarrow$

hydrostatic equilibrium

vertical disk structure, steady state

$$\frac{1}{\rho}\frac{dP}{dz} = \frac{\partial}{\partial z}\left[\frac{GM}{(R^2+z^2)^{1/2}}\right] \qquad \text{mom eqn}$$

non-self gravitating disk

for thin disk $z \ll R$

$$\Rightarrow \frac{1}{\rho}\frac{dP}{dz} = -\frac{1}{2}\frac{GM\,2z}{(R^2+\cancel{z^2})^{3/2}}$$

$$\simeq -\frac{GMz}{R^3}$$

typical scale height of disk is H

$$u_\phi = \left(\frac{GM}{R}\right)^{1/2}$$

So $\Rightarrow$

$$\frac{1}{\rho}\frac{P}{H} \simeq -\frac{GMH}{R^3}$$

$$\Rightarrow |c_s^2| \simeq \left|\frac{GM}{R}\right|\frac{H^2}{R^2} \simeq v_\phi^2\frac{H^2}{R^2}$$

$$\Rightarrow \boxed{c_s^2 \ll v_\phi^2 \iff H^2/R^2 \ll 1}$$