Steady Thin Disks

- radial structure in thin disk evolves on viscous time scales $\simeq R^2/\nu = t_{visc}$
- this presents another way turbulent viscosity is motivated:
  even knowing nothing about disk properties, viscous time must be less than or equal age of system presumed to be an accretor
- star forming disk ages $\simeq$ few $\times 10^7$ years
- if accretion is to explain observed features then $t_{age} > t_{visc}$
  but for molecular viscosities
  $R \simeq 10^{14}$ cm, $c_s \simeq 10^5$ cm/s, $l_{mfp} \simeq 10$ cm
  $\Rightarrow R^2/c_s l_{mfp} \simeq 3 \times 10^{14}$ yr! too long

$\Rightarrow$ at least in YSO systems, accretion models require turbulent diffusion

- In many systems we can assume mass transfer rate changes on timescales longer than t_{visc}
- System will adjust to steady state structure

- In steady state $-u_R \Sigma R = \text{constant} = \frac{\dot{M}}{2\pi}$

from ∡ momentum eqn (143)

$$\frac{\partial}{\partial R}\left(\Sigma u_R R^3 \Omega\right) = \frac{1}{2\pi}\frac{\partial G}{\partial R}$$, now integrate

$$\Rightarrow \quad \Sigma u_R R^3 \Omega = \frac{G}{2\pi} + \tilde{C}$$

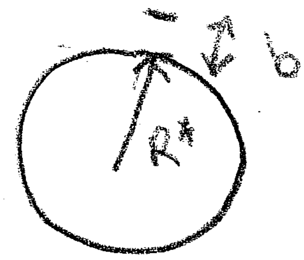
$$= \nu \Sigma R^3 \frac{\partial \Omega}{\partial R} + \tilde{C}$$

$$\Rightarrow \quad \overbrace{-\nu\Sigma}^{\text{note combo}} \frac{\partial \Omega}{\partial R} = -u_R \Sigma \Omega + \frac{\tilde{C}}{R^3} \qquad (147)$$

now, for $R \simeq R^*$ (radius of star)

Ω, which is $\simeq \Omega_K$ in disk must slow down to match to "star" which is rotating below break-up ($\Omega < \Omega_K$). There is a thin boundary layer where $\partial\Omega/\partial R \to 0$, this allows us to determine $\tilde{C}$ →

at $R = R_* + b$, $b \ll R$



$\frac{\partial \Omega}{\partial R} \to 0$ since

$$\Omega(R_*+b) \simeq \left(\frac{GM}{R_*^3}\right)^{1/2}\left[1 + O(b/R_*)\right]$$

$$\Rightarrow \breve{C} \simeq R_*^3 \Sigma u_R \Omega \Big|_{R_*+b}$$ from (147)

$$\Rightarrow \breve{C} = -\frac{\dot M (GMR_*)^{1/2}}{2\pi},$$ (since $\dot M = 2\pi R \Sigma H u_R$, $u_R < 0$)

then plugging into (147)

$$\Rightarrow -R\nu\Sigma \frac{\partial \Omega}{\partial R} = -V_R \Sigma \Omega R - \frac{\dot M (GMR_*)^{1/2}}{2\pi R^2}$$

for $\Omega = \Omega_K$, $\frac{\partial \Omega}{\partial R} = -\frac{3}{2}\frac{\Omega_K}{R}$, then divide by Ω:

$$\Rightarrow \frac{3}{2}\nu\Sigma = \underbrace{-V_R \Sigma R}_{\frac{\dot M}{2\pi}} - \frac{\dot M R_*^{1/2}}{2\pi R^{1/2}} \cancel{\frac{\Omega_K}{\Omega}}$$

$$\Rightarrow \nu\Sigma = \frac{2\dot M}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right] \qquad (148.)$$

Important

recall

$$D(R) = \frac{1}{2} \underbrace{\nu \Sigma} \Big(\underbrace{R \frac{\partial \Omega}{\partial R}}_{-\frac{3}{2}\frac{\Omega}{R}} \Big)^2$$ (from page 145)

$$= \frac{1}{2} \nu \Sigma \frac{9\Omega^2}{4} = \frac{9}{8} \nu \Sigma \Omega^2$$

$\Rightarrow$ using (148)

$$D(R) = \frac{3 G M \dot{M}}{4 \pi R^3} \left[1 - \left(\frac{R_*}{R} \right)^{1/2} \right] \qquad (149)$$

= energy loss rate per area from dissipation

notice it does not depend on viscosity except in combination $\nu\Sigma$, why?

$$\dot{M} = -2\pi V_R \Sigma R \simeq -2\pi \frac{\nu}{\cancel{R}} \Sigma \cancel{R}$$

constant accretion rate $\Rightarrow$ $\nu\Sigma$ constant.
Since $\dot{M}$ determines energy dissipated a low ν can be compensated for by high Σ. (Amazing result really ...)

luminosity emitted from $R_1 < R < R_2$

$$L(R_1, R_2) = 2\int_{R_1}^{R_2} D(R)\, 2\pi R\, dR$$

2 sides of disk

$$-\frac{1}{R} + \frac{2R_*^{1/2}}{3}\frac{1}{R^{3/2}}\Bigg|_{R_*}^{\infty}$$

$$= \frac{3GM\dot{M}}{2}\int_{R_1}^{R_2}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]\frac{dR}{R^2}$$

$$\left[-\frac{1}{R} + \frac{2R_*^{1/2}}{3R^{3/2}}\right]_{R_1}^{R_2}$$

let $R_1 \to \infty$, $R_2 \to R^*$

gravitation energy available

$$\simeq \frac{GM\dot{M}}{2R_*} = \frac{1}{2}\frac{dU_g}{dt}$$

(other half is left for boundary layer dissipation; $\frac{1}{2}$ availble energy dissipated in disk, $\frac{1}{2}$ dissipated in boundary layer)

Formal

Check that $V_R \sim \nu/R$, & $\Omega \simeq \Omega_K$ from momentum eqn

consider radial component of momentum equation

$$V_R \frac{\partial V_R}{\partial R} - \frac{V_\phi^2}{R} + \frac{1}{\rho}\frac{\partial P}{\partial R} + \frac{GM}{R^2} = \nu_T \nabla^2 V_R \qquad (150)$$

- hydrostatic equilib tells us $c_s^2 = \frac{H^2}{R^2} V_\phi^2$ ✓
- viscous term is assumed small. then:

$$\Rightarrow V_R \frac{\partial V_R}{\partial R} - \frac{V_\phi^2}{R} + \frac{GM}{R^2} \simeq 0$$

but $V_R = -\frac{\dot{M}}{2\pi R \Sigma} = -\frac{\cancel{\dot{M}}\, \nu\, 3\cancel{\pi}}{2\cancel{\pi} R \cancel{\dot{M}}}\left(1-\left(\frac{R_*}{R}\right)^{1/2}\right)^{-1}$ from (148) (151)

$$= -\frac{\nu}{R}\left(1-\left(\frac{R_*}{R}\right)^{1/2}\right)^{-1} \text{ so indeed ✓}$$

$\rightarrow -V_R \simeq \mathcal{O}\left(\frac{\nu}{R}\right)$, then (150)

$\Rightarrow \nu = \alpha c_s H \Rightarrow \frac{V_R^2}{R} \ll \frac{V_\phi^2}{R}$ so

$V_\phi^2 \simeq \frac{GM}{R}$ - Keplerian ✓

from (150)

Spectrum from acc disk (opt thick)

Assume acc disk is optically thick:

$\tau_{eff} \equiv \Sigma \kappa_{eff} \gg 1$; $\kappa_{eff} \equiv \frac{\sigma_{eff}}{m_p}$ = cross section for photon absorption and scattering per mass.

Σ: surface density

τ_{eff}: optical depth to abs. + scattering

(see Rybicki & Lightman)

then at each radius it radiates as blackbody: $\sigma T^4(R) = D(R)$

(recall $D(R)$ is energy/time·area = flux)

then from (149):

$$T(R) = \left(\frac{D(R)}{\sigma}\right)^{1/4} = \left\{\frac{3GM\dot{M}}{8\pi R^3 \sigma}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]\right\}^{1/4}$$

for $R \gg R_*$ $\Rightarrow$ $T(R) = T_i \left(R/R_*\right)^{-3/4}$

$$T_i = \left(\frac{3GM\dot{M}}{8\pi\sigma R_*^3}\right)^{1/4} \simeq 4\times10^4 \left(\frac{\dot{M}}{10^{16}\,\mathrm{g/s}}\right)^{1/4}\left(\frac{M}{M_\odot}\right)^{1/4}\left(\frac{R}{10^9\,\mathrm{cm}}\right)^{-3/4} \mathrm{K}$$

WD

$$= 10^7 \left(\frac{\dot{M}}{10^{17}\,\mathrm{g/s}}\right)^{1/4}\left(\frac{M}{M_\odot}\right)^{1/4}\left(\frac{R}{10^6\,\mathrm{cm}}\right)^{-3/4} \mathrm{K}$$

NS

WD should be UV sources ✓

NS should be X-ray Sources ✓

Spectrum emitted by each element of disk is (note: $dE = I_\nu(\hat{k}\cdot\hat{n})\,d\Omega\,dt\,d\nu\,dA$) (Rybicki & Lightman or Shu vol 1)

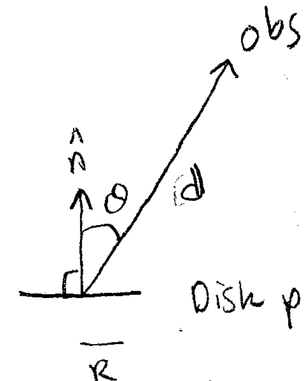
$d\Omega \equiv$ solid ∡ element

specific intensity $= I_\nu$

$$I_\nu = B_\nu(T(R)) = \frac{2h\nu^3}{c^2\left(e^{h\nu/kT(R)}-1\right)} \quad \frac{\text{erg}}{\text{cm}^2\cdot\text{s}\cdot\text{Hz}\cdot\text{ster}}$$

solid angle

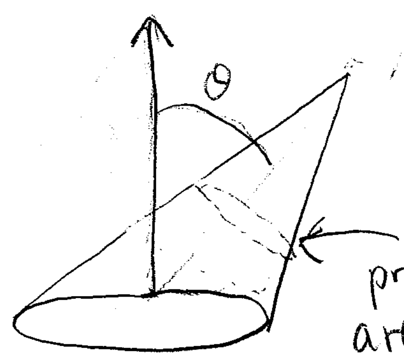
$$F_\nu = \int I_\nu \cos\theta \, d\Omega$$



$$\simeq \int_{R_*}^{R_{out}} I_{\nu_f} \cos\theta \, \frac{2\pi R\, dR}{d^2}$$

(using

$$d^2\, d\Omega = 2\pi R\, dR)$$



$$\simeq \frac{2\pi\cos\theta}{d^2}\int_{R_*}^{R_{out}} I_\nu \, R\, dR$$

→

$$F_{\nu_f} = \frac{2\pi\cos\theta}{D^2}\,\frac{4\pi h}{c^2}\,\nu_f^3 \int_{R_*}^{R_{out}} \frac{R\,dR}{e^{h\nu_f/kT(R)}-1}$$

$$T(R) = T_i\left(\frac{R}{R_*}\right)^{-3/4}$$

$$x = \frac{h\nu_f}{kT(R)} = \frac{h\nu_f}{kT_{out}}\left(\frac{R}{R_{out}}\right)^{3/4}$$

$$\frac{dx}{dR} = \frac{3}{4}\,\frac{h\nu_f\,R_{out}}{kT_{out}}\left(\frac{R}{R_{out}}\right)^{-1}$$

$$R = x^{4/3}R_{out}\left(\frac{kT_{out}}{h\nu_f}\right)^{4/3}$$

$$R\,dR = \frac{4}{3}\,\frac{kT_{out}}{h\nu_f R_{out}}\left(\frac{R}{R_{out}}\right)^{1/4} R\,dx$$

$$= \frac{4}{3}\,\frac{kT_{out}\,R^{5/4}}{h\nu_f R_{out}^{5/4}}\,dx$$

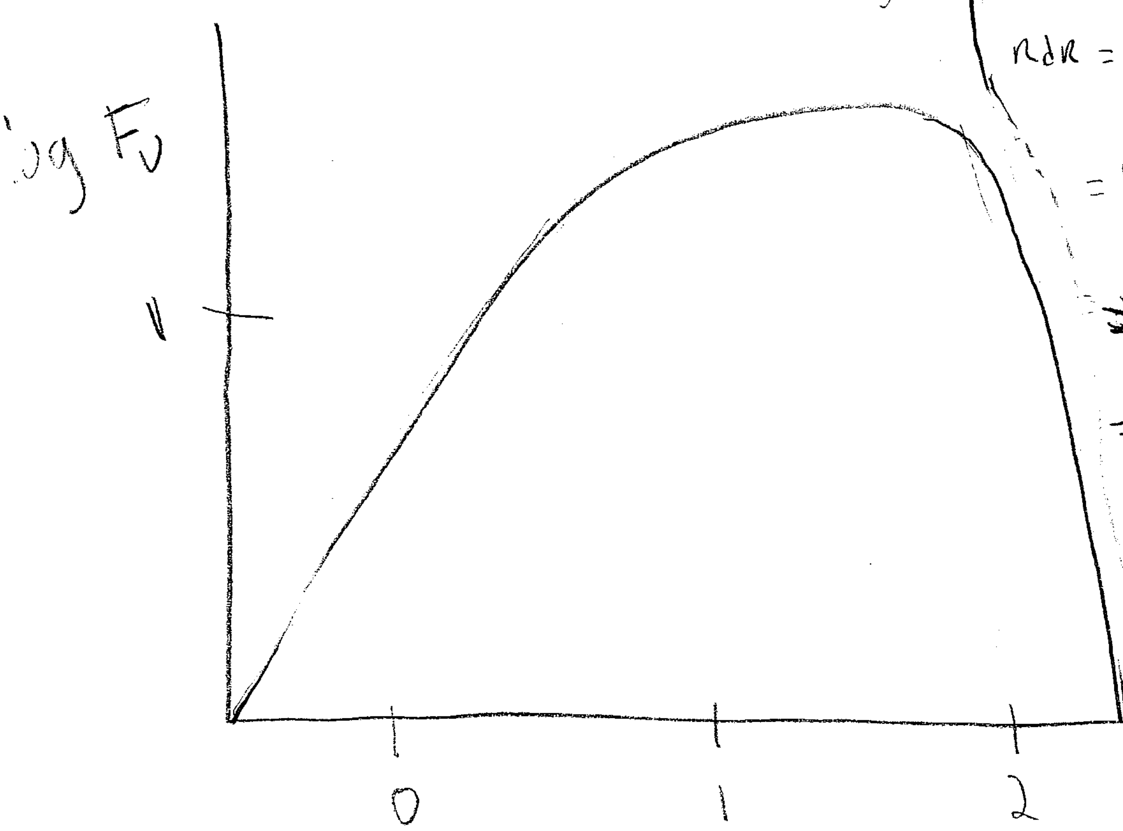
$$= \frac{4}{3}\,\frac{kT_{out}}{h\nu_f}\,x^{5/3}\left(\frac{kT_{out}}{h\nu_f}\right)^{5/3}dx$$

$$= \frac{4}{3}\left(\frac{kT_{out}}{h\nu_f}\right)^{8/3} x^{5/3}\,dx$$

$$\Rightarrow \nu_f^3\int_{R_*}^{R}\frac{R\,dR}{e^{h\nu_f/kT}-1}$$

$$\propto \nu_f^{1/3}\int_{h\nu_f/kT_{in}}^{h\nu_f/kT_{out}}\frac{x^{5/3}}{e^x-1}\,dx$$

plot $F_{\nu_f}(\nu_f)$



$\log\left(h\nu/kT_{out}\right)$

(Stretched out black body)