Hydrodynamics and Rotation

Need to consider rotating fluids, since most astrophysical objects have ∡ momentum.

Most astrophysical rotators posess differential rotation: two reasons:

(1) viscosity may not be able to act fast enough to smooth out differential rotation

(2) some physical mechanism present to maintain diff rotation.

Consider centrifugal force in rotating body:

Assume axisymmetric steady rotation: $\Rightarrow \partial_t = 0$, $\partial_\theta = 0$ and $V_r = 0$, in the r-component of Navier-Stokes equation in cylindrical coordinates (see Appendix of Shu or Choudhuri)

$$-\frac{V_\theta^2}{r} = g_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \qquad (196)$$

when radial pressure support is unimportant (e.g. thin acc dishs)

$V_\theta \simeq \sqrt{r|g_r|}$; centrifugal force balances grav.

→

In stars, pressure is not negligible, so a balance of rotation & pressure balances gravity.

In general the rotation is differential because of ① the gravitational force law ② the ineffectiveness of microphysical viscosity to make the flow uniform and ③ the fact that turbulent "viscosity" in rotating systems, although much stronger than microphysical viscosity may not strong enough and because turbulent viscosity can be highly anisotropic in rotating flows and contribute to sustaining diff rotation as in the solar convection zone.

Rayleigh criterion: not all diff. rot flows are stable:

Consider fluid annulus at distance r_0 from axis rotates with velocity v_0, and this ring is interchanged with ring at $r_1 > r_0$ rotating with v_1. System is stable when displaced ring wants to return to original position.

→

conserving χ momentum, ring displaced to r_1 acquires velocity

$V_1 = \frac{V_0 r_0}{r_1}$. The centrifugal acceleration at this new position is then $\frac{V_0^2 r_0^2}{r_1^3}$, whereas a ring in equilibrium at r_1 would be $\longrightarrow$ $\frac{V_1^2}{r_1}$. Thus if

$$\frac{V_0^2 r_0^2}{r_1^3} < \frac{V_1^2}{r_1} \qquad (197)$$

the system is stable and ring will return to original radius. When the inequality is not satisfied, system is unstable to turbulence. Eq (197) can also be written

$$(r_0^2 \Omega_0)^2 < (r_1^2 \Omega_1)^2 \quad \text{or}$$

$$\frac{d}{dr}(\Omega r^2)^2 > 0 \qquad (198)$$

Rayleigh's criterion

Hydrodynamics in Rotating Frame

Navier-Stokes equation in rotating frame (e.g. Marion, or Goldstein)

$$\frac{d\vec{v}}{dt} \rightarrow \frac{d\vec{v}}{dt} + 2\vec{\Omega}\times\vec{v} + \Omega\times(\Omega\times r) \qquad (198)$$

(Here, assume that Ω = constant.)

$\Rightarrow$

$$\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\nabla p + \vec{F} + \nu\nabla^2\vec{v} - \underbrace{2\Omega\times V}_{\text{coriolis force}} - \underbrace{\Omega\times(\Omega\times r)}_{\text{centrifugal force}}$$

note that $-\Omega\times(\Omega\times r) = -\frac{1}{2}\nabla(\vec{\Omega}\times\vec{r})^2$ for Ω = constant

(of:

$$\Downarrow \qquad\qquad \Downarrow$$

$$-(\Omega\cdot r)\vec{\Omega} + \Omega^2\vec{r} = -\frac{1}{2}\nabla(\Omega\cdot r)^2 + \frac{1}{2}\nabla(\Omega^2 r^2)$$

$$\text{if } \frac{\partial\Omega}{\partial r} = 0 \Rightarrow \quad = -\Omega(\Omega\cdot r) + \Omega^2 r$$

)

for Ω = constant

thus:

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v = -\frac{\nabla p}{\rho} - \nabla\underbrace{\left(\Phi - \frac{1}{2}|\vec{\Omega}\times\vec{r}|^2\right)}_{\Phi_{eff}} + \nu\nabla^2\vec{v} - 2\Omega\times\vec{v} \qquad (199)$$

Since centrifugal force can be written as potential, it is easier to deal with than the coriolis term $-2\Omega\times\vec{v}$

$\rightarrow$

For slowly rotating object like Earth, centrifugal term is small. But large scale flows, hurricanes and ocean currents are influenced by coriolis force. To assess when its important, compare $V \cdot \nabla V$ to $\Omega \times V$:

$V \cdot \nabla V \sim V^2/L$ (L = scale of structure)

$\Omega \times V \sim \Omega V$

earth rotation (Ω) velocity of structure (V)

$$\Rightarrow \quad R_0 \equiv \frac{V^2}{\Omega V L} = \frac{V}{\Omega L} \leq 1$$

$\Rightarrow$ coriolis force is important when the Rossby Number $R_0 < 1$, where V, L are typical velocity and length scales of the flow.

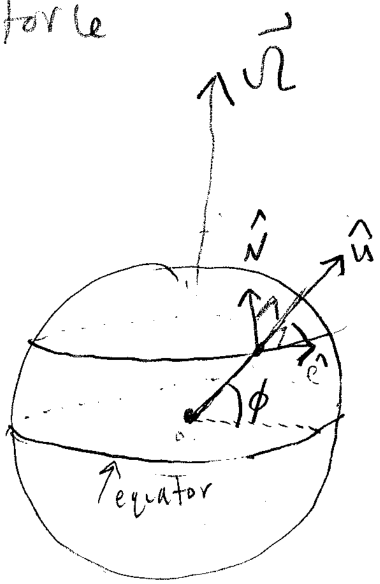
For fluid phenomena in the lab, $R_0 >> 1$, but in atmosphere and oceans $R_0 < 1$. Studying large scale atmospheric or ocean flows is Geophysical Fluid Dynamics. Usually one assumes thin spherical fluid shell with small Rossby number.

More detailed

Calculation of Rossby number

$$R_0 = \frac{\text{ratio of inertial terms in Navier-Stokes}}{\text{Coriolis force}}$$

$$= \frac{|v \cdot \nabla v|}{|2\vec{\Omega} \times v|}$$



Consider Earth:

$\hat{u}$ = "up" normal to plane
$\hat{N}$ north
$\hat{e}$ = east
ϕ = lattitude

coords defined at a point on sphere

in coordinate system $(\hat{e}, \hat{n}, \hat{u})$:

$$\vec{\Omega} = (0, |\Omega|\cos\phi, |\Omega|\sin\phi), \quad \vec{V} = (V_e, V_n, V_u)$$

$$-2\vec{\Omega} \times \vec{V} = -2(\cancel{V_u|\Omega|\cos\phi} - |\Omega| V_n \sin\phi,\ |\Omega| V_e \sin\phi,\ \cancel{-V_e|\Omega|\cos\phi})$$

(V_u usually small)

"up" force usually small compared to other terms in Nav Stokes

$$\Rightarrow |2\Omega \times \vec{V}| \simeq |2|\Omega| V_n \sin\phi| \Rightarrow$$

$$R_0 = \frac{|v \cdot \nabla v|}{|2|\Omega| V \sin\phi|} \simeq \left|\frac{V^2}{2L\cancel{V}\Omega\sin\phi}\right| = \left|\frac{V}{2L\Omega\sin\phi}\right|$$

Geostrophic approximation

In atmospheric or ocean applications assume flows are nearly "horizontal" (non-radial) in a thin layer (much thinner than earth's radius).

For low Rossby Number flows, left hand terms of (199) are small compared to leading terms on right side. We can also neglect viscosity and centrifugal terms (slow rotation, nearly inviscid flow). Then (199) =>

$$-\frac{\nabla p}{\rho} - \vec{g}\hat{e}_r - 2\vec{\Omega}\times\vec{v} = 0 \qquad (g = -\nabla\phi) \qquad (200)$$

Usually, coriolis force is small compared to gravity in vertical direction so the $\hat{r}$ component becomes

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = g \qquad (201)$$

but horizontal direction:

$$\nabla_h p = -2\rho(\vec{\Omega}\times\vec{v})_h$$

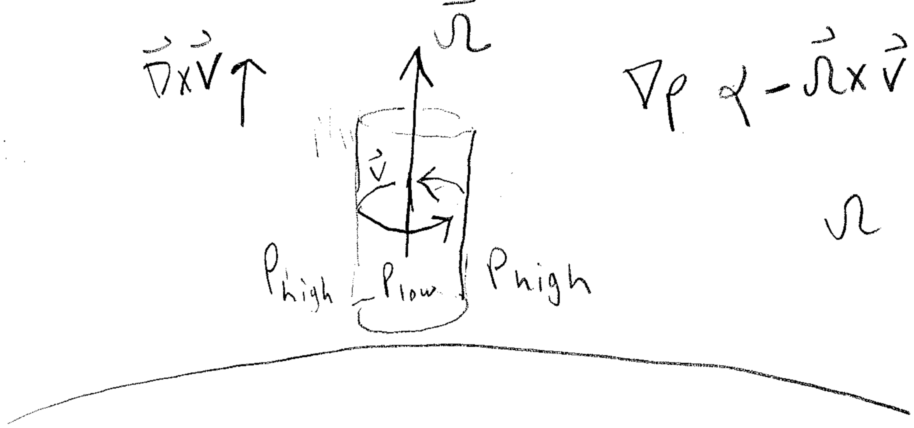
Geostrophic Approximation

Horizontal pressure gradient balanced by coriolis force

$\rightarrow$

Note the interesting fact that the velocity is $\perp$ to the gradient in pressure if cross product with Ω balances $\nabla_h p$.

This means that if there is a low pressure region in the atmosphere, velocity does not flow into low pressure region, but flows around it.:



$$\nabla p \propto -\vec{\Omega}\times\vec{v}$$

Ω is Earth rotation, v is flow velocity

vortex $\vec{\nabla}\times\vec{v}$ is $\parallel$ to underlying rotation $\vec{\Omega}$

$\Rightarrow$ Cyclonic circulation around low pressure region in a vortex tube

Vorticity in Rotating Frame

Using vector identity for $-v\cdot\nabla v$ we can write (199)

$$\frac{\partial \vec{v}}{\partial t} = \vec{v}\times\vec{\omega} - \nabla\left(\frac{p}{\rho} + \frac{1}{2}v^2 + \phi - \frac{1}{2}(\vec{\Omega}\times\vec{r})^2\right) - 2\vec{\Omega}\times\vec{v} \qquad (202)$$

taking curl $\Rightarrow$ $$\frac{\partial w}{\partial t} = \nabla\times(\vec{v}\times\vec{\omega}) + \nabla\times(\vec{v}\times 2\vec{\Omega}) \qquad (203)$$

assuming $\Omega =$ constant we can write (203) as

$$\frac{\partial}{\partial t}(\omega + 2\vec{\Omega}) = \nabla \times [v \times (\omega + 2\Omega)] \quad (204)$$

This is of the form $\frac{\partial \vec{Q}}{\partial t} = \nabla \times \vec{v} \times \vec{Q}$ with $Q = (\vec{\omega} + 2\vec{\Omega})$ thus we know from derivation of Kelvin Circulation theorem that

$$\frac{d}{dt}\int (\omega + 2\Omega)\cdot dS = 0 \quad \text{(Bjerknes Theorem)} \quad (205)$$

This generalization implies that if Ω is increased, local vorticies must increase <u>OPPOSITELY</u> to underlying rotation to satisfy the theorem.

<u>Self-gravitating masses</u> - Maclaurin & Jacobi ellipsoids:

Consider an initially spherically symmetric, gravitating fluid of uniform density and start it rotating: Flattening near the poles is expected. Assume Ω is constant, then move to frame in which $v = 0$ (rotating frame) and consider the equilibrium configurations. <u>Maclaurin</u> = bi-axial ellipsoid, <u>Jacobi</u> = tri-axial ellipsoid