Toward MHD from a two-fluid approach

consider multiparticle phenomena in a plasma on scales much larger than Debye Length and time scales much larger than plasma frequency; then charge separation in plasma can be neglected.

$\lambda_D = \left(\frac{kT}{8\pi n e^2}\right)^{1/2}$, $\omega_{p,e}^2 = \frac{4\pi n e^2}{m_e}$)

But: when considering long time scales collisions cannot be neglected. Here I will address the derivation of collisional MHD starting from two-fluid approach, and also modeling the collision contributions.

In a two fluid approach, we consider a fully ionized plasma of protons and electrons. The protons are treated as one fluid and the electrons as another fluid. →

Consider e^- fluid first:
Collisions between them do not change i.e. e^- fluid momentum: only when e^- and ions collide is momentum transferred.

Thus, the equations for the electron fluid (to first order in V_e) is given by (assuming $n = n_- = n_+$)

$$m_e n \frac{\partial \vec{V}_e}{\partial t} = -\nabla p_e - ne\left(E + \frac{\vec{V}_e}{c} \times B\right) - m_e n \nu_c (\vec{V}_e - \vec{V}_i) \quad (251)$$

where n is number density, ν_c is a collision frequency between electrons and ions. For the moment we neglect the fluid viscosity and will restore it later. Eqn (251) is basically the fluid equation like an Euler eqn for electrons with the extra ν_c term and dropping the $V_e \cdot \nabla V_e$ term on the grounds that it is second order.

Since the current density: $j \equiv ne(\vec{V}_i - \vec{V})$ (251a)
the last term in (251) is proportional to the current density, thus:

$$m_e n \frac{\partial V_e}{\partial t} = -\nabla p_e - ne\left(\vec{E} + \frac{\vec{V}_e}{c} \times \vec{B}\right) + ne\eta \vec{j} \quad (252)$$

with $\eta = \dfrac{m_e \nu_c}{ne^2}$ (253)

η can be explained as follows:

consider homogeneous (uniform) plasma in steady-state with $\vec{E}$-field driving a current $\vec{J}$. Then (252) becomes

$$-ne\vec{E} + ne\eta\vec{J} = 0 \qquad (254)$$

So that

$$\vec{E} = \eta\vec{J} \qquad (255)$$

$\Rightarrow$ η is the plasma resistivity.

To calculate it, we need expression for ν_c.

To get the collision frequency, consider an approximate approach: If the impact parameter between e^- and p is large then minimal deflection takes place. Thus define impact parameter r_0 for which e^- deflection is sufficiently large to change its momentum by of order its original momentum.

$\longrightarrow$

to estimate r_0, let u be typical relative velocity between the proton & e^- so that r_0/u is effective interaction time. Since strongest interaction force is $\frac{e^2}{r_0^2}$, the impulse ($= F \cdot \Delta t$)

$$= \frac{e^2}{r_0^2}\frac{r_0}{u} = \frac{e^2}{r_0 u} \simeq \Delta p \qquad (256)$$

by definition of r_0, $\Delta p = p = m_e u$

so that $r_0 \simeq \frac{e^2}{m_e u^2}$ (257)

Moreover, the effective collision cross section is given by πr_0^2, thus the collision frequency $\simeq$

$$\nu_c \simeq n \sigma u = n \pi r_0^2 u = \frac{\pi n e^4}{m_e^2 u^3}, \qquad (258)$$

using (257). For thermal velocities, plug in $u = \left(\frac{k_B T}{m_e}\right)^{1/2}$ and (258) gives:

$$\nu_c = \frac{\pi n e^4}{m_e^{1/2}(k_B T)^{3/2}}, \qquad (259)$$

Using (259) in (253)

$$\rightarrow \quad \eta \simeq \frac{\pi m_e^{1/2} e^2}{(k_B T)^{3/2}} \qquad (260)$$

A rigorous calculation (Spitzer & Härm 1953)

gives

$$\boxed{\text{Spitzer resisitivity}} \rightarrow \eta = \underbrace{\left(\frac{Z\pi^{1/2} \ln\Lambda}{g(Z)\, 4\sqrt{2}}\right)}_{\text{extra factor}} \underbrace{\left(\frac{\pi m_e^{1/2} e^2}{(k_B T)^{3/2}}\right)}_{(260)} \qquad (261)$$

[Z = charge number for + ions)

The extra factor is typically of order 1-10.

(ie: $\ln\Lambda = \ln\left(\frac{3}{2Ze^2} \frac{k_B^{3/2} T^{3/2}}{\pi^{1/2} n^{1/2}}\right)$, $g(Z=1) \simeq 0.6$).

Thus the rough treatment at least gets the basic scalings reasonably well.

For $Z=1$, proton-electron plasma, the sptizer resistivity is

$$\eta = 7.3 \times 10^{-9} \frac{\ln\Lambda}{T^{3/2}} \ \text{sec} \qquad (262)$$

Now, having obtained η, let us consider equation of motion for the ion fluid $\rightarrow$

The collision term in the ion equation should be equal and of opposite sign to that in the e^- equation, since momentum is exchanged between the two. Thus

$$m_i n \frac{\partial \vec{V}_i}{\partial t} = -\nabla P_i + ne\left(\vec{E} + \frac{\vec{V}_i}{c} \times \vec{B}\right) - ne\eta \vec{J} \qquad (263)$$

(where again we worked to first order in $\vec{V}$ so $\vec{V}_i \cdot \vec{\nabla} V_i$ is neglected).

Now we combine (263) with (252) to get a 1 fluid model:

The total density and net fluid velocity are

$$\rho = n(m_i + m_e) \qquad (264)$$

$$\vec{V} = \frac{m_i \vec{V}_i + m_e \vec{V}_e}{m_i + m_e} \qquad (265)$$

Then adding (263) and (252) $\Rightarrow$

$$n \frac{\partial}{\partial t}(m_i V_i + m_e V_e) = \frac{ne(\vec{V}_i - \vec{V}_e)}{c} \times \vec{B} - \nabla(P_i + P_e)$$

$\rightarrow$ or, using (264), (265) & $j = ne(\vec{V}_i - \vec{V}_e)$:

$$\rho \frac{\partial \vec{V}}{\partial t} = \frac{\vec{J} \times \vec{B}}{c} - \nabla P \qquad (266)$$

where $p \equiv p_i + p_e$.

Now, actually the pressure is really a tensor, as we discussed early in the course in deriving the hydrodynamic equations. the pressure tensor was given by

$$P_{ij} = nm\langle v_i v_j \rangle - nm\bar{v}_i\bar{v}_j$$

this term was neglected in our present approach. When it is not neglected, (266) becomes

$$\rho\frac{\partial\vec{v}}{\partial t} = -\vec{v}\cdot\nabla\vec{v} - \nabla p + \underbrace{\vec{J}\times\vec{B}}_{\text{magnetic term}} \qquad (267)$$

magnetic term now added to Euler equation

Note $\vec{E}$ force has disappeared. Note also that the Navier-Stokes eqn had is like the Euler eqn but with the added viscosity term that resulted from deviations from Maxwellian. The derivation of that term would proceed the same had we started from the Boltzmann eqn for e^- and protons separately in deriving (252) and (263). Thus I will add the term without a rigorous derivation Thus the momentum equation for single fluid MHD is given by

$$\rho \frac{\partial \vec{v}}{\partial t} = -\vec{v}\cdot\nabla\vec{v} - \nabla p + \vec{J}\times\vec{B} + \nu \nabla^2 \vec{v} \qquad (268)$$

(for non-uniform ν, this term should be, in general $= -\nabla\times(\nu\nabla\times v)$)

Now (268) was derived by taking the sum of e^- & ion fluid eqns. By taking the difference, we get an expression for the electric field in terms of the $\vec{v}$ & $\vec{B}$. Such a relation is needed when the magneto-fluid equations are combined with maxwell's equations.

Multiplying (252) by m_i and subtracting (263) multiplied by m_e gives:

$$m_i m_e n \frac{\partial}{\partial t}(\vec{v}_i - \vec{v}_e) = ne(m_i + m_e)\vec{E} + \frac{ne}{c}(m_e\vec{v}_i + m_i\vec{v}_e)\times\vec{B} - m_e\nabla p_i + m_i\nabla p_e - (m_i+m_e)ne\eta\vec{J} \qquad (269)$$

Using (264), (265), (251a) and

$$m_e\vec{v}_i + m_i\vec{v}_e = m_i\vec{v}_i + m_e\vec{v}_e + (m_e - m_i)(\vec{v}_i - \vec{v}_e) = \frac{\rho}{n}\vec{v} + \frac{m_e - m_i}{ne}\vec{J} \qquad (270)$$

we get from (269)

$$\vec{E} + \frac{\vec{V}}{c} \times \vec{B} = \eta \vec{J} + \frac{1}{e\rho}\left[\frac{m_e m_i n}{e}\frac{\partial}{\partial t}\left(\frac{\vec{J}}{n}\right) + (m_i - m_e)\frac{\vec{J}\times\vec{B}}{c} + m_e \nabla p_i - m_i \nabla p_e\right] \quad (270)$$

This is the <u>Generalized Ohm's law</u>.

when system changes on time scales long compared to collision time the $\frac{\partial}{\partial t}\left(\frac{J}{n}\right)$ term is small compared to $\eta \vec{J}$ term.

then (270) =>

$$\vec{E} + \frac{\vec{V}}{c} \times \vec{B} = \eta \vec{J} + (m_i - m_e)\frac{\vec{J}\times\vec{B}}{c} + m_e \nabla p_i - m_i \nabla p_e \quad (271)$$

often in astro the <u>Hall Effect</u> term $(m_i - m_e)\frac{\vec{J}\times\vec{B}}{c}$ and the pressure gradient terms are ignorable compared to $\eta \vec{J}$. Thus in many cases

$$\vec{E} + \frac{\vec{V}}{c} \times \vec{B} = \eta \vec{J} \quad (272)$$

is the appropriate Ohm's law.

(Note that for pair plasma (e^-, e^+), there is no Hall effect or ∇p terms in (271)).

Collecting the important equations:

(268) →

$$\rho \frac{\partial \vec{v}}{\partial t} = -\vec{v}\cdot\nabla\vec{v} - \nabla p + j \times B + \nu \nabla^2 \vec{v}$$

(272) →

$$\vec{E} + \frac{\vec{v}}{c} \times \vec{B} = \eta j$$

mass continuity (same as for unmagnetized fluids)

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho v) = 0 . \qquad (273)$$

Recall that for incompressible flows ($\nabla\cdot v = 0$), we dont need to worry about the energy equation. Most of the MHD we will consider will be incompressible

We thus have time evolution eqn for v, ρ and we have eqn that relates E to B, but we need equation for $\partial_t \vec{B}$. This comes from combining (272) with maxwell's equations:

Note that from Maxwell's eqns:

$$\frac{1}{c}\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0 \qquad (274)$$

$$\nabla \times \vec{B} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 4\pi \frac{\vec{J}}{c} \qquad (275)$$

combining (273) & (274)

$$\Rightarrow \quad \frac{\partial B}{\partial t} = \nabla \times (v \times B) - c \nabla \times \eta \vec{J} \qquad (276)$$

we then use (275) :

$$\nabla \times \vec{J} = \underbrace{\frac{c}{4\pi} \nabla \times \nabla \times B}_{\mathcal{O}\left(\frac{c}{\ell^2} B\right) \; ①} + \underbrace{\frac{1}{4\pi} \frac{\partial (\nabla \times E)}{\partial t}}_{\mathcal{O}\left(\frac{v}{\ell c t} B\right) \; ②} \qquad (277)$$

this ratio of ① / ② $\simeq \frac{c^2 t}{\ell v} \simeq \frac{c^2}{v^2}$

thus we ignore ② $\Rightarrow$

$\Rightarrow$ (277)

$$\simeq \frac{c}{4\pi} \nabla \times \nabla \times B = -\frac{c}{4\pi} \nabla^2 \vec{B} \qquad (278)$$

(278) into (276) $\Rightarrow$

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \frac{\eta c^2 \nabla^2 \vec{B}}{4\pi}$$

$$= \nabla \times (v \times B) + \nu_m \nabla^2 \vec{B} \qquad (279)$$

for $\nabla \eta = 0$. (279) is <u>Magnetic</u> <u>Induction</u> <u>Eqn</u>