equations

$\underline{\chi}$ is microscopic quantity and

$n\langle\underline{\chi}\rangle$ is macroscopic. thus

previous equation (where $\langle\underline{\chi}\rangle = \frac{1}{n}\int \chi f d^3u$)

$$\partial_t\left(n\langle\underline{\chi}\rangle\right) + \frac{\partial}{\partial x_i}\left(n\langle u_i\chi\rangle\right) - n\left\langle u_i\frac{\partial\chi}{\partial x_i}\right\rangle - \frac{n}{m}\left\langle F_i\frac{\partial\chi}{\partial u_i}\right\rangle - \frac{n}{m}\left\langle\frac{\partial F}{\partial u_i}\chi\right\rangle = 0 \quad (14b)$$

provides a link between micro & macro quantities. Fluid equations are macro equations so (14b) is fundamental.

Recall that (14b) applies for any conserved quantity. Classically, mass is conserved, so lets first consider

$\underline{\chi} = m$ in (14b) $\longrightarrow$

for $\vec{F}$ indendent of u, and all particles of same mass m:

$$\frac{\partial}{\partial t}(mn) + \frac{\partial}{\partial x_i}(nm\langle u_i\rangle) = 0 \qquad (15)$$

if we write $\rho = mn$ and $V_i \equiv \langle u_i\rangle$

then we have continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho V_i) = 0 \qquad (16)$$

(or $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0$). This is one of the fundamental fluid equations.

Second Now let $\chi = mu_i$ in (14)

Since u_i, x_i are independent variables and $\frac{\partial \vec{F}}{\partial u_i} = 0$ by assumption.

$$\Rightarrow \frac{\partial}{\partial t}(nm\langle u_j\rangle) + \frac{\partial}{\partial x_i}(nm\langle u_i u_j\rangle) - nF_j = 0 \qquad (17)$$

now define $P_{ij} = nm\langle(u_i - V_i)(u_j - V_j)\rangle$ with $V_i = \langle u_i\rangle$ (18)

$$= nm\langle u_i u_j\rangle + nm V_i V_j - nm\underbrace{\langle u_i\rangle}_{V_i} V_j - nm\underbrace{\langle u_j\rangle}_{V_j} V_i$$

$$= nm\langle u_i u_j\rangle - nm V_i V_j$$

thus (18) in (17) $\Rightarrow$

$$\frac{\partial}{\partial t}(\rho V_j) + \frac{\partial}{\partial x_i}P_{ij} + \frac{\partial}{\partial x_i}(\rho V_i V_j) - \frac{\rho}{m}F_j = 0 \qquad (19)$$

Eqn (19) is the <u>momentum</u> equation with a pressure tensor.

third let $\chi = \frac{1}{2} m |\vec{u} - \vec{v}|^2$ in (14b)

this corresponds to conserved energy in collisions for monatomic gas, and constant mean velocity $\vec{v}$.

The result is then:

$$\partial_t (\rho \varepsilon) + \frac{\partial}{\partial x_i} (\rho \varepsilon v_i) + \frac{\partial q_i}{\partial x_i} + P_{ij} \Lambda_{ij} = 0 \qquad (20)$$

(energy eqn)

$\varepsilon \equiv \frac{1}{2} \langle |\vec{u} - \vec{v}|^2 \rangle$ = internal energy per mass

$q \equiv \frac{\rho}{2} \langle (u-v) |u-v|^2 \rangle$ = energy flux (units: $\frac{\text{energy}}{\text{Area} \cdot \text{time}}$)

$$\Lambda_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

Now simplify (19) & (20) using (16)

The results are:

$$(19) \rightarrow \quad \rho \left(\frac{\partial v_j}{\partial t} + v_i \frac{\partial v_j}{\partial x_i} \right) = - \frac{\partial P_{ji}}{\partial x_i} + \frac{\rho}{m} F_j \qquad (21)$$

$$(20) \rightarrow \quad \rho \left(\frac{\partial \varepsilon}{\partial t} + v_i \frac{\partial \varepsilon}{\partial x_i} \right) + \frac{\partial q_i}{\partial x_i} + P_{ij} \Lambda_{ij} = 0 \qquad (22)$$

eqn (16), (21), (22) do represent mass, momentum, and energy conservation but these represent 5 eqns with 14 unknowns(!): $\vec{V}$ (3-components)

P_{ij} (6 components, since symmetric)

ρ (1-component)

q_i (3-components)

ε (1-component)

Thus we need relations between these quantities to close system of equations

eqns (16), (21), (22) are called the "moment" equations since they arise from multiplying Boltzmann eqn by powers of 0, 1, 2 velocities and integrating over velocity.

Note distinction between u_i & V_i

u_i: velocity of individual particle

V_i: mean velocity of overall flow

Alternatively:

$\langle \vec{u}_i \rangle = \vec{V}$, $\vec{u} = u_{fluct} + V_i$ →

u_{fluct}: random component

V_i: mean component

We argued before, that collisions set up a Maxwellian distribution when frequent enough. Now let us see what this implies for reducing the number of variables, and a "simple" set of eqn:

Assume that distribution function is Maxwellian:

$$f^{(0)}(\vec{x},\vec{u},t) = n(\vec{x},t)\left[\frac{m}{2\pi k_b T(x,t)}\right]^{3/2} \exp\left[-\frac{m(\vec{u}-\vec{v}(x,t))^2}{2k_b T(\vec{x},t)}\right] \quad (23)$$

where we write x, t dependencies explicitly.

Using (23) we have

$$P_{ij} = \rho\left(\frac{m}{2\pi k_b T}\right)^{3/2}\int d^3U\, U_i U_j \,\mathrm{Exp}\left[-\frac{mU^2}{2k_b T}\right]$$

$$\vec{U} = \vec{u} - \vec{v}$$

Integral vanishes when integrand is odd $\Rightarrow$

$$P_{ij} = p\delta_{ij} = nk_b T \quad (24)$$

which comes from integrating

$$\int_0^\infty U^2 e^{-AU^2}\,dU = \frac{1}{4}\sqrt{\frac{\pi}{A^3}}$$

$\rightarrow$

→ We can also see that the flux $\vec{q}$ satisfies

$$\vec{q} = 0, \quad \text{since it is odd integral.} \tag{25}$$

⇒ From definition of $\mathcal{E} = \frac{1}{2}\langle |U|^2 \rangle$

we also have that

$$\mathcal{E} = \frac{3}{2}\frac{k_b T}{m} \tag{26}$$

thus; using 24, 25, 26 we have eliminated 3 variables of $\vec{q}$, 5 variables of the original P_{ij} tensor, and $\mathcal{E}$ can be written as function of ρ, thus $14 - 9 = 5$ variables left and 5 equations!

using $P_{ij} = p\,\delta_{ij}$ we also have

$$P_{ij}\Lambda_{ij} = \frac{1}{2}\,p\,\delta_{ij}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = p\,\vec{\nabla}\cdot\vec{v} \tag{27}$$

from defn of Λ_{ij} below eqn (21).

Using (24), in (21) gives

momentum: $$\boxed{\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V} = -\frac{1}{\rho}\nabla p + \frac{\vec{E}}{m}} \tag{28}$$

using (25), (26) & (27) in (22) gives

[illegible]gy: $$\boxed{\rho\left(\frac{\partial \mathcal{E}}{\partial t} + \vec{v}\cdot\vec{\nabla}\mathcal{E}\right) + p\,\nabla\cdot V = 0} \tag{29}$$

continuity eqn was

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0} \tag{30}$$

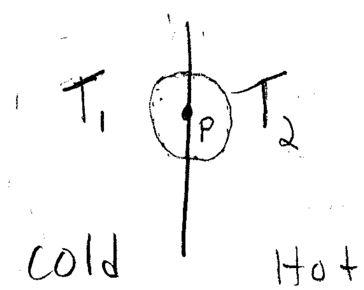
→

Transport Processes:

→ In previous derivation $\vec{q} = 0$ so no
(A) heat flow.

→ we also had P_{ij} being diagonal;
(B) this means that momentum cannot be transported from one layer of fluid to another.
This implies no **shear forces**

Both (A) & (B) resulted from assumption of Maxwellian Distribution: can immediately see that some departure from Maxwellian is required for transport:



Heat flows from Hot to Cold; in neighborhood of P distribution is not isotropic and not Maxwellian!

we need to consider perturbations around Maxwellian distribution:

$$f(x,u,t) = f^{(0)}(\vec{x},u,t) + g(\vec{x},u,t) \qquad (31)$$

↑ maxwellian ↑ small departure

putting (31) in Boltzmann equation (page 6 of Jan 21 notes)

∴ collision integral is

$$\int d^3u_1 \int d\Omega \, |\vec{u}-\vec{u}_1| \, \sigma(\Omega) \, (f'f_1' - ff_1)$$

$$= \int d^3u_1 \int d\Omega \, |\vec{u}-\vec{u}_1| \, \sigma(\Omega) \left(f^{(0)\prime} g_1' + f_1^{(0)\prime} g' - f^{(0)} g_1 - f_1^{(0)} g \right)$$

to first order.

A typical term has magnitude

$$-\int d^3u_1 \int d\Omega \, |\vec{u}-\vec{u}_1| \, \sigma(\Omega) \, (f_1^{(0)} g) \simeq -\vec{u}_{rel} \, n \, \sigma \, g(x,u,t)$$

$|\vec{u}_{rel} n \sigma|$ is a collision frequency with units $\frac{1}{\tau}$ ⇒ collision integral is roughly

$-\frac{g}{\tau}$ ⇒ Boltzmann eqn:

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla} + \frac{\vec{F}}{m}\cdot\nabla_u \right) f = -\frac{(f - f^{(0)})}{\tau} \qquad (32)$$

this term is responsible for g
when there are strong spatial gradients.

To order of mag

$$\frac{|u| f^{(0)}}{L} \sim \frac{|g|}{\tau}$$

where L is gradient scale over which properties change.

mean free path

$$\Rightarrow \quad \frac{|g|}{f^{(0)}} \simeq \frac{|u|\tau}{L} \sim \frac{\lambda_{mfp}}{L} \equiv \alpha$$

$$\Rightarrow \quad f = f^{(0)} + \alpha f_1^{(0)} + \alpha^2 f_2^{(0)} \ldots$$

Chapman–Ensog expansion.

To compute "corrections" use lowest order in (32)

$$\Rightarrow$$

$$g = -\tau\left(\partial_t + u_i \partial_i + \frac{F_i}{m}\partial_{u_i}\right) f^{(0)} \qquad (33)$$

From (23), $$f^{(0)} = \frac{m^{3/2}\, n(x,t)}{\left(2\pi k_B T(x,t)\right)^{3/2}} \operatorname{Exp}\left[-\frac{m\left(\vec{u} - \vec{V}(\vec{x},t)\right)^2}{2k_B T(\vec{x},t)}\right], \quad f = f(n, T, \vec{V})$$

so

chain rule for terms in (33):

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial n}{\partial t}\frac{\partial f^{(0)}}{\partial n} + \frac{\partial T}{\partial t}\frac{\partial f^{(0)}}{\partial T} + \frac{\partial V_i}{\partial t}\frac{\partial f^{(0)}}{\partial V_i}$$

use (30)

use (28)

use (29)

$$\frac{\partial f^{(0)}}{\partial x} = \frac{\partial n}{\partial x}\frac{\partial f^{(0)}}{\partial n} + \frac{\partial T}{\partial x}\frac{\partial f^{(0)}}{\partial T} + \frac{\partial V_i}{\partial t}\frac{\partial f^{(0)}}{\partial V_i}$$

<u>Using</u> (23) for $f^{(0)}$ in (33) and using the $f^{(0)}$ "moment" equations for continuity (30), momentum (28) and energy density (29); we get (set $F_i = 0$ to simplify)

$$g = -\tau\left(\frac{1}{T}\frac{\partial T}{\partial x_i} U_i \left(\frac{m}{2k_B T} U^2 - \frac{5}{2}\right) + \frac{m}{k_B T}\Lambda_{ij}\left(U_i U_j - \frac{1}{3}\delta_{ij} U^2\right)\right] f^{(0)} \qquad (34)$$

with $\Lambda_{ij} \equiv \partial_i V_j + \partial_j V_i$; $\vec{U} \equiv \vec{u} - \vec{V}$

$\longrightarrow$

→ That g depends linearly on velocity and temperature gradients is expected, based on our simple argument before, for deviations from Maxwellian dist. → gradients imply deviation from Maxwellian.

→ Linear dependence on τ implies that the longer the time between collisions, the more the deviation from Maxwellian can be sustained, and thus a larger correction g. (collisions tend to make f closer to $f^{(0)}$).

→ Now we can calculate P_{ij}, $\vec{q}$, and ε for the non-Maxwellian distribution $f = f^{(0)} + g$ with $\langle A \rangle \equiv \frac{1}{n}\int A f\, d^3u$ as defn for averaging of quantity A,

from before:

only g contributes from $f = f^{(0)} + g$

$$\vec{q} = \frac{nm}{2}\langle \vec{U} U^2 \rangle = \frac{\rho}{2}\int d^3U\, \vec{U} U^2 g$$

Only even powers contribute to integrand so only 1st term on right of (34) contributes:

$$\vec{q} = -K \nabla T, \quad \left(\text{where } K = \frac{m\tau}{6T}\int d^3U\, U^4\left(\frac{m}{2k_BT}V^2 - \frac{5}{2}\right) f^{(0)} = \frac{5}{2} n\tau \frac{k_B^2 T}{m}\right) \tag{35}$$

$\frac{erg}{cm\, s\, K}$ $\frac{erg}{cm\, s\, K}$

That $\vec{q} = -\mathbb{K}\nabla T$ is a familiar form of heat transport equation (which we have derived from a "bottom up" approach).

Also:

$P_{ij} = nm\langle V_i V_j \rangle$, is no longer diagonal

instead:

$$= p\delta_{ij} + \Pi_{ij}, \qquad (35a)$$

with $\Pi_{ij} \equiv m\int d^3V\, V_i V_j\, g$,

from (34) we then have

$$\Pi_{ij} = -\frac{\tau m^2}{k_B T}\Lambda_{kl}\int d^3V\, V_i V_j \left(V_k V_l - \frac{1}{3}\delta_{kl}V^2\right) f^{(0)}$$

but for this integral, only isotropic contributions survive, since $f^{(0)}$ is isotropic (no dependence on vector $\vec{V}$ only its magnitude).

this means $\rightarrow$

$$\langle V_i V_j V_k V_l \rangle = a\,\delta_{ij}\delta_{kl} + b\,\delta_{ik}\delta_{jl} + c\,\delta_{il}\delta_{jk}$$

$$\langle V_i V_j \delta_{kl} V^2 \rangle = d\,\delta_{ij}\delta_{kl}$$

$\longrightarrow$

to find a, b, c : need 3 equations.
Multiply by each separate δ combination:

$$\langle U^4 \rangle = 9a + 3b + 3c \qquad (35)$$

$$\langle U^4 \rangle = 3a + 9b + 3c \qquad (36)$$

$$\langle U^4 \rangle = 3a + 3b + 9c \qquad (37)$$

$$\Rightarrow \quad 0 = 6a - 6b$$

$$0 = 6a - 6c$$

$$0 = 6b - 6c$$

$$\Rightarrow \quad b = c = a = \frac{\langle U^4 \rangle}{15}$$

also $\langle U_i U_j U^2 \delta_{ke} \rangle = d\, \delta_{ij} \delta_{ke}$

$$\Rightarrow 3\langle U^4 \rangle = 9d \Rightarrow d = \frac{\langle U^4 \rangle}{3}$$

$$\Rightarrow \Lambda_{ke} \left\langle U_i U_j U_k U_e - \frac{U_i U_j U^2 \delta_{ke}}{3} \right\rangle$$

note the interesting equality

$$= \Lambda_{ke} \langle U^4 \rangle \left(\frac{\delta_{ij}\delta_{ke}}{15} + \frac{\delta_{ik}\delta_{je}}{15} + \frac{\delta_{ie}\delta_{jk}}{15} - \frac{1}{9}\delta_{ij}\delta_{ke} \right)$$

$$= \frac{2}{15} \langle U^4 \rangle \Lambda_{ij} - \frac{6}{135} \Lambda \delta_{ij} = \frac{2}{15} \langle U^4 \rangle \left(\Lambda_{ij} - \frac{1}{3} \delta_{ij} \Lambda \right)$$

thus

$$\Pi_{ij} \propto \left(\Lambda_{ij} - \frac{1}{3}\delta_{ij}\Lambda\right)$$

we can write

$$\Pi_{ij} = -2\mu\left(\Lambda_{ij} - \underbrace{\frac{1}{3}\delta_{ij}\nabla\cdot V}_{=\frac{1}{3}\Lambda\delta_{ij}}\right) \qquad (38)$$

to get μ evalute one component of Π_{ij}: (from p 35)

$$\Pi_{12} = \frac{\tau m^2}{k_B T}\Lambda_{kl}\int d^3U\, U_1 U_2\left(U_k U_l - \frac{1}{3}\delta_{kl}U^2\right)f^{(1)}$$

$$= -2\frac{\tau m^2}{k_B T}\Lambda_{12}\int d^3U\, U_1^2 U_2^2 f^{(0)}$$ since only even powers to contribute.

thus: $$\mu = \frac{m^2\tau}{k_B T}\int d^3U\, U_1^2 U_2^2 f^{(0)} = \tau n k_B T \qquad (38a)$$

from (38) since $\langle u_1^2 u_2^2\rangle = \frac{k^2T^2}{m^2}$, use: $\int_{-\infty}^{\infty} q^2 e^{-aq^2}dq = \frac{\pi^{1/2}}{2a^{3/2}}$ and $\int_{-\infty}^{\infty} e^{-aq^2}dq = \frac{\pi^{1/2}}{a^{1/2}}$ and $\langle Q\rangle = \frac{\int Q f d^3u}{n}$

The off diagonal component of P_{ij} thus has coefficent μ, this is viscosity! density means momentum transport is possible between different flows moving at different velocities. More on this later.

with expressions for $\vec{q}$ and P_{ij}

we put them into the moment equations:

using P_{ij} and $\Lambda_{ij} \equiv \frac{1}{2}(\partial_j v_i + \partial_i v_j)$; $\Pi_{ij} = -2\mu(\Lambda_{ij} - \frac{1}{3}\delta_{ij}\vec{\nabla}\cdot\vec{v})$

$$\Rightarrow \frac{\partial P_{ij}}{\partial x_j} = \frac{\partial p}{\partial x_i} - \mu\left[\nabla^2 v_j + \frac{1}{3}\frac{\partial}{\partial x_j}(\nabla\cdot v)\right]$$

then plugging into (19)

$$\rho\left(\frac{\partial v_j}{\partial t} + v_i\frac{\partial v_j}{\partial x_i}\right) = -\frac{\partial p}{\partial x_j} + \mu\left[\nabla^2 v_j + \frac{1}{3}\frac{\partial}{\partial x_j}(\vec{\nabla}\cdot\vec{v})\right] + \frac{\rho}{m}F_j \quad (39)$$

from (38), (35a) & defn of Λ_{ij}, we also have

$$P_{ij}\Lambda_{ij} = p\,\nabla\cdot v - 2\mu\left[\Lambda_{ij}\Lambda_{ij} - \frac{1}{3}(\nabla\cdot v)^2\right], \quad (39a)$$

plugging (39) and (38) for Π_{ij} into energy moment eqn (20)

$$\Rightarrow \rho\left(\frac{\partial \varepsilon}{\partial t} + v\cdot\nabla\varepsilon\right) - \nabla\cdot(K\nabla T) + p\,\nabla\cdot v - \underbrace{2\mu\left[\Lambda_{ij}\Lambda_{ij} - \frac{1}{3}(\nabla\cdot v)^2\right]}_{\text{heat production by viscous damping}} = 0 \quad (40)$$

now, μ term in (40) and $(\nabla\cdot v)$ term in (39)

are often small, if we neglect them

→

=> momentum

$$\frac{\partial v}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\nabla p + \vec{F} + \frac{\mu}{\rho}\nabla^2 v \qquad (41)$$

=> energy

$$\rho\left(\frac{\partial \varepsilon}{\partial t} + v\cdot\nabla\varepsilon\right) - \nabla\cdot(\overline{k}\,\nabla T) + p\,\nabla\cdot\vec{v} = 0 \qquad (42)$$

and mass continuity:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \qquad (43)$$

are the fluid equations, and we have now used $\vec{F}$ to represent force density.