Fluid Equations

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho v) = 0 \qquad \text{(continuity)} \qquad (44)$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + F + \frac{\mu}{\rho}\nabla^2\vec{v} \qquad \text{(momentum)} \qquad (45)$$

(for constant viscosity)

$$\rho\left(\frac{\partial \varepsilon}{\partial t} + v\cdot\nabla\varepsilon\right) - \nabla\cdot(K\nabla T) + p\nabla\cdot V = 0 \qquad \text{(energy)} \qquad (46)$$

(45) is called Navier-Stokes eqn.

These <u>5 equations</u> constitute a dynamical theory:

$\vec{V}$: 3 quantities

ρ, P, ε : 4 quantities

but $P \propto \varepsilon$, and $P = nkT$ so we eliminate 2/7 quantities and are left with 5 equations and 5 variables.

- Can also derive the fluid equations from macroscopic stress consideration. I won't do that here.

Vorticity Equation & incompressible flow

take curl of Nav. Stokes equation:

$\vec{\omega} = \nabla \times V \Rightarrow$

assume $\vec{F} = -\nabla \phi$ (conservative force)

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla \times (\vec{v}\cdot\nabla v) = -\nabla \times \frac{1}{\rho}\nabla p + \cancel{\nabla \times \vec{F}} + \frac{\mu}{\rho}\nabla^2 \vec{\omega}$$

but $(\vec{v}\cdot\vec{\nabla}\vec{v}) = \frac{1}{2}\vec{\nabla}(\vec{v}\cdot\vec{v}) - \vec{v}\times(\vec{\nabla}\times\vec{v})$

$$\Rightarrow \frac{\partial \vec{\omega}}{\partial t} = \nabla \times (\vec{v}\times\vec{\omega}) + \frac{1}{\rho^2}\nabla\rho \times \nabla p + \mu \nabla^2 \vec{\omega} \qquad (47)$$

now, consider an incompressible flow: in such a flow the density remains constant in space and time. The continuity equation then gives

$$\cancel{\frac{\partial \rho}{\partial t}}^{\,0} + \rho \nabla\cdot v + \cancel{v\cdot\nabla\rho}^{\,0} = 0 \Rightarrow \nabla\cdot V = 0$$

For incompressible flow:

$$\frac{\partial \vec{\omega}}{\partial t} = \nabla \times (\vec{v}\times\vec{\omega}) + \mu \nabla^2 \vec{\omega} \qquad (48)$$

compare to magnetic induction equation in incompressible MHD:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (v \times \vec{B}) + \nu_m \nabla^2 \vec{B} \qquad (49)$$

where ν_m is magnetic diffusivity. Note similarity between (48) and (49)!

The similarity of (48) & (49) implies "deep" connections etween behavior of vorticity, or vortex lines and magnetic field lines in incompressible MHD.

more on incompressibility

when is flow incompressible?

We will later see that disturbances in a fluid propagate at the sound speed, c_s. Thus in general, unless the agent causing the disturbance moves faster than c_s, the density will smooth out on time scales short compared to the evolution of the quanties of interest => systems with subsonic material velocities are largely incompressible.

(Note also that for barotropic flows, defined by $P = P(\rho)$, the third term of (47) also vanishes. these flows can be compressible. More on these later)

For incompressible flow, energy equation is <u>redundant</u>:

Since $\nabla \cdot V = 0$; and using $\nabla \times V \equiv \omega$

$\partial_t \bar{\omega} = \nabla \times \vec{V} \times \vec{\omega} + \mu \nabla^2 \bar{\omega}$ we can fully solve for $\vec{v}$. If a vector field's divergence and curl are known, then we can solve for vector field. Thus $\nabla \cdot V$ and (48) are enough to solve for $\vec{v}$. Once we have v, we get ρ om (44) and p from (45). Thus (46) is never needed, since p and ε are related.

Thus energy equation is not needed for incompressible flows.

→ This is not true when radiation is important.
⇒ radiative transfer and energy equation are needed. extra terms in the energy equation corresponding to radiation stress are required.

(There is a formal analogy between $f(\vec{x}, v, t)$ and $I_\nu(\vec{x}, \hat{n}, \nu, t)$)

specific intensity

Hydrostatic Equilibrium

onsider a fluid at rest, so that $\vec{v}=0$

then momentum equation with $\vec{v}=0$

$$\Rightarrow \quad F = +\frac{1}{\rho}\nabla p \tag{50}$$

energy equation

$$\nabla\cdot(\underline{K}\nabla T) = 0 \tag{51}$$

Consider fluid in gravitational field in equilibrium

$\vec{F} = -g\hat{e}_z$ where $\hat{e}_z$ is vertical direction

ne $\hat{z}$ component of (50) then gives

$$-\rho g = \frac{\partial p}{\partial z} \tag{52}$$

for incompressible flow this completely describes the system

$\Rightarrow \quad p = p_0 - \rho g z$, where $p_0 = p(z=0)$

and thus p increases as z decreases below 0

for incompressible flow.

$\rightarrow$

Now consider a compressible flow and consider the isothermal soln to (51). Since $p = \frac{k_B}{m} \rho T$ (which we derived from Boltzmann eqn).

For constant T, (52) gives

$$\frac{k_B T}{m} \frac{d\rho}{dz} = -\rho g$$

$$\Rightarrow \rho = \rho_0 \, \mathrm{Exp}\left[\frac{-mgz}{k_B T}\right], \quad \rho_0 \equiv \rho(0) \qquad (53)$$

$\Rightarrow$ density falls off exponentially in an isothermal atmosphere

Note that (50) is a fundamental equation of stellar structure. However (51) would have convection and radiative transport terms in addition to the conduction terms present here.

But consider now the solar corona rather than the solar interior

$\rightarrow$

Solar Corona

coronal temperature is hotter than solar surface by factor $\simeq 1000$.

Assume spherical symmetry as a crude approximation. take boundary condition $T = T_0$ at base of corona $r = r_0$.

(we will later discuss more about heating corona)

Mass of corona is negligible so it is under gravitational influence of the sun.

in spherical geometry, hydrostatic equilib momentum eqn

$$\Rightarrow \frac{dP}{dr} = -\frac{GM}{r^2}\rho = -\frac{GM}{r^2}\frac{mP}{k_B T} \qquad (54)$$

energy equation:

$$\frac{d}{dr}\left(K r^2 \frac{dT}{dr}\right) = 0 \qquad (55)$$

K = thermal conductivity $\propto T^{5/2}$ (derived later)

so (55) $\Rightarrow$

$$r^2 T^{5/2}\frac{dT}{dr} = \text{constant}$$

$$\Rightarrow T = T_0\left(\frac{r_0}{r}\right)^{2/7}, \quad T(\infty) = 0, \quad T_0 = T(r_0) \qquad (56)$$

Using this for T in (54)

$$\Rightarrow \quad \frac{dp}{p} = \frac{-GM_\odot m \; dr}{r^{12/7} r_0^{2/7} k_B T_0}$$

soln is

$$p = p_0 \, \mathrm{Exp}\left[\frac{7}{5}\frac{GM\,m}{k_B T_0 r_0}\left\{\left(\frac{r_0}{r}\right)^{5/7} - 1\right)\right. \qquad (57)$$

where $p(r_0) = p_0$.

Note that at $r \to \infty$ $p \neq 0$!

No solution with both $T(\infty)$ AND $p(\infty)$ vanishing.

Significance is that hot solar corona can only be in equilibrium if there is a pressure at infinity to keep it from expanding. But since the pressure available is not enough, Parker (1958) used this argument to predict the solar wind! It was detected several years after the prediction!

...ature dependence of $\mathcal{K}$, the thermal conduction coefficient (Quick derivation needed for page 46)

u = typical relative velocity between particles.

r_0/u = time during which particles are close enough to make "collision" by Coulomb interaction. Then

$$\Rightarrow \frac{e^2}{r_0^{\cancel{2}}}\frac{\cancel{r_0}}{u} = \Delta p = \text{change in momentum from interaction}$$

we set $\Delta p = p$ to define a "collision"

for non-relativistic electrons (at fixed $T_i = T_e$, electrons conduct the heat rather than ions because electrons are more mobile)

$$r_0 \simeq \frac{e^2}{m_e u^2}$$

collision cross section is then πr_0^2

and collision frequency is

$$(\pi r_0^2)(n)(u) = \frac{\pi n e^4}{m_e^2 u^3} \equiv \nu_c$$

use $u \sim \left(\frac{k_b T}{m_e}\right)^{1/2}$

$$\Rightarrow \nu_c = \frac{\pi n e^4}{m_e^{1/2} k_b^{3/2} T^{3/2}}$$

so collision time

$$\tau \simeq \frac{1}{\nu_c} = \frac{(k_b T)^{3/2} m_e^{1/2}}{\pi n e^4}.$$

Then from eqn (35) (page 34)

$$\mathcal{K} \propto \tau T \propto \underline{\underline{T^{5/2}}}$$

Bernouillis Principle

moving Beyond Hydrostatics a simple hydrodynamic problems involve steady flows. (time independent)

Define a streamline as the curve tangent to the velocity $\vec{v}$ at every point.
when a flow is steady, the streamlines trace the paths of all fluid parcels.

lets write $F = -\nabla\phi$ (for conservative force)

Euler equation (= eqn(45) without the viscous term) steady state is:

$$\nabla\left(\tfrac{1}{2}v^2\right) - \vec{v}\times(\vec{\nabla}\times\vec{v}) = -\frac{1}{\rho}\nabla p - \nabla\phi \qquad (58)$$

Integrate along streamline:

$$\int d\vec{\ell}\cdot\left[\nabla\tfrac{1}{2}(v^2) - \underbrace{\cancel{\vec{v}\times(\vec{\nabla}\times\vec{v})}}_{0 \text{ since } d\vec{\ell}\,\|\,\vec{v}} + \frac{1}{\rho}\nabla p + \nabla\phi\right] = 0$$

$$\Rightarrow \int \frac{dp}{\rho} + \tfrac{1}{2}v^2 + \phi = \text{constant} \quad = \underline{\text{Bernouillis Principle}} \qquad (59)$$

where the integral is along a streamline.

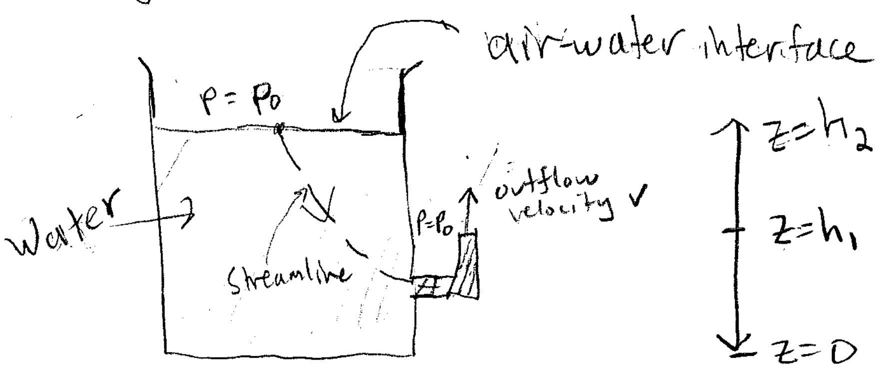
→

compressible flows, ρ = constant

(49) becomes

$$\frac{v^2}{2} + \frac{p}{\rho} + gh = \text{constant} \qquad (60)$$

apply this to a tank with outlet:



consider a streamline that extends from air-water interface at top of container to the nozzle.

$p = p_0$ both at top interface and just external to the nozzle

we have from Bernoulli's principle for incompressible flow:

$$\underbrace{\cancel{\frac{V_{in}^2}{2}}}_{\text{small}} + \underbrace{\frac{p_0}{\rho} + gh_2}_{\text{at air water interface}} = \underbrace{\frac{V_{out}^2}{2} + \frac{p_0}{\rho} + gh_1}_{\text{at nozzle}}$$

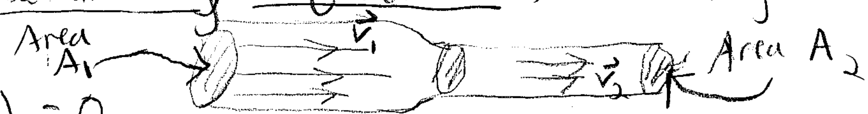
$$\Rightarrow \frac{V_{out}^2}{2} = g(h_2 - h_1) = g\Delta h$$

$$\Rightarrow |\mathbf{V}_{out}| = (2g\Delta h)^{1/2} \qquad (61)$$

Note this is independent of the nozzle's direction!

Bernouilli's Theorem also implies that pressure drops when velocity of flow increases:

consider continuity equation for steady flow in pipe:



$$\Rightarrow \cancel{\frac{\partial \rho}{\partial t}} + \nabla\cdot(\rho v) = 0$$

0 integrating $\Rightarrow \int \rho \mathbf{v}\cdot d\vec{s} =$ constant.

For constant density (incompressible) & constant $\vec{V}$ over the cross sectional area, this implies

$V_1 A_1 = V_2 A_2$. Thus as A decreases V increases:

$$\frac{V_2}{V_1} = \frac{A_1}{A_2} > 1$$

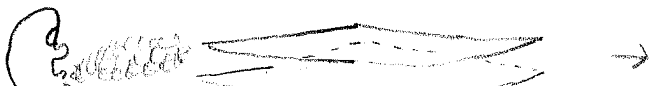
Now from Bernouillis principle we have:

$$\frac{V_1^2}{2} + \frac{P_1}{\rho} = \frac{V_2^2}{2} + \frac{P_2}{\rho}$$

$$\Rightarrow P_1 = \frac{\rho(V_2^2 - V_1^2)}{2} + P_2 > P_2$$

thus as flow is constricted in a pipe its velocity increases and pressure decreases.

By this reasoning, what happens if you hold two pieces of paper parallel and try to separate them by blowing?

 ?