Kelvin Circulation theorem (Helmholtz 1858; Kelvin 1869)

For ideal incompressible flow

$$\partial_t \vec{w} = \nabla \times (\vec{v} \times \vec{w}) \qquad (62)$$

Define flux of vorticity through a surface at time t_1 and t_2 as

$\int_{S_1} \vec{w} \cdot d\vec{S}$ and $\int_{S_2} \vec{w} \cdot d\vec{S}$. I will show that (62) $\Rightarrow$ Kelvin Circulation theorem:

$$\frac{d\Phi}{dt} \equiv \frac{d}{dt} \int \vec{w} \cdot d\vec{S} = 0 \quad \text{or} \quad \int_{S_1} \vec{w} \cdot d\vec{S} = \int_{S_2} \vec{w} \cdot d\vec{S}. \qquad (63)$$

The proof also applies to the magnetic field $\vec{B}$, as the induction equation has the same form as (62).

(64)

Derivation of the time evolution of Flux

F(1) Blachman 2/7/10

(51a)

Let $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}$ be the material derivative that follows the time evolution of a quantity as fluid moves with velocity $\vec{v}(\vec{x},t)$. For a vector $\vec{A}$

$$\frac{D}{Dt}\left(\int_S \vec{A}\cdot d\vec{S}\right) = \frac{D}{Dt}\int_{S_0} \vec{A}\cdot\vec{J}\, dS_0, \qquad \text{(F1)}$$

where $d\vec{S}$ is a surface element on the surface that evolves as the result of the fluid motion and $d\vec{S}_0$ is surface element of a fixed control surface (e.g. at $t=0$) surface. $\vec{J}$ is the Jacobian for the coordinate transformation. For a surface integral, this is given by

$$J_k = \varepsilon_{kij} \frac{\partial x_i}{\partial \sigma_1} \frac{\partial x_j}{\partial \sigma_2} \qquad \text{(F2)}$$

where σ_1, σ_2 are local cartesian coordinates of the fixed surface element dS_0 and x_1, x_2, x_3 are local cartesian coordinates of evolving surface element dS

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Because the right side of (1) is an integral over a fixed surface, we can take the $\frac{D}{Dt}$ inside the integral in (1):

$$\frac{D}{Dt}\int_S \vec{A}\cdot d\vec{S} = \frac{D}{Dt}\int_{S_0} \vec{A}\cdot\vec{J}\,dS_0 = \int\left(\vec{J}\cdot\frac{D\vec{A}}{Dt} + \vec{A}\cdot\frac{D\vec{J}}{Dt}\right)dS_0 \qquad (F3)$$

we need an expression for $\frac{D\vec{J}}{Dt}$. On page (F4) I derive this explicitly. For the moment I just take the result:

$$\frac{DJ_q}{Dt} = J_m\left(\delta_{mq}(\nabla\cdot V) - \partial_q V_m\right) \qquad (F4)$$

Using (4) and

$$\frac{DA_i}{Dt} = \frac{\partial A_i}{\partial t} + \left(\vec{V}\cdot\nabla\vec{A}\right)_i \qquad \left[\text{Note } \left(\vec{V}\cdot\nabla\vec{A}\right)_i = \vec{V}\cdot\vec{\nabla}A_i \text{ in cartesian coords}\right] \qquad (F5)$$

in equation (3), we obtain

$$\frac{D}{Dt}\int_S \vec{A}\cdot d\vec{S} = \int\Big[\underbrace{\left(\frac{\partial A_i}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\vec{A}\right)_i\right)J_i}_{(a)} + \underbrace{A_q J_m\left(\delta_{mq}(\vec{\nabla}\cdot\vec{V}) - \partial_q V_m\right.}_{(b)} dS_0\Big) \qquad (F6)$$

F(3)

(F1c)

the contributions (a) and (b) can be combined in equation (F6) giving

$$\frac{D}{Dt}\int_S \vec{A}\cdot d\vec{S} = \int_{S_0}\left(\frac{\partial A_i}{\partial t} + \vec{V}\cdot\nabla A_i + A_i\,\vec{\nabla}\cdot\vec{V} - \vec{A}\cdot\nabla V_i\right) J_i\, dS_0 \quad (F7)$$

since $J_i\, dS_0 = dS_i$ we now have an integral over the moving surface in (F7).

In addition, we have the vector identity

$$\vec{\nabla}\times(\vec{v}\times\vec{A}) = \vec{A}\cdot\nabla\vec{V} - \vec{v}\cdot\nabla\vec{A} - \vec{A}(\vec{\nabla}\cdot\vec{v}) + \vec{v}(\nabla\cdot\vec{A}) \quad (F8)$$

Using $J_i\, dS_0 = dS_i$ and (8) in (7), we have

$$\frac{D}{Dt}\int_S \vec{A}\cdot d\vec{S} = \int_S \left[\left(\frac{\partial\vec{A}}{\partial t} - \nabla\times(\vec{v}\times\vec{A})\right) + \vec{v}(\nabla\cdot\vec{A})\right]\cdot d\vec{S} \quad (F9)$$

thus if $\frac{\partial\vec{A}}{\partial t} = \nabla\times(\vec{v}\times\vec{A})$ AND $\nabla\cdot\vec{A} = 0$, then

$$\frac{D}{Dt}\int_S \vec{A}\cdot d\vec{S} = 0$$

F(4)

(51d)

Derivation of $\dfrac{D\vec{J}}{Dt}$ (needed in eqn (4))

$$J_k = \varepsilon_{kij} \frac{\partial X_i}{\partial s_1} \frac{\partial X_j}{\partial s_2}$$

$$\frac{DJ_k}{Dt} = \varepsilon_{kij}\left(\left(\frac{\partial}{\partial s_1}\frac{DX_i}{Dt}\right)\frac{\partial X_j}{\partial s_2} + \frac{\partial X_i}{\partial s_1}\left(\frac{\partial}{\partial s_2}\frac{DX_j}{Dt}\right)\right) \qquad (F10)$$

but $\dfrac{DX_j}{Dt} = V_j$ and $\dfrac{DX_i}{Dt} = V_i$

$$\frac{DJ_k}{Dt} = \varepsilon_{kij}\left(\frac{\partial V_i}{\partial s_1}\frac{\partial X_j}{\partial s_2} + \frac{\partial X_i}{\partial s_1}\frac{\partial V_j}{\partial s_2}\right) \qquad (F11)$$

$$= \varepsilon_{kij}\left(\frac{\partial V_i}{\partial X_m}\frac{\partial X_m}{\partial s_1}\frac{\partial X_j}{\partial s_2} + \frac{\partial X_i}{\partial s_1}\frac{\partial V_j}{\partial X_m}\frac{\partial X_m}{\partial s_2}\right)$$

Use antisymmetry of ε_{kij} in index interchange between $i \leftrightarrow j$ to obtain:

$$= \varepsilon_{kij}\frac{\partial V_i}{\partial X_m}\left(\frac{\partial X_m}{\partial s_1}\frac{\partial X_j}{\partial s_2} - \frac{\partial X_j}{\partial s_1}\frac{\partial X_m}{\partial s_2}\right) \qquad (F12)$$

now consider each component of (F12):

(5le)

the $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ components of (12) obtained by setting $k =$ to 1, 2, and 3 respectively : and using $\varepsilon_{kij} = -\varepsilon_{kji}$ along with the fact that ε_{ijk} vanishes when any two indices are the same. Thus :

$$\frac{DJ_1}{Dt} = \frac{\partial V_2}{\partial X_q}\left(\frac{\partial X_q}{\partial S_1}\frac{\partial X_3}{\partial S_2} - \frac{\partial X_q}{\partial S_2}\frac{\partial X_3}{\partial S_1}\right) - \frac{\partial V_3}{\partial X_q}\left(\frac{\partial X_q}{\partial S_1}\frac{\partial X_2}{\partial S_2} - \frac{\partial X_q}{\partial S_2}\frac{\partial X_2}{\partial S_1}\right) \quad (F$$

- though q indices are summed, $q = 3$ does not contribute to first term and $q = 2$ does not contribute to second term.
- recognize that

$$\left.\begin{aligned} \frac{\partial X_1}{\partial S_1}\frac{\partial X_3}{\partial S_2} - \frac{\partial X_1}{\partial S_2}\frac{\partial X_3}{\partial S_1} &= -J_2 \\ \frac{\partial X_2}{\partial S_1}\frac{\partial X_3}{\partial S_2} - \frac{\partial X_2}{\partial S_2}\frac{\partial X_3}{\partial S_1} &= J_1 \\ \frac{\partial X_1}{\partial S_1}\frac{\partial X_2}{\partial S_2} - \frac{\partial X_2}{\partial S_1}\frac{\partial X_1}{\partial S_2} &= J_3 \end{aligned}\right\} \quad (F14$$

using the equations of (F14), (F13) $\Rightarrow$

$$\frac{DJ_1}{Dt} = -\frac{\partial V_2}{\partial X_1}J_2 + \frac{\partial V_2}{\partial X_2}J_1 - \frac{\partial V_3}{\partial X_1}J_3 + \frac{\partial V_3}{\partial X_3}J_1$$

$$= (\vec{\nabla}\cdot\vec{V})J_1 - \vec{J}\cdot\frac{\partial \vec{V}}{\partial X_1} \quad (F15)$$

analagously to (13) we have

$$\frac{DJ_2}{Dt} = \frac{\partial V_3}{\partial x_q}\left(\frac{\partial x_q}{\partial s_1}\frac{\partial x_1}{\partial s_2} - \frac{\partial x_q}{\partial s_2}\frac{\partial x_1}{\partial s_1}\right)$$

$$- \frac{\partial V_1}{\partial x_q}\left(\frac{\partial x_q}{\partial s_1}\frac{\partial x_3}{\partial s_2} - \frac{\partial x_q}{\partial s_2}\frac{\partial x_3}{\partial s_1}\right) \qquad (F16)$$

similarly to the treatment below (F13), using (F14) this reduces to:

$$\frac{DJ_2}{Dt} = -\frac{\partial V_3}{\partial x_2}J_3 + \frac{\partial V_3}{\partial x_3}J_2 + \frac{\partial V_1}{\partial x_1}J_2 - \frac{\partial V_1}{\partial x_2}J_1$$

$$= (\nabla\cdot\vec{V})J_2 - \vec{J}\cdot\frac{\partial \vec{V}}{\partial x_2} \qquad (F17)$$

Finally we have:

$$\frac{DJ_3}{Dt} = \frac{\partial V_1}{\partial x_q}\left(\frac{\partial x_q}{\partial s_1}\frac{\partial x_2}{\partial s_2} - \frac{\partial x_q}{\partial s_2}\frac{\partial x_2}{\partial s_1}\right) - \frac{\partial V_2}{\partial x_q}\left(\frac{\partial x_q}{\partial s_1}\frac{\partial x_1}{\partial s_2} - \frac{\partial x_q}{\partial s_2}\frac{\partial x_1}{\partial s_1}\right)$$

$$= \frac{\partial V_1}{\partial x_1}J_3 - \frac{\partial V_1}{\partial x_3}J_1 - \frac{\partial V_2}{\partial x_2}J_3 - \frac{\partial V_2}{\partial x_3}J_2$$

$$= (\vec{\nabla}\cdot\vec{V})J_3 - \vec{J}\cdot\frac{\partial \vec{V}}{\partial x_3} \qquad (F18)$$

Combining (F15), (F17) & (F18) :

$$\frac{DJ_q}{Dt} = (\nabla \cdot \vec{V}) J_q - \vec{J}_m \nabla_q \vec{V}_m$$

$$= J_m \left(\delta_{mq} (\vec{\nabla} \cdot \vec{V}) - \partial_q V_m \right) \qquad (F19)$$

which is eqn (4) //

→

So eqn (F9) shows that only when
$\nabla\cdot\vec{A}=0$ AND $\frac{\partial\vec{A}}{\partial t}-\nabla\times(\vec{v}\times\vec{A})=0$ does
the material derivative of flux transport
hold. When $\vec{A}=\vec{\omega}$ this is the
Kelvin circulation theorem for vorticity lines.
When $\vec{A}=\vec{B}$ this is Alfvén theorem
(though not his proof)

Consider the case when $\nabla\cdot\vec{A}\neq 0$. This
means a source of monopoles: (eg if $\vec{A}=\vec{B}$)
the magnetic monopole density would be
proportional to $\nabla\cdot\vec{B}$ (by analogy to $\vec{\nabla}\cdot\vec{E}$ for electric
charge). If we set $\vec{A}=\vec{B}$ in F9 and take
advantage of the induction equation $\frac{\partial B}{\partial t}=\nabla\times\vec{v}\times\vec{B}$, then

$$\frac{D}{Dt}\int\vec{B}\cdot d\vec{S}=\int\underbrace{\vec{v}(\vec{\nabla}\cdot\vec{B})}_{\text{like an advection of magnetic charge}}d\vec{S} \qquad (F20)$$

(There is more to be said about the interpretation
of F20, we will discuss...)

Analogy between vorticity & B-field

Maxwells equations:

$$\frac{1}{c}\frac{\partial B}{\partial t} + \nabla \times E = 0 \qquad (70)$$

$$\frac{1}{c}\frac{\partial E}{\partial t} + \nabla \times B = \frac{4\pi J}{c} \qquad (71)$$

$$\nabla \cdot E = 4\pi\rho \qquad (72)$$

$$\nabla \cdot B = 0 \qquad (73)$$

Ohm's Law:
results from subtracting momentum equations for positive and negative charges. That is one integrates Boltzmann equation for "+" and "−" charges in presence of Electromagnetic force. The result is

$$\boxed{\vec{E} + \frac{\vec{V}\times\vec{B}}{c} = \eta J} + \left(\frac{1}{nec}\vec{J}\times\vec{B} - \frac{1}{ne}\nabla P_e - \frac{m_e}{ne}\left(\frac{\partial \vec{J}}{\partial t} + \vec{V}_e\cdot\nabla\vec{J}\right) + \eta_2\nabla^2 J\right) \qquad (74)$$

↓ MHD Ohm's Law

(Generalized Ohms Law also includes)

these plasma terms are "usually" small for "colder denser" plasmas of astrophysics; they are important for "hot diffuse" plasmas in the lab

Using MHD Ohm's law in (70):

$$\Rightarrow \frac{\partial B}{\partial t} = \nabla \times (\vec{V}\times\vec{B}) - \eta\,\nabla \times J \qquad (75)$$

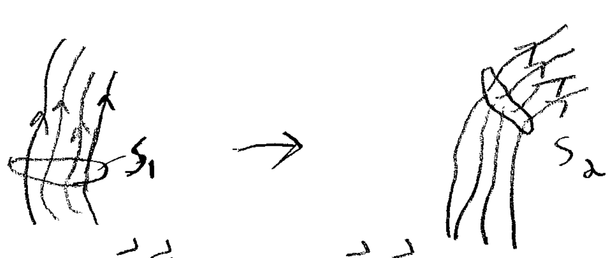
$$\simeq \nabla \times (V \times \vec{B}) \quad \text{for} \quad \eta = 0$$

thus

$$\frac{B}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) \quad \text{in ideal MHD}$$

So that $\frac{d}{dt} \int \vec{B} \cdot dS = 0$ by analogy to the ideal circulation theorem.

<u>This is flux freezing</u> since $\int \vec{B} \cdot dS = \phi$ is magnetic flux. The interpretation for magnetic flux freezing is identical to vortex line freezing: As we follow fluid elements that consitute a surface S_1 at t_1 as they evolve to surface S_2 at t_2, the normal component of vorticity or magnetic field adjusts to conserve the respective fluxes. E.g.



How far can one take the analogy between vorticity and magnetic field?

Topic of discussion...

$$\vec{S}_1 \cdot \vec{\omega}_1 = \vec{S}_2 \cdot \vec{\omega}_2$$

$$\vec{S}_1 \cdot \vec{B}_1 = \vec{S}_2 \cdot \vec{B}_2$$

Viscous flows

Ideal fluids are assumed to have no viscosity. This means that $P_{ij} = p\delta_{ij} + \underset{0}{\Pi_{ij}}$ and that the forces on a fluid surface are normal to that surface.

Force density $\propto \nabla_i P_{ij}$ so jth component of force is $F_j \propto \int \nabla_i P_{ij} \, dV = \int P_{ij} \, dS_i$

but if $\Pi_{ij} = 0$, then $F_j \propto \int p \, dS_j$

so that force points $\perp$ to the surface dS since vector area has direction $\perp$ surface.

But this violates our common experience: moving your hand through water you feel a "drag" force which is a force between different layers in a fluid, not a normal force. Thus Π_{ij} cannot in general be zero. We showed earlier that the fluid momentum equation with viscosity can be written

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = \vec{F} - \frac{1}{\rho}\nabla p + \nu \nabla^2 v \qquad (76)$$

where $\nu \equiv \frac{\mu}{\rho}$ and we have ignored both spatial dependence of ν and $\nabla\cdot v$ term in Π_{ij}.

this is the <u>Navier-Stokes equation</u> (reduces to Euler equation with $\nu = 0$) ν is called the kinematic viscosity.

The presence of the ν term means that vorticity is no longer conserved:

taking curl of (76) =>

$$\frac{\partial \vec{\omega}}{\partial t} = \nabla\times(\vec{v}\times\vec{\omega}) + \underline{\nu\nabla^2\vec{\omega}}$$

allows vorticity flux dissipation

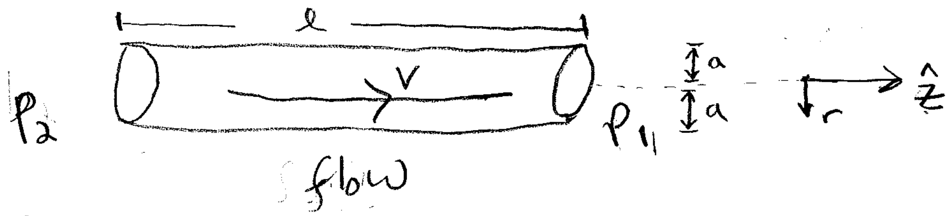
(similarly non-ideal MHD

$$\frac{\partial B}{\partial t} = \nabla\times(\vec{v}\times\vec{B}) + \underbrace{\nu_M \nabla^2 \vec{B}}$$

allows magnetic dissipation)

Viscous flow through pipe

Consider steady flow of incompressible viscous fluid through pipe of circular cross section.



cylindrical coords

$P_2 - P_1 = \Delta p$ pressure drives the flow

For incompressible flow, v should not depend on z: $\rho V A$ = constant, where A is oss sectional area. $\Rightarrow$ V = constant for ρ, A constant.

Thus we have $V_z = V_z(r)$

From steady Nav-Stokes eqn:

$$-\nabla p = \cancel{\vec{V}\cdot\vec{\nabla}\vec{V}} + \mu \nabla^2 \vec{V}$$

$(V_z \partial_z V_z = 0)$

$$\Rightarrow \quad -\frac{\Delta p}{l} = \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) \quad \text{in cylindrical coords.} \quad (77)$$

2nd order equation so we need 2 boundary conditions $\rightarrow$

First boundary condition is that $V_z = 0$ at wall of pipe. For pipe of radius a, $v = 0$ at $r = a$

Second boundary condition is that $v(r)$ profile is symmetric and smooth around $r = 0$

$\Rightarrow \frac{dv}{dr} = 0$ at $r = 0$.

Integrating (77) with the two bdry conds

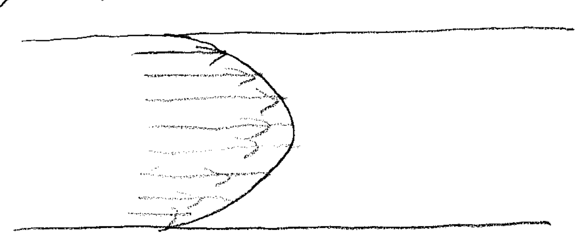
integrate once $\quad -\frac{\Delta P}{2l} r^2 = \mu r \frac{dv}{dr} + \cancel{C_1}^{\,0}$ since $\frac{dv}{dr} = 0$ at $r = 0$

integrate again : $\quad -\frac{\Delta P r^2}{4 l \mu} + C_2 = v$

$C_2 = \frac{a^2 \Delta P}{4 \mu l} \Rightarrow$

$$\boxed{v(r) = \frac{\Delta P}{4 \mu l}(a^2 - r^2)} \qquad (78)$$

the velocity profile is parabolic!



Mass flux through pipe is given by

$$Q = \int_0^a \rho\, v(r) \cdot 2\pi r\, dr \quad , \quad \text{using (78)}$$

$$\Rightarrow \boxed{Q = \frac{\pi \Delta p}{8 \nu l} a^4} \qquad (79)$$

where $\nu = \frac{\mu}{\rho}$

(79) is Poiseuille's formula and can be used to measure viscosity of liquids! :

① measure Δp ② measure Q ③ l, a are known from pipe shape. ①, ②, ③ imply viscosity can be measured.

The parabolic shape just described is valid for <u>laminar</u> flows which occur at relatively slow velocities but not valid for <u>turbulent</u> flows which occur at larger velocities. The sense of "large" and "small" needs to be made precise $\longrightarrow$