Gas Dynamics: the role of compressibility

- incompressible limit was considered
- compressibility required for sound waves and for shocks
- Basics of compressible flow can be considered for perfect gas.

Perfect gas $p = n k_B T = R \rho T$ (83)

$$R = \frac{k_B}{m}$$

$\mathcal{E}$, internal energy per mass

$$\mathcal{E} = C_v T = \frac{RT}{\gamma - 1} \quad (84)$$

$\gamma = \frac{C_p}{C_v}$, ratio of specific heats at constant pressure and volume

for monatomic gas: $\gamma = 5/3$, $C_v = 3/2$, $C_p = 5/2$

Entropy per unit mass:

$$T ds = d\mathcal{E} + p\, d\left(\frac{1}{\rho}\right) \quad (85)$$

from (83), (84) & (85) $\Rightarrow$

$$s = C_v \ln\left(\frac{p}{\rho^\gamma}\right) + s_0 \quad (86)$$

$$ds = C_v \frac{dT}{T} + \frac{p}{T} d\left(\frac{1}{\rho}\right)$$
$$= \frac{C_v}{(p/\rho)} d\left(\frac{p}{\rho}\right) + (\gamma - 1) C_v \rho\, d\left(\frac{1}{\rho}\right)$$
$$= C_v\, d\ln p/\rho - C_v\, d\ln(\rho^{\gamma-1})$$
$$= C_v\, d\ln(p/\rho^\gamma)$$

We also know that heat dQ

satisfies $dQ = dU + p\,dV$ (87)

and for adiabatic gas ($dU = m_p \mu \, d\mathcal{E}$) μ = mean mol weight

$dQ = 0 = dU + p\,dV$ or

$\frac{ds}{dt} = 0$ along a streamline from (85),
(ie. $\rightarrow$ $\frac{ds}{dt} = \partial_t s + \vec{v}\cdot\vec{\nabla} s = 0$)

so that (86) $\Rightarrow$

$$\frac{d(P/\rho^{\gamma})}{dt} = 0 \quad \text{for adiabatic gas} \tag{88}$$

The enthalpy per unit mass, is given by

$$w \equiv \mathcal{E} + \frac{P}{\rho} = \frac{\gamma}{\gamma - 1} RT \tag{89}$$

so that

$$T\,ds = d\mathcal{E} + p\,d\left(\frac{1}{\rho}\right) = dw - \frac{1}{\rho}\,dP$$

or $$dw = T\,ds + \frac{1}{\rho}\,dP \tag{90}$$

for adiabatic flow then

$\int \frac{dP}{\rho} = w + \text{constant}$, so Bernoulli's principle

$$\frac{1}{2}v^2 + \int \frac{dP}{\rho} + \phi = \text{constant}_2 = \frac{1}{2}v^2 + \frac{\gamma}{\gamma - 1} RT + \phi \tag{91}$$

along a streamline, for adiabatic flow. $\rightarrow$

Note that adiabatic flows are assumed to have negligible dissipation (negligible heat generation). Thus such flows are intrinsically non-viscous and to focus on effects of compressibility we go back to consideration of ideal fluid (ignoring viscosity and heat conduction) for the moment, under assumption that these transport processes operate on time scales long compared to the compressible effects considered.

Sound Waves

Lets derive sound speed:

Consider homogeneous, initially stationary flow with density ρ_0 and pressure P_0 in absence of external forces

Perturb pressure such that

$$P = P_0 + P_1(\vec{x},t) \qquad (92)$$

and density responds with perturbation such that

$$\rho = \rho_0 + \rho_1(\vec{x},t) \qquad (93)$$

velocity

$\vec{v} = \vec{v}_1(x,t)$ where subscript 1 indicates perturbed quantities

$\rightarrow$

We can write: $\left(\frac{\Delta P}{\Delta \rho} = \frac{P - P_0}{\rho - \rho_0} = \frac{P_1}{\rho_1} = \frac{dP}{d\rho} \right)$ (71)

$$P_1 = \frac{dP}{d\rho} \rho_1$$

$$\equiv C_s^2 \rho_1 \quad \text{where} \quad \frac{dP}{d\rho} \equiv C_s^2 \qquad (94)$$

assuming perturbations evolve on times short compared to viscous or conductive transport the flow is adiabatic, then (88) & (94) imply:

$$C_s = \sqrt{\frac{\gamma P_0}{\rho_0}} \qquad (95)$$

which is the adiabatic sound speed.

perturbed quantities satisfy continuity eqn:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot V_1 = 0, \qquad (96)$$

(where we assume $\rho_1 \ll \rho_0$, $P_1 \ll P_0$, $|V_1| \ll |V_0|$ and neglect quadratic terms in perturbed quantities.)

Momentum (Euler) eqn:

$$(\rho_0 + \underbrace{\cancel{\rho_1}}_{\text{small}}) \left[\frac{\partial \vec{V}_1}{\partial t} + \overbrace{\cancel{(V_1 \cdot \nabla) V_1}}^{\text{small}} \right] = -\nabla P_1 \qquad (96a)$$

after linearizing: $\Rightarrow \rho_0 \frac{\partial \vec{V}_1}{\partial t} = -\nabla P_1$

$$= -C_s^2 \nabla \rho_1, \qquad (97)$$

using (94).

combining (96) and (97)

$$\Rightarrow \quad \frac{\partial^2 \rho_1}{\partial t^2} - c_s^2 \nabla^2 \rho_1 = 0 \qquad (98)$$

which is a wave equation for acoustic waves propagating at speed c_s.

Note: isothermal case, $P_0 =$ constant ρ_0

$$\Rightarrow c_s = \sqrt{\frac{dP}{d\rho}} = \left(\frac{P_0}{\rho_0}\right)^{1/2}$$

1 atm $\simeq$ 1 bar $\simeq 10^5$ pascals

1 dyn $\simeq$ 0.1 pascals

$\Rightarrow$ 1 atm $\simeq 10^6$ dyn = erg/cm^3

in air, at $0°C$ and atmospheric pressure, $c_s = 2.8 \times 10^4 \frac{cm}{s}$ but this is lower that what is measured experimentally (Newton 1689)

Laplace (1816) was first to take adiabatic case, $\gamma = 1.4 \Rightarrow c_s = 3.32 \times 10^4$ cm/s which agreed with experiment $\left(c_s = \sqrt{\frac{\gamma P_0}{\rho_0}}\right)$

- In liquids do you expect sound speeds to be higher or lower than in gasses?

higher since they are harder to compress so for given dP, $d\rho$ is smaller and $c_s^2 = \frac{dP}{d\rho}$ is then larger.

For linear perturbation analysis, superposition holds and we can decompose perturbation into Fourier components

$$\rho_1 = \tilde{\rho}_1 \exp[i(\vec{k}\cdot\vec{x} - \omega t)] \quad (99)$$

plugging into (98) => dispersion relation

$$\omega^2 = c_s^2 k^2 \quad (100)$$

which applies only for simple, initially homogeneous medium (remember we ignored quadratic terms and $\vec{v}_0$)

Note that acoustic waves of all frequencies travel with same speed, and group & phase velocities are equal (=> non-dispersive waves)

For stratified atmosphere, group & phase velocities are not equal, and sound waves are dispersive.

Note that $\vec{k}$ gives direction of wave propagation.

(97) => $\vec{v}_1 \parallel \vec{k}$ so sound waves are longitudinal. Since $\nabla\cdot\vec{v} \propto k^2\rho \propto e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

$\nabla\cdot\vec{v}$ represents alternating compressions & rarefactions

Shocks

For large amplitude waves, quadratic perturbation terms cannot be neglected, in particular the term $\mathbf{v}_1 \cdot \nabla \mathbf{v}_1$ [in fluid momentum eqn. (96a)]

More specifically, perturbation approach does not really apply for large amplitude waves.

Consider the 1-D Euler equation; and consider x-direction as both wave propagation and direction of fluid velocities:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{101}$$

To assess the influence of the second term we drop the last term and solve the simpler equation for $v(x,t)$:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0 \tag{102}$$

Consider the curves $dx/dt = v$ in the x,t plane.

note that $\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\frac{dx}{dt}$ so (102) $\Rightarrow$

$\frac{dv}{dt} = 0$ along any curve for which $v = \frac{dx}{dt}$.

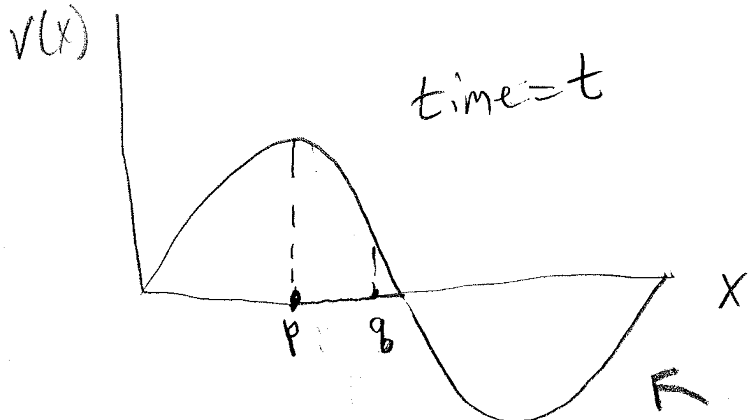
$\rightarrow$

we started with partial dif eqn but ended with ordinary diff eqn:

$$\frac{d\vec{v}}{dt} = 0 \qquad (103)$$

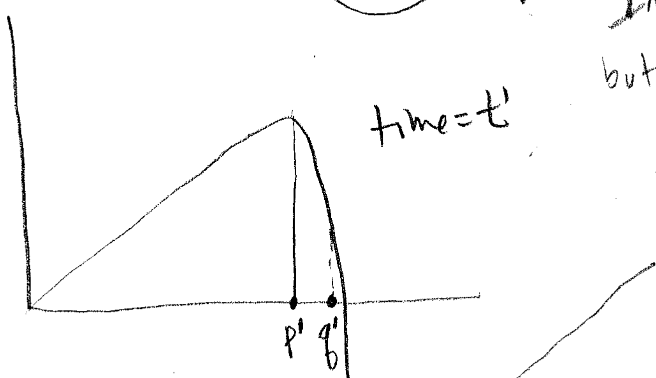
(Curves given by (103) are called "characteristic curves" of eqn (102), and $\vec{v}$ is the Riemann invariant)

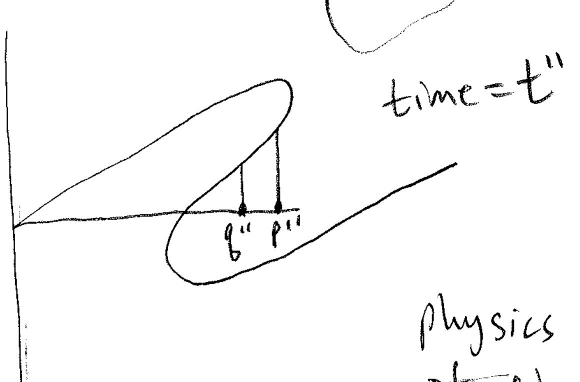
Consider an initial velocity profile:



under the action of (102), profile evolves as sequence on the left.

Initially p moves faster than q but comoving velocity is constant so $p-q > p'-q'$: points become closer together along their trajectories.





eventually p'' overtakes q'' suggesting velocity profile becomes multivalued.

This is unphysical and more physics is need, but represents the onset of steepening of waves to form shocks

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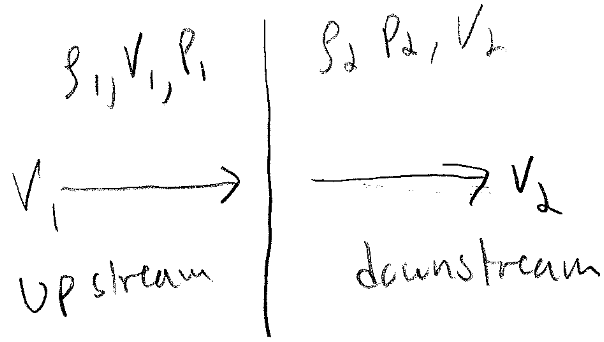
Shock structure

note how even initially smooth wave can steepen to form a shock. Shock is small region over which fluid variables change dramatically. To see what the change is like we can think of it as discontinuity and solve the "jump conditions"

consider shock propagating in undisturbed medium of density ρ_1, P_1 and lets move into frame at which shock is at rest.



Now we can appeal to <u>flux conservation</u> equations to understand how quantities change across the shock:

Continuity equation: flux of particles

$$\partial_t \rho + \nabla\cdot(\rho \vec{V}) = 0 \qquad (103)$$

with use of (103) the Euler momentum equation can be written as flux of momentum eqn:

$$\partial_t(\rho\vec{V}) + \partial_i(\rho V_i \vec{V}) = -\vec{\nabla} p \qquad (104)$$

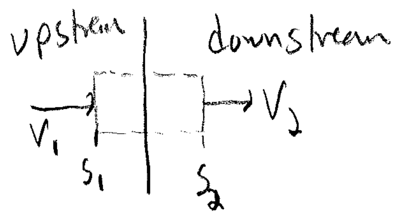
Energy equation can be manipulated by use of (103) & (104) to also take form of flux conservation eqn:

$$\partial_t\left(\rho\varepsilon + \tfrac{1}{2}\rho v^2\right) + \nabla\cdot\left[\rho\vec{v}\left(\tfrac{1}{2}v^2 + \overbrace{\frac{p\gamma}{(\gamma-1)\rho}}^{\text{enthalpy}}\right)\right] = 0 \qquad (105)$$

.. (103), (104), (105) we have ignored viscous terms and thermal conduction. In steady state,

$$(103) \Rightarrow \int \rho\vec{v}\cdot d\vec{S} = \text{constant}, \qquad (106)$$

Consider a small box which spans across the discontinuity:



then $(106) \Rightarrow \rho_1 V_1 = \rho_2 V_2$ for $S_1 = S_2$ (107)

similarly, for steady state (104) and (105) imply that for 1-D flow:

$$P_1 + \rho_1 V_1^2 = P_2 + \rho_2 V_2^2 \quad (108)$$

$$\frac{1}{2} V_1^2 + \frac{\gamma P_1}{(\gamma-1)\rho_1} = \frac{1}{2} V_2^2 + \frac{\gamma P_2}{(\gamma-1)\rho_2} \quad (109)$$

Now we have 3 equations (107, 108, 109) for 6 variables ($\rho_1 V_1 P_1$; $\rho_2, V_2 P_2$)

Eliminating P_2 & V_2, + algebra =>

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M^2}{2 + (\gamma-1) M^2}, \text{ where} \quad (110)$$

$$M \equiv \frac{V_1}{\sqrt{\gamma P_1/\rho_1}} = \frac{V_1}{C_{s_1}} \text{ is the } \underline{\text{Mach Number}}$$

and measures the speed of the upstream flow, or the speed at which shock is propagating into upstream flow as measured in the frame where upstream is at rest. Note we expect $M > 1$ for shock: if $M < 1$, then shock could produce acoustic waves moving with sound speed, and there would be no pile up of wave fronts, so any discontinuity would not survive.

we can also write (110)

as $\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{(\gamma-1) + 2/M^2}$ thus, for $M > 1$ (110)

$\rho_2/\rho_1 > 1$ and the ratio increases with M.

Thus a faster shock $\leftrightarrow$ more compression

As $M \to 1$ $\rho_2/\rho_1 \to 1$: no shock.

The above calculation was done in the shock frame. For non-relativistic flows the compression ratio (111) is unchanged in the lab frame.
In the lab frame, the shock advances into the undisturbed medium, compressing the flow behind it. Note for a monatomic gas $\gamma = 5/3$ and $\frac{\rho_2}{\rho_1} = 4$ as $M \to \infty$. This is maximum compression for a non-relativistic non-radiating gas.

We have neglected viscosity, assuming shock is infinitely thin, but indeed it is the viscosity and transport that determine the shock thickness.
Also radiative shocks can have larger compression ratios. Why?