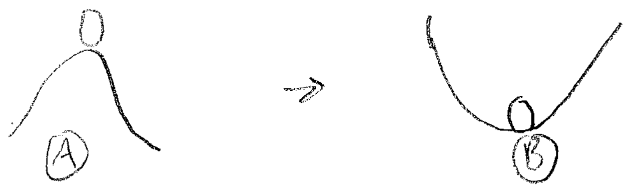
Linear Theory of instabilities

- Equilibrium vs. stable equilibrium: Consider simple system:



ball rolling down hill.

Though both positions are equilibria, only second is stable. Thus (A) equilibrium is unstable to formation of (B)

- ball may incur oscillations about the stable equilibrium (B) position (corresponds to waves in a fluid system)

- To find equilibria of fluid set ∂_t of all quantities to zero, and solve. But to find stable equilibria and instabilities one must perturb around the equilibrium and see how the perturbations evolve

- when we looked at sound waves we ignored non-linear terms $v \cdot \nabla v$ and we found waves.

Arbitrary perturbations can be constructed from superposition of fourier modes for linear problems.

- One common example of instability is convection when you heat water in pot, conduction transports heat first, then changing over to convection.

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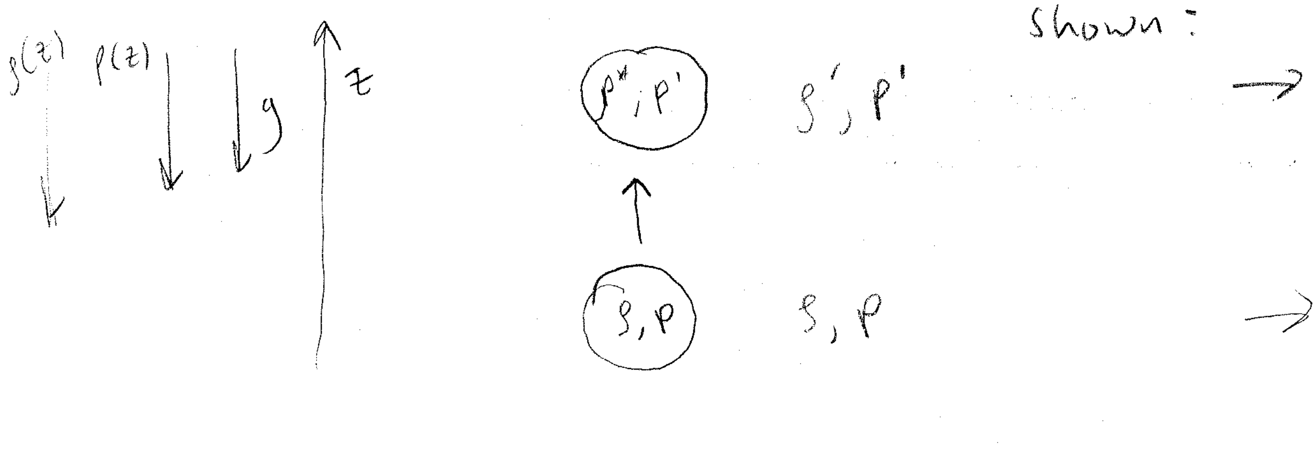
→ system becomes convectively unstable when the the temperature gradient from top to bottom exceeds a certain value. Transition to instability also known as bifurcation

→ as perturbations grow, non-linearities ensue in fluids → turbulence. Linear theory not valid for turbulence, but can be useful at least to determine which configurations can be expected to incur transition to turbulence.

→ by considering simple systems one can gain intuition about which systems tend to be unstable.

Convective Instability

Consider perfect gas in hydrostatic equilibrium in uniform gravity. If z axis is chosen such that gravity is in negative z direction then $\rho(z)$ & $p(z)$ decrease with z. Consider vertical displacement of blob as shown:



where initially p and ρ have same density as surroundings. External density and pressure at new position are ρ' & p'. Pressure balance inside and outside is maintained swiftly by acoustic waves, but heat imbalance/exchange takes longer when mediated by conduction. We can consider the blob to be displaced adiabatically, then let ρ^* be its new density. If $\rho^* < \rho'$, the blob will be buoyant and continue upward, implying instability. If $\rho^* > \rho'$ then the blob will tend to return, making the system stable. So we need to determine ρ^*/ρ:

For adiabatic flow, $\rho^* = \rho \left(\frac{p'}{p}\right)^{1/\gamma}$ (122)

If $\frac{dp}{dz}$ is pressure gradient we can substitute

$$p' = p + \frac{dp}{dz}\Delta z \qquad (123)$$

and using $\rho^* = \rho \frac{\left(p + \frac{dp}{dz}\Delta z\right)^{1/\gamma}}{p^{1/\gamma}}$ expanding to lowest order in Δz:

$$\Rightarrow \quad \rho^* = \rho + \frac{\rho}{\gamma p}\frac{dp}{dz}\Delta z \qquad (124)$$

but for ambient medium

$$\rho' = \rho + \frac{d\rho}{dz}\Delta z \qquad (125)$$

then using $\rho = P/RT$

$$\Rightarrow \rho' = \rho + \frac{\rho}{P}\frac{dP}{dz}\Delta z - \frac{\rho}{T}\frac{dT}{dz}\Delta z \qquad (126)$$

where $d\rho/dz$ & dT/dz are density and temp gradients.

(124) and (126) $\Rightarrow$

$$\rho^* - \rho' = \left[-\left(1-\frac{1}{\gamma}\right)\frac{\rho}{P}\frac{dP}{dz} + \frac{\rho}{T}\frac{dT}{dz}\right]\Delta z \qquad (127)$$

since $\frac{dT}{dz}$ and $\frac{dP}{dz}$ are both negative,

stable atmosphere $(\rho^* > \rho')$ requires

$$\left|\frac{dT}{dz}\right| < \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\left|\frac{dP}{dz}\right| \qquad (128)$$

This is Schwarzschild stability condition

important for stellar modeling.

$\rightarrow$

Since Force per unit volume acting inside displaced blob is $(\rho^* - \rho)(-g)$

equation of motion is approximately:

$$\rho^* \frac{d^2}{dt^2} \Delta z = -(\rho^* - \rho')g \qquad (129)$$

Substituting from (127)

$$\Rightarrow \quad \rho^* \frac{d^2}{dt^2} \Delta z = -g\left(\frac{\rho}{T}\frac{dT}{dz} - \left(1-\frac{1}{\gamma}\right)\frac{\rho}{P}\frac{dP}{dz}\right)\Delta z$$

to lowest order in Δz: we replace ρ^* by ρ (from 124)

and then obtain

$$\frac{d^2}{dt^2}\Delta z + N^2 \Delta z = 0$$

where $N = \sqrt{\frac{g}{T}\frac{dT}{dz} - \left(1-\frac{1}{\gamma}\right)\frac{g}{P}\frac{dP}{dz}}$

is the Brunt-Väisälä frequency. Thus for stable stratification blob will oscillate.

In reality such motions give rise to internal gravity waves by disturbing the surrounding medium.

We ignored internal gravity waves by ignoring the effect of blobs' motion on external medium. Full treatments account for these waves when full perturbative treatment is developed.