I. Fluids and Plasmas: The Big Picture

Both fluid dynamics and plasma dynamics are important for astrophysics. Let us discuss why, and the relation between the two.

- Examples of Fluids: a river, car exhaust, air these fluids are composed of neutral particles. The bu dynamics are modeled by equations that treat these systems as continuous media without having to worry about dynamics of individual particles. We will be more precise about this later.

- If we heat a fluid to high enough temperature, the neutral particles ionize. Even if the net charge of the system is zero the fluid particles themselves are charged. Depending on the amount of ionization we have a "partially ionized plasma" or "fully ionized plasma"

- Fluid Books often focus on bulk properties of flows without considering individual particles. This is appropriate when the inverse of the collision frequency = (w_c^{-1}) is short compared to the time scale of evolution t_{sys} - of the multi-particle system under study.

- Similarly for a highly ionized plasma: when (collision frequency)$^{-1}$ = $w_c^{-1}/t_{bulk} << 1$ the plasma can be treated as a "fluid". However, the charged particles of the plasma can carry currents and thus sustain magnetic fields. Magnetohydrodynamics represents the "simplest" generalization of fluid mechanics to include charged particles and electromagnetic fields.

- when the collison frequency is sufficiently small, the dynamics of individual particles become increasingly more important in modeling the system. Magnetohydrodynamics (MHD) is thus a special limit of plasma physics which is the subject of the dynamics of charged particle systems when the collision frequencies are not necessarily large. Kinetic Theory is used for the latter.

- Most astrophysical objects are made of plasma: gas with a significant ionized fraction.
- >90% of the luminous material in the universe can be classified as plasma
- sometimes the neutral fluid equations can be used even for plasmas in astrophysics. This depends on the problem being considered. For other problems MHD and/or kinetic theory is required.

In short:

plasma physics
↓ ← dynamics of individual particles unimportant
MHD
↓ ← E & M unimportant
Fluid Mechanics

[MHD & fluid mechanics are special cases of plasma physics]

Dynamical Theories

- dynamical theory implies time evolution theory
- Mechanics, E&M, QM examples
- common features: ① way of expressing state of system ② Eqns for time evolution

Mechanics: $\vec{x}_i(t)$, $\vec{p}_i(t)$: Newtons laws

E&M : $\vec{E}(\vec{x},t)$, $\vec{B}(x,t)$: Maxwells Eqn

QM : $\psi(x,t)$: Time dependent Schrodinger Eqn

- fluids & plasma need ① & ② as well
- These requirements can be expressed geometrically using concept of "phase space": the space such that each of the variables needed to define the state of the system corresponds to 1 dimension. For state functions that are continuous the phase space is infinite dimensional. The state of a system at any time corresponds to 1 point in the phase space, and equations describe a trajectory through this space.

Levels of Dynamical theory

Neutral Fluids

Level	Description of state	Dynamical Eqns
0: N Quantum particles	$\psi(\vec{x}_1 \dots \vec{x}_N)$	Schrodinger Eqns
1: N Classical particles	$(\vec{x}_1 \dots \vec{x}_N, \vec{p}_1 \dots \vec{p}_N)$	Newton's Laws
2: Distribution function	$f(\vec{x}, \vec{p}, t)$	Boltzmann Eqn
3: Continuum Model	$\rho(\vec{x},t), T(\vec{x},t), \vec{v}(\vec{x},t)$	Hydrodynamic Eqns

plasmas

Level	Description of State	
0: N Quantum particles	$\psi(\vec{x}_1, \dots \vec{x}_N)$	Schrodinger Eqn
1: N Classical Particles	$(\vec{x}_1, \dots \vec{x}_N, \vec{p}_1 \dots \vec{p}_N)$	Newtons laws
2: Distribution Function	$f(\vec{x}, \vec{p}, t)$	Vlasov equation
2.5 Two-fluid Model (ions + electrons)	$\begin{cases} \rho_e(x,t) \quad T_e(\vec{x},t), \vec{V}_e(\vec{x},t) \\ \rho_i(x,t) \quad T_i(\vec{x},t), \vec{V}_i(\vec{x},t), \end{cases} \vec{E}(x,t), \vec{B}(x,t)$	two-fluid plasma equations
3: One fluid model	$\rho(\vec{x},t) \quad T(\vec{x},t), \ V(\vec{x},t), \ \vec{B}(x,t)$	MHD

What determines when we can use a particular level?

$\left.\begin{array}{l}\text{More rigor later} \\ \text{but a physical description} \\ \text{follows}\end{array}\right\} \rightarrow$

Explanation of the Levels of Dynamical Theories

All microscopic systems obey Quantum mechanics. However we can treat a collection of N particles classically when the characteristic distance between particles is large enough so that there is little interference between their wavepackets. condition for classical treatment is that

$$n^{-1/3} \gg \lambda_d = \frac{h}{p} = \frac{h}{\sqrt{m k_b T}} \qquad (1)$$

where n is particle density and λ_d is the De Broglie wavelength. The quantity $n^{-1/3}$ is just the typical interparticle spacing and momentum $p = m V_{th} = m\sqrt{\frac{k_b T}{m}}$ for a thermal gas. Eqn (1) implies that each individual wave packet is isolated and expectation values can be treated classically. (Ehrenfest's theorem). This explains moving from level 0 to 1. on page 5.

If N is large then it is too impractical to solve equations for all individual particles. Then one moves to Level 2 and uses $f(\vec{x},\vec{u},t)$ distribution function, which is

→

the particle number density in space $(\vec{x},\vec{u})$ at time t.

A dynamical theory then requires an equation for f. (Vlasov or Boltzmann equation)

Level 3 treats the fluids as continua.

Since a gas is a flowing fluid and we know that a gas whose cender of mass is at rest can be described by 2 variables (e.g. ρ, T) a moving gas requires 3 $(\rho(\vec{x},t), T(x,t), \vec{v}(\vec{x},t))$.

For a plasma we must also have an equation for $\vec{B}(x,t)$ since magnetic fields can be embedded. Since astrophysical plasmas are usually good conductors, on large enough scales $\vec{E}=-\vec{v}\times\vec{B}$ as the plasma shorts out "microscopic" electric fields from currents. This is the MHD regime. On smaller scales, there are charge separations, and $\vec{E}_{\text{microscopic}}$ must be considered. This two-fluid regime is between MHD & Vlasov theory nd is thus Level 2.5.

Comment on turbulence:

when fluid or plasma systems are subject to violent disturbances they can become turbulent : ie. incur motions which appear to be chaotic and seem unpredictable. we will see how even though system in principle has dynamical equations, in practice they cannot easily be solved for turbulent flows.

Liouville's Theorem

consider a dynamical system whose state is prescribed by position & momentum coords. $(q_s, p_s, s = 1, \ldots 6N)$ and satisfies Hamilton's eqn of motion

$$\dot{p}_{s,i} = -\frac{\partial H}{\partial q_s} \qquad H = T + V \qquad (2)$$

$$\dot{q}_{s,} = \frac{\partial H}{\partial p_s} \qquad (3)$$

(a classical system of N particles satisfies this system of eqns)

→

Define **ensemble**: set of many replicas of "identical" systems except being at different micro states at some given time. each member of the ensemble is represented by a point in the phase space.

e.g. snapshot of a collection of particles

$\vec{q}_s(t), \vec{P}_s(t)$ → a point in 6N+1 dimensional phase space for N particles $(1 \leq s \leq N)$

e.g. fluid

$\underbrace{\vec{v}(\vec{x},t)}_{\text{velocity}}, \underbrace{T(\vec{x},t)}_{\text{temp}}, \underbrace{\rho(\vec{x},t)}_{\text{mass density}}$ → a point in infinite dimensional phase space

we can define **ensemble density** ρ_{ens} as the density of ensemble points at a given location in phase space: e.g. $\rho_{ens}(\vec{q}_s, \vec{P}_s, t)$ for our system of particles

→ consider $\overset{q_s, P_s, t}{\text{one}}$ member of the ensemble (6N+1 values needed to specify) $\vec{q}_s, \vec{P}_s$ and its trajectory $\vec{q}_s(t), \vec{P}_s(t)$.

If we measure density as a function of time varying on this trajectory Liouville's theorem is

$$\frac{D\rho_{ens}}{Dt} = 0 \qquad (4)$$

where $\frac{D}{Dt}$ is time derivative along the trajectory. To prove →

proof of Liouvilles thm

If $(q_s p_s)$ and $(q_s+\delta q_s,\ p_s+\delta p_s)$ denote the system at times t and $t+\delta t$ then

$$\frac{D\rho_{ens}}{Dt} = \lim_{\delta t \to 0} \frac{\rho_{ens}(q_s+\delta q_s,\ p_s+\delta p_s,\ t+\delta t) - \rho_{ens}(q_s p_s, t)}{\delta t} \qquad (5)$$

expansion in Taylor series:

sums over particles

$$\rho_{ens}(q_s+\delta q_s,\ p_s+\delta p_s,\ t+\delta t) = \rho_{ens}(q_s, p_s, t) + \sum_{s=1}^{3N} \delta q_s \frac{\partial \rho_{ens}}{\partial q_s} + \sum_{s=1}^{3N} \delta p_s \frac{\partial \rho_{ens}}{\partial p_s} + \delta t \frac{\partial \rho_{en}}{\partial t}$$

plugging into (5) gives:

$$\boxed{\frac{D\rho_{ens}}{Dt} = \frac{\partial \rho_{ens}}{\partial t} + \sum_s \dot{q}_s \frac{\partial \rho_{ens}}{\partial q_s} + \sum_s \dot{p}_s \frac{\partial \rho_{ens}}{\partial p_s}} \qquad (6)$$

now let us derive another result that we will use in conjunction with (6) to show why right side vanishes.:

The <u>continuity equation</u> applies to any mass conserving system & states that

$$\underbrace{\frac{\partial}{\partial t}\int \rho\, dV}_{\text{time derivative of mass}} = -\underbrace{\int \rho \vec{v} \cdot d\vec{S}}_{\text{outward mass flux}} \qquad (7)$$

→

∴ using Gauss' theorem

$$-\int \rho\,\vec{v}\cdot d\vec{s} = -\int \nabla\cdot(\rho\vec{v})\,dV$$ so

⇒ (7) →

$$\int \left[\partial_t \rho + \nabla\cdot(\rho\vec{v})\right] dV = 0$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

$$\frac{\partial\rho}{\partial t} + \rho\,\underset{0}{\cancel{\nabla\cdot\vec{v}}} + \vec{v}\cdot\nabla\rho = 0$$

$$= \frac{\partial\rho}{\partial t} + \vec{v}\cdot\nabla\rho = 0$$

$$\frac{d\rho}{dt} = 0$$

since it must be true for any volume :

$$\partial_t \rho + \nabla\cdot(\rho\vec{v}) = 0 \qquad (8)$$

this applies for a density in regular 3-space OR for an ensemble density in a volume of phase space. Thus

✓ $$\boxed{\partial_t \rho_{ens} + \underbrace{\sum_s \frac{\partial}{\partial q_s}(\rho_{ens}\dot{q}_s) + \sum_s \frac{\partial}{\partial p_s}(\rho_{ens}\dot{p}_s)}_{\text{generalized divergence using all coordinates in phase space.}} = 0} \qquad (9)$$

plugging into (6) gives

$$\Rightarrow \frac{\partial\rho_{ens}}{\partial t} + \sum_s \dot{q}_s \frac{\partial\rho_{ens}}{\partial q_s} + \sum_s \dot{p}_s \frac{\partial\rho_{ens}}{\partial p_s} + \rho_{ens}\sum_s\left(\cancel{\frac{\partial\dot{q}_s}{\partial q_s}} + \cancel{\frac{\partial\dot{p}_s}{\partial p_s}}\right) = 0$$

0 by Eqn! (2) & (3)

$$\rightarrow \boxed{\frac{\partial\rho_{ens}}{\partial t} + \sum_s \dot{q}_s \frac{\partial\rho_{ens}}{\partial q_s} + \sum_s \dot{p}_s \frac{\partial\rho_{ens}}{\partial p_s} = 0} \qquad (10)$$

as desired. QED

Collisioness Boltzmann Equation

Consider N classical particles, all of same type.

- $6N$ position & velocity coords. Call this space "Γ-space", and a point in Γ represents a state of the system
- Define "μ-space" as the 6 dimensional position & velocity space. Each particle is represented by a point in μ space at some time. The system state can be determined by N points in μ space. Note correspondence:

  [1 point in Γ-space ↔ N points in μ space]
  Both describe state of system

- A trajectory in Γ-space gets mapped to N trajectories in μ-space

Define distribution function in μ space

$$f(\vec{x},\vec{u},t) = \lim_{\delta V \to 0^+} \frac{\delta N}{\delta V}, \quad V \text{ is } \overset{\mu}{\text{~~[illegible]~~}} \text{ space volume}$$

$\delta V \to 0^+$ means take δV to small volume compared to system size, but still containing many particles

$f(\vec{x},\vec{u},t)$ is density of points in μ-space

→

We can derive a similar equation to Liouville's thm for $f(\vec{x}, \vec{u}, t)$ if point trajectories in μ space satisfy

$$\dot{\vec{u}} = -\vec{\nabla} H \; ; \; \vec{\nabla} = \hat{e}_x \partial_x + \hat{e}_y \partial_y + \hat{e}_z \partial_z \qquad (11$$

$$\dot{\vec{x}} = \vec{\nabla}_u H \; ; \; \vec{\nabla}_u = \hat{e}_x \frac{\partial}{\partial u_x} + \hat{e}_y \frac{\partial}{\partial u_y} + \hat{e}_z \frac{\partial}{\partial u_z} \qquad (12]$$

(since we needed such relations in the proof)

→ in Γ space, Hamiltonian H is fn of 6N+1 variables (6N coords + time)

→ in μ-space H is function of 7 variables
6 coords + time

→ when the N particles are non-interacting

$$H(\vec{u}, \vec{x}, t) = \tfrac{1}{2} u^2 + \phi(\vec{x})$$ for particle with coords $(\vec{u}, \vec{x})$

if particles interact, there are problems: suppose particle has coords $(\vec{u}, \vec{x})$ and interacts with nearby particle at $(\vec{u}', \vec{x}')$. This interaction can be described by potential $\phi(\vec{x}, \vec{x}')$ (and this incorporating the 6N+1 dimensions of Γ-space is possible by considering a different x' for each particle interacting with the original)

But $\phi(\vec{x}, \vec{x}')$ cannot be written as only a function of $\vec{x}$, so <u>cannot</u> be incorporated into an H that satisfies (11) & (12)

→

$\Rightarrow$ Hamiltonian dynamics of N particles is always possible in Γ-space (phase space) but only possible in μ space when particles are not interacting, i.e. <u>collisionless</u>

For a collisionless system then

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \dot{\vec{x}} \cdot \nabla f + \dot{\vec{u}} \cdot \nabla_u f = 0$$

we can write this as

$$\frac{\partial f}{\partial t} + \dot{x}_i \partial_i f + \dot{u}_i \partial_{u_i} f = 0 \quad \text{(repeated indices are summed)} \qquad (13)$$

<u>collisionless Boltzmann equation</u>

when interactions are present, (13) must be modified

<u>Note:</u> To derive 13 most directly, start with conservation of mass:

$$\int \frac{\partial f(\vec{x}', \vec{u}', t)}{\partial t} d^3x' \, d^3\vec{u}' + \int f(\vec{x}', \vec{u}', t) \, \vec{u}' \cdot d\vec{S}' \, d^3\vec{u}' + \int f(\vec{x}', \vec{u}', t) \, \vec{g} \cdot d\vec{S}_u \, d^3\vec{x}' = 0 \qquad (*)$$

(where $\vec{u}' = \frac{d\vec{x}'}{dt}$; $\vec{q}' = \frac{d\vec{u}'}{dt}$; $\vec{S} = (S_i, S_j, S_k)$

depends on forces

$S_i = n_i \, dx_j \, dx_k$

$\vec{S}_u = (S_{u,i}, S_{u,j}, S_{u,k})$

$S_{u,i} = n_i \, du_j \, du_k$)

$$\Rightarrow (*) \Rightarrow \int \left[\frac{\partial f}{\partial t} + \underbrace{\vec{\nabla}\cdot(f\vec{u}')} + \underbrace{\vec{\nabla}_u\cdot(f\vec{q}')} \right] d^3x' \, d^3u'$$

$$\Rightarrow \frac{\partial f}{\partial t} + \vec{u}\cdot\nabla f + f\cancel{\vec{\nabla}\cdot\vec{u}} + \vec{q}\cdot\vec{\nabla}_u f + \cancel{\underbrace{f\vec{\nabla}_u\cdot\vec{q}}} = 0$$

0 since $\vec{x}, \vec{u}$ are independent variables

$= f \cancel{\frac{\partial}{\partial u}} \frac{d\vec{u}}{dt}$

$= 0$ if

if forces per unit mass are independent of particle velocity

$$\frac{F}{m} = \frac{d\vec{u}}{dt} = \vec{a}$$

$$\Rightarrow \frac{\partial f}{\partial t} + \vec{u}\cdot\vec{\nabla} f + \vec{q}\cdot\vec{\nabla}_u f = 0$$

$$\frac{Df}{Dt} = 0$$

$$\frac{Df}{Dt} \, d^3x \, d^3u = 0$$

Note: if $\vec{F}$ does depend on $\vec{u}$ but is $\perp$ to $\vec{u}$ such as $\vec{u}\times\vec{B}$: then $\vec{\nabla}_u\cdot\vec{F} = 0$: to see

$$\vec{\nabla}_u\cdot\vec{F} = \frac{\partial}{\partial u_i}(\varepsilon_{ijk} u_j B_k)$$

$$= \varepsilon_{ijk} u_j \underbrace{\cancel{\frac{\partial B_k}{\partial u_i}}}_{0} + B_k \underbrace{\cancel{\varepsilon_{ijk} \frac{\partial u_j}{\partial u_i}}}_{0}$$

$\frac{\partial u_j}{\partial u_i} = \delta_{ij}$

Important Invariants

In Γ space, suppose that ensemble points at time t fill phase space $d^n q_s d^n p_s$ and at time t' fill phase space $d^n q_s' d^n p_s'$

$q_s = x_s$

the conservation of ensemble points in phase space implies

$$\rho_{ens} d^n q_s' d^n p_s' = \rho_{ens}' d^n q_s d^n p_s \qquad (13b)$$

and Liouvilles theorem => $\boxed{\rho_{ens}(q_s, p_s, t) = \rho_{ens}'(q_s', p_s', t')}$ (13c)

$$\Rightarrow \quad d^n q_s' d^n p_s' = d^n q_s d^n p_s$$

(assuming elastic collisions)

For μ space, Liouvilles theorem holds only for collisionless systems. In this case

$$f\, d^3x\, d^3p = f' d^3x' d^3p'$$

(number conservation of particles)

Liouvilles thm => $f = f'$

$$\Rightarrow \quad d^3x\, d^3p = d^3x' d^3p' \qquad (13d)$$

(volumes in which fixed # particles are contained)

We will use these later

Collisional Boltzmann Equation

Need to modify purely collisionless Boltzmann equation to include interactions between particles

We consider the case of a dilute gas

$n a^3 << 1$ (small particle radius = a compared to interparticle spacing).

and no long-range interactions between particles.

Now the collisionless Boltzmann equation says that $f(\vec{x}, \overset{= m\vec{u}}{\boxed{\vec{p}}}, t)$ does not change along the trajectory of a particle. Collisions can change this by bumping particles to different velocities, thus increasing or decreasing the number of particles in a given element of μ space

$$\frac{Df}{Dt} d^3x\, d^3u = C_{in} - C_{out}$$

$C_{in}, C_{out} \equiv$ rates at which particles enter or leave $d^3x\, d^3u$ from collisions

Consider elastic collisions:

(a) $\vec{u} + \vec{u}_1 = \vec{u}' + \vec{u}_1'$ (momentum cons.)

($\vec{u}, \vec{u}_1$ = particle velocities before collisions
$\vec{u}', \vec{u}_1'$ = velocities after collision)

(b) $\frac{1}{2}|\vec{u}|^2 + \frac{1}{2}|\vec{u}_1|^2 = \frac{1}{2}|\vec{u}'|^2 + \frac{1}{2}|\vec{u}_1'|^2$ (energy cons)

(a) & (b)
$\Rightarrow$
$2\vec{u}\cdot\vec{u}_1 = 2\vec{u}'\cdot\vec{u}_1'$
$\Rightarrow (\vec{u}-\vec{u}_1)^2 = (\vec{u}'-\vec{u}_1')^2$

these equations provide $\underline{4}$ equations for $\underline{6}$ unknowns $(\overset{\swarrow 3}{\vec{u}'}, \overset{\swarrow 3}{\vec{u}_1'})$; final velocities, given initial velocities

The remaining constraints come from:

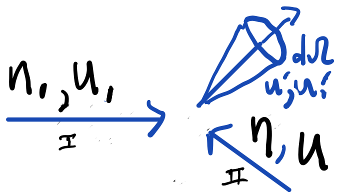
1) $\underline{\text{coplanarity of } \vec{u}', \vec{u}_1', \vec{u}, \vec{u}_1}$ for radial force of interactions (e.g. coulomb collisions) - eliminates

2) impact parameter, which gives the ∡ of deflection. This comes from microphysics of interaction

Statistically, # 2) is modeled by differential cross section. We assume its given and show how dynamics of system can then be studied:

- consider beam of particles with number density n_1 and velocity $\vec{u}_1$, colliding with beam having number density n and velocity $\vec{u}$. The latter beam sees particle flux $I = |\vec{u}-\vec{u}_1| n_1$ from first beam
(number per area per time) →

Define $\delta_t n_c \equiv \frac{\#\text{ collisions}}{\text{time}\cdot\text{volume}}$ that deflect particles from second beam into solid angle $d\Omega$, by interaction with first beam :



$$\delta_t n_c = \underbrace{(n)}_{\text{n of second beam}}\underbrace{(I_1)}_{\text{flux of } n_1 \text{, that n-beam is exposed to}}\underbrace{d\Omega}_{\text{solid } \measuredangle}\underbrace{\sigma(\vec{u},\vec{u}_1|\vec{u}',\vec{u}_1')}_{\text{differential scattering cross section } [\frac{d\sigma}{d\Omega}]} = n I_1 d\Omega \frac{d\sigma}{d\Omega} = n I_1 d\sigma$$

(individual interactions are reversible for elastic scattering so that

$$\sigma(\vec{u},\vec{u}_1|\vec{u}_1',\vec{u}') = \sigma(\vec{u}_1',\vec{u}'|\vec{u},\vec{u}_1)$$)

Now

since $n = f(\vec{x},\vec{u},t)\,d^3\vec{u}$ = number per volume

and $I = |\vec{u}-\vec{u}_1|n_1 = |\vec{u}_1-\vec{u}_1|\, f(\vec{x},\vec{u}_1,t)\,d^3u_1$

$$\delta_t n_c = \sigma(u,u_1|u',u_1')\,|\vec{u}-\vec{u}_1|\, f(x,u,t)\, f(x,u_1,t)\, d\Omega\, d^3u\, d^3u_1$$

Since $C_{out} = \frac{\#\text{ collisions}}{\text{time}}$ in 6-D volume $d^3x\, d^3u$, $\Rightarrow$

$$C_{out} = d^3x\, d^3u \int d^3u_1 \int d\Omega\, \sigma(u,u_1|u',u_1')\,|\vec{u}-\vec{u}_1|\, f(x,\vec{u},t)\, f(x,\vec{u}_1,t)$$

= rate at which particles leave $d^3x\, d^3u$ from collisions $\rightarrow$

To get C_{in} consider reverse collisions; that is replace $u' \leftrightarrow u$ and $u_1' \leftrightarrow u_1$ straight away we have:

$$C_{in} = d^3x\, d^3u' \int d^3u_1' \int d\Omega\, \sigma(u,u_1|u',u_1')|u'-u_1'|f(x,u',t)f(x,u_1',t)$$

But: ① conservation of momentum & energy for collisions $\Rightarrow$ $|u-u_1| = |u'-u_1'|$ (see page 17 above)

and ② Earlier we proved that phase space $\rightarrow$(eqn 13b,c) measures at any times are equal (from Liouville's Thm + conservation of particle number & elastic collision assumption) thus for 2-particle phase space

$$\cancel{d^3x}\, d^3u\, d^3u_1 = d^3u'\, d^3u_1'\, \cancel{d^3x}$$ ③ we also argued

$$\sigma(u,u_1|u',u_1') = \sigma(u',u_1'|u,u_1).$$ Thus ①,②,③

$$\Rightarrow C_{in} = d^3x\, d^3u \int d^3u_1 \int d\Omega\, \sigma(u',u_1'|u,u_1)|u-u_1| f(x,u',t)f(x,u_1',t)$$

comparing to C_{out} we then combine to get:

$$\frac{Df}{Dt} d^3x\, d^3u = C_{in} - C_{out} = d^3x\, d^3u \int d^3u_1 \int d\Omega\, \sigma(\Omega)(f' f_1' - f f_1)|\vec{u}-\vec{u}_1|$$

(where $f' \equiv f(u')$ and $f_1' \equiv f(u_1')$
$f_1 \equiv f(u)$; $f_1 = f(u_1)$) $\longrightarrow$

we thus have

$$\boxed{\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \vec{u}\cdot\nabla f + \frac{\vec{F}}{m}\cdot\nabla_{u} f = \int d^3u_1 \int d\Omega\, |\vec{u}-\vec{u}_1|\, \sigma(\Omega)\,(f'f_1' - ff_1)} \quad (14)$$

Collisional Boltzmann eqn

$\vec{F} = m\dot{\vec{u}}$ is any force field that particles experience e.g. gravity

to recap: right side measures effects of collisions on distribution function for a dilute gas. (dilute because we assumed only binary collisions)

Maxwellian Distribution

Uniform classical gas relaxes to maxwell dist. This can be derived from above collisional Boltz. eqn: Consider case when $\vec{F}$ term is negligible, and f is independent of time and space (ie. in equilibrium).

Boltz eqn ⇒

$$f f_1 = f'f_1'$$

or

$$\log f(u) + \log f(u_1) = \log f(u') + \log f_1(u_1') \quad (*)$$

→

Suppose $\chi(u)$ is a conserved quantity,

then $$\underbrace{\chi(u) + \chi(u_1)}_{\text{before}} = \underbrace{\chi(u') + \chi(u_1')}_{\text{after collision}}$$

Since this has same form of previous equation ⊛ we must be able to write $\log f(u)$ as a linear combination of $\chi(u)$

That is :

sum over all conserved quantities

$$\log f(u) = C_0 + \sum_s C_s \chi_s(\vec{u}) \qquad (C_0, C_s \text{ are constants})$$

∴ energy & the 3 momenta are the complete set of relevant quantities here:

$$\log_e f(\vec{u}) = C_0 + C_1 \vec{u}^2 + C_{2x} u_x + C_{2y} u_y + C_{2z} u_z$$

$$\Rightarrow \log_e f(\vec{u}) = -B(\vec{u} - \vec{u}_0)^2 + \log_e A$$

where $C_0, C_1, C_{2x}, C_{2y}, C_{2z}$, have been replaced by $B, A, u_{0x}, u_{0y}, u_{0z}$,

exponentiate

$$\Rightarrow f(u) = A e^{-B(\vec{u} - \vec{u}_0)^2}$$

$$n = \int_{-\infty}^{\infty} d^3u\, f(u) \Rightarrow A = \left(\frac{B}{\pi}\right)^{3/2} n$$

↑ number density

→

$\Rightarrow$

$$f(u) = \left(\frac{B}{\pi}\right)^{3/2} n\, e^{-B(\vec{u}-\vec{U}_0)^2}$$

Note that

$$\frac{\int u\, f(u)\, d^3u}{\int f(u)\, d^3u} = \langle u \rangle$$

$n \leftrightarrow$

$$\langle \vec{u} \rangle = \frac{1}{n}\int_{-\infty}^{\infty} f(u)\, \vec{u}\, d^3u = \left(\frac{B}{\pi}\right)^{3/2} \int_{-\infty}^{\infty} d^3\vec{u}'(\vec{u}'+\vec{U}_0)\, e^{-B\vec{u}'^2}$$

(where $\vec{u} \to \vec{u}' + \vec{U}_0$ change of variables was used)

$$= \vec{U}_0 \left(\frac{B}{\pi}\right)^{3/2} \int d^3u\, e^{-Bu^2} = \vec{U}_0$$

$$= \left(\frac{B}{\pi}\right)^{3/2} \int du_x du_y du_z\, e^{-Bu_x^2} e^{-Bu_y^2} e^{-Bu_z^2}$$

$\Rightarrow$ non-zero $\vec{U}_0$ implies a mean streaming motion

if we go to frame in which $\vec{U}_0 = 0$ and consider system of temperature T, then $B = \frac{m}{2k_BT}$

and $f(u) = n\left(\frac{m}{2\pi k_BT}\right)^{3/2} \exp\left[-\frac{m\vec{u}^2}{2k_BT}\right]$

$$I_n = \int_{-\infty}^{\infty} e^{-Bu^2} u^n\, du$$

$$I_{n+2} = -\frac{\partial I_n}{\partial B}$$

$$I_0 = \sqrt{\frac{\pi}{B}}$$

$$I_2 = \frac{1}{2}\sqrt{\frac{\pi}{B^3}}$$

Maxwell Boltzmann is a soln to steady-state Boltzmann equation

NOT SURPRISING!

22a

$$\langle u^2\rangle = \left(\frac{B}{\pi}\right)^{3/2} \frac{3}{2}\sqrt{\frac{\pi}{B^3}}\cdot\frac{\pi}{2}$$

$$= \frac{3}{2B} = \frac{3kT}{m}, \quad \underline{\text{for } B = \frac{m}{2kT}}$$

$$\frac{1}{2} m \langle u^2\rangle = \frac{3}{2} kT$$

$$\langle u^2\rangle = \frac{\int du_x du_y du_z (u_x^2+u_y^2+u_z^2) e^{-Bu_x^2} e^{-Bu_y^2} e^{-Bu_z^2}}{\int du_x du_y du_z \, e^{-Bu_x^2} e^{-Bu_y^2} e^{-Bu_z^2}}$$

$$= \frac{3\cdot\frac{1}{2}\left(\frac{\pi}{B^3}\right)^{1/2}\cdot\left(\frac{\pi}{B}\right)}{\left(\frac{\pi}{B}\right)^{3/2}} = \frac{3}{2B}$$

Conservation equations

$$\underline{\chi} + \underline{\chi}_1 = \underline{\chi}' + \underline{\chi}_1' \qquad (*)$$

for conserved quantity $\chi(\vec{x},\vec{u})$ before & after collisions

now let us derive the equation for the averaged $\underline{\chi}$

this is important for eventually deriving the hydrodynamic fluid eqns:

we multiply the collisional Boltzman equation by $\underline{\chi}$. The result for the RHS after integrating is

$$= \int d^3u \int d^3u_1 \int d\Omega\, \sigma(\Omega)\, |\vec{u}-\vec{u}_1|\, (f'f_1' - ff_1)\, \underline{\chi}(\vec{u},\vec{x})$$

$$= \frac{1}{2} \int d^3u \int d^3u_1 \int d\Omega\, \sigma(\Omega)\, |\vec{u}-u_1|\, (f'f_1' - ff_1)(\underline{\chi}(\vec{u},x) + \underline{\chi}(\vec{u}_1,x))$$

Since the RHS of collision Botz. Eqn is symmetric in $\vec{u}_1 \leftrightarrow \vec{u}$.

We can also go further:

$$= \frac{1}{4} \int d^3u \int d^3u_1 \int d\Omega\, \sigma(\Omega)\, |\vec{u}-\vec{u}_1|\, (f'f_1' - ff_1)(\underline{\chi}(\vec{u},x) + \underline{\chi}(\vec{u}_1,x) - \underline{\chi}(\vec{u}',x) - \underline{\chi}(u_1',x))$$

because the collision integral in Boltzmann eqn is antisymmetric in $u' \leftrightarrow u$, $u_1' \to u'$. But from $(*)$ this RHS now $= 0$!

→

the left side of (14) when multiplied by $\underline{\chi}$ & integrated is then = 0 ⇒ we have

$$\int d^3u\, \underline{\chi}\left(\frac{\partial f}{\partial t} + u_i\frac{\partial f}{\partial x_i} + \frac{F_i}{m}\frac{\partial f}{\partial u_i}\right) = 0$$

manipulation using chain rule gives, using $\partial_t \underline{\chi} = 0$:

$$\partial_t \int d^3u\, \underline{\chi} f + \frac{\partial}{\partial x_i}\int d^3u\, \underline{\chi} u_i f - \int d^3u\, u_i f \frac{\partial \underline{\chi}}{\partial x_i}$$

$$+ \frac{1}{m}\int d^3u \underbrace{\frac{\partial}{\partial u_i}\cancel{(\underline{\chi}F_i f)}}_{\substack{0 \\ \text{(surface term} \\ \text{by Gauss thm)}}} - \frac{1}{m}\int d^3u \frac{\partial \underline{\chi}}{\partial u_i}F_i f - \frac{1}{m}\int d^3u\,\underline{\chi}\frac{\partial F_i}{\partial u_i} f = 0 \qquad (14a)$$

Using the notation $\langle\underline{\chi}\rangle = \frac{1}{n}\int f\underline{\chi}\, d^3u = \frac{\int f\chi\, d^3u}{\int f\, d^3u}$ with $n = \int f\, d^3u$, we can write (14a) as

$$\partial_t(n\langle\underline{\chi}\rangle) + \frac{\partial}{\partial x_i}(n\langle u_i\underline{\chi}\rangle) - n\left\langle u_i \frac{\partial \underline{\chi}}{\partial x_i}\right\rangle - \frac{n}{m}\left\langle F_i\frac{\partial\underline{\chi}}{\partial u_i}\right\rangle$$

$$- \frac{n}{m}\left\langle\frac{\partial F_i}{\partial u_i}\underline{\chi}\right\rangle = 0$$

this tells us how the volume density of any quantity $n\langle\underline{\chi}\rangle$ evolves with time

→

fluid equations first for maxwellian particle distributions

$\underline{\chi}$ is microscopic quantity and

$n\langle\underline{\chi}\rangle$ is macroscopic. thus

previous equation (where $\langle\underline{\chi}\rangle = \frac{1}{n}\int \underline{\chi} f d^3u$)

$$\partial_t \left(n\langle\underline{\chi}\rangle\right) + \frac{\partial}{\partial x_i}\left(n\langle u_i \underline{\chi}\rangle\right) - n\left\langle u_i \frac{\partial \underline{\chi}}{\partial x_i}\right\rangle - \frac{n}{m}\left\langle F_i \frac{\partial \underline{\chi}}{\partial u_i}\right\rangle - \frac{n}{m}\left\langle \frac{\partial F_i}{\partial u_i}\underline{\chi}\right\rangle = 0 \quad (14b)$$

provides a link between micro & macro quantities. Fluid equations are macro equations so (14b) is fundamental.

Recall that (14b) applies for any conserved quantity. Classically, mass is conserved, so lets first consider

$\underline{\chi} = m$ in (14b)

$\longrightarrow$

for $\vec{F}$ indendent of u, and all particles of same mass m:

$$\frac{\partial}{\partial t}(mn) + \frac{\partial}{\partial x_i}(nm\langle u_i\rangle) = 0 \qquad (15)$$

if we write $\rho = mn$ and $V_i \equiv \langle u_i\rangle$

then we have <u>continuity equation</u>

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho V_i) = 0 \qquad (16)$$

(or $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0$). This is one of the fundamental fluid equations.

Second Now let $\underline{\chi} = mu_i$ in (14b)

Since u_i, x_i are independent variables and $\frac{\partial \vec{F}}{\partial u_i} = 0$ by assumption.

$$\Rightarrow \frac{\partial}{\partial t}(nm\langle u_j\rangle) + \frac{\partial}{\partial x_i}(nm\langle u_iu_j\rangle) - nF_j = 0 \qquad (17)$$

now define $P_{ij} = nm\langle \overbrace{(u_i - V_i)}^{U_i}\overbrace{(u_j - V_j)}^{U_j}\rangle$ $\boxed{\text{with } V_i \equiv \langle u_i\rangle}$ (18)

$\langle V_j\rangle = V_j$

$$= nm\langle u_iu_j\rangle + nm V_iV_j - nm\underbrace{\langle u_i\rangle}_{V_i}V_j - nm\underbrace{\langle u_j\rangle}_{V_j}V_i$$

$$P_{ij} = nm\langle u_iu_j\rangle - nm V_iV_j$$

Thus (18) in (17) ⇒

$$\frac{\partial}{\partial t}(\underbrace{\rho V_j}_{}) + \frac{\partial}{\partial x_i}P_{ij} + \frac{\partial}{\partial x_i}(\rho V_iV_j) - \frac{\rho}{m}F_j = 0 \qquad (19)$$

with 20 → $\rho\partial_t v_j - \cancel{v_j\partial_i(\rho v_i)} - \partial_i P_{ij} + \cancel{v_j\partial_i(\rho v_i)} + \rho v_i\partial_i v_j - \frac{\rho}{m}F_j = 0$

Eqn (19) is the momentum equation with 1 pressure tensor.

third let $\chi = \frac{1}{2}m|\vec{u}-\vec{v}|^2$ in (14b)

this corresponds to conserved energy in collisions for monatomic gas, and constant mean velocity $\vec{v}$.

the result is then:

exercise at home ↓

$$\partial_t(\rho\varepsilon) + \frac{\partial}{\partial x_i}(\rho\varepsilon v_i) + \frac{\partial Q_i}{\partial x_i} + P_{ij}\Lambda_{ij} = 0 \qquad (20)$$

(energy eqn)

$\varepsilon \equiv \frac{1}{2}\langle|\vec{u}-\vec{v}|^2\rangle$ = internal energy per mass

$\vec{Q} \equiv \frac{\rho}{2}\langle(\vec{u}-\vec{v})|\vec{u}-\vec{v}|^2\rangle$ = energy flux (units: $\frac{\text{energy}}{\text{Area}\cdot\text{time}}$)

$$\Lambda_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

Now simplify (19) & (20) using (16)

The results are:

(19) → $$\rho\left(\frac{\partial v_j}{\partial t} + v_i\frac{\partial v_j}{\partial x_i}\right) = -\frac{\partial P_{ji}}{\partial x_i} + \frac{\rho}{m}F_j \qquad (21)$$

(20) → $$\rho\left(\frac{\partial\varepsilon}{\partial t} + v_i\frac{\partial\varepsilon}{\partial x_i}\right) + \frac{\partial Q_i}{\partial x_i} + P_{ij}\Lambda_{ij} = 0 \qquad (22)$$

eqn (16), (21), (22) do represent mass, momentum, and energy conservation but these represent 5 eqns with 14 unknowns(!): $\vec{V}$ (3- components)

P_{ij} (6 - components, since symmetric)

ρ (1 - component)

Q_i (3 - components)

ε (1 - component)

Thus we need relations between these quantities to close system of equations

eqns (16), (21), (22) are called the "moment" equations since they arise from multiplying Boltzmann eqn by powers of 0, 1, 2 velocities and integrating over velocity.

Note distinction between $\underset{\substack{\uparrow \\ \text{velocity of individual particle}}}{u_i}$ & $\underset{\nwarrow \text{ mean velocity of overall flow}}{V_i}$

Alternatively:

$$\langle \vec{u} \rangle = \vec{V} \; , \quad \vec{u} = \underset{\uparrow \text{random component}}{u_{fluct}} + \overset{\swarrow \text{mean component}}{V_i} \longrightarrow$$

We argued before, that collisions set up a Maxwellian distribution when frequent enough. Now let us see what this implies for reducing the number of variables, and a "simpler" set of eqns:

Assume that distribution function is Maxwellian:

$$f^{(0)}(\vec{x},\vec{u},t) = n(\vec{x},t)\left[\frac{m}{2\pi k_b T(x,t)}\right]^{3/2} \exp\left[-\frac{m(\vec{u}-\vec{v}(x,t))^2}{2k_b T(\vec{x},t)}\right] \qquad (23)$$

where we write x, t dependencies explicitly.

Using (23) we have

$$P_{ij} = \rho\left(\frac{m}{2\pi k_b T}\right)^{3/2}\int d^3U\, U_i U_j\, \mathrm{Exp}\left[-\frac{mU^2}{2k_b T}\right]$$

$$\vec{U} = \vec{u} - \vec{v}$$

Integral vanishes when integrand is odd $\Rightarrow$

$$P_{ij} = p\,\delta_{ij} \Rightarrow 3P = P_{ij}\delta_{ij} \Rightarrow P = \tfrac{1}{3} P_{ij}\delta_{ij} = nkT \qquad (24)$$

which comes from integrating

$$\int_{-\infty}^{\infty} U^2 e^{-AU^2}\, dU = \frac{1}{2}\sqrt{\frac{\pi}{A^3}}$$

→

- We can also see that the flux $\vec{Q}$ satisfies

$\vec{Q}=0$, since it is odd integral. (25)

- From definition of $\mathcal{E} = \frac{1}{2}\langle |U|^2 \rangle$ we also have that

$$\mathcal{E} = \frac{3}{2}\frac{k_b T}{m} = \frac{3}{2}\, p/\rho \qquad (26)$$

thus; using 24, 25, 26 we have eliminated 3 variables of $\vec{Q}$, 5 variables of the original P_{ij} tensor, and $\mathcal{E}$ can be written is function of p, thus $14-9=5$ variables left and 5 equations!

- using $P_{ij} = p\delta_{ij}$ we also have

$$P_{ij}\Lambda_{ij} = \frac{1}{2}p\delta_{ij}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_j}\right) = p\vec{\nabla}\cdot\vec{V} \qquad (27)$$

from defn of Λ_{ij} below eqn (21).

Using (24), in (21) gives

momentum: $$\boxed{\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V} = -\frac{1}{\rho}\vec{\nabla}p + \frac{\vec{E}}{m}} \qquad (28)$$

Using (25), (26) & (27) in (22) gives

rgy: $$\boxed{\rho\left(\frac{\partial \mathcal{E}}{\partial t} + \vec{V}\cdot\vec{\nabla}\mathcal{E}\right) + p\vec{\nabla}\cdot\vec{V} = 0} \qquad (29)$$

continuity eqn was

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0} \qquad (30)$$

→

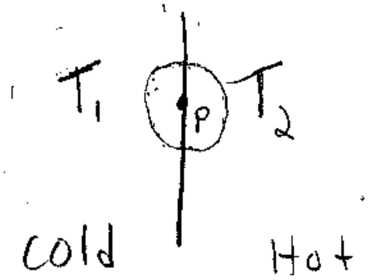
Transport Processes:

Ⓐ → In previous derivation $\vec{Q} = 0$ so no heat flow.

→ we also had P_{ij} being diagonal;

Ⓑ this means that momentum cannot be transported from one layer of fluid to another. This implies no shear forces

Both Ⓐ & Ⓑ resulted from assumption of Maxwellian Distribution: can immediately see that some departure from Maxwellian is required for transport:



Heat flows from Hot to Cold; in neighborhood of P distribution is not isotropic and not Maxwellian!

we need to consider perturbations around Maxwellian distribution

$$f(x, u, t) = f^{(0)}(\vec{x}, u, t) + g(\vec{x}, u, t) \qquad (31)$$

↑ maxwellian ↑ small departure

$$g = f - f^{(0)}$$

putting (31) in Boltzmann equation, (page 6 above) ㉜

∴ collision integral is

$$\int d^3u_1 \int d\Omega\, |\vec{u}-\vec{u}_1|\, \sigma(\Omega)\, (f'f_1' - f f_1)$$ (function of $\vec{u}, \vec{x}$)

$$= \int d^3u_1 \int d\Omega\, |\vec{u}-\vec{u}_1|\, \sigma(\Omega)\, (f^{(0)\prime} g_1' + f_1^{(0)\prime} g' - f^{(0)} g_1 - f_1^{(0)} g)$$

to first order.

A typical term has magnitude

$$-\int d^3u_1 \int d\Omega\, |\vec{u}-\vec{u}_1|\, \sigma(\Omega)\, (f_1^{(0)} g) \simeq -\vec{u}_{rel}\, n\, \sigma\, g(x,u,t)$$

$|\vec{U}_{rel} n \sigma|$ is a collision frequency with units $\frac{1}{\tau}$ ⇒ collision integral is roughly $-\frac{g}{\tau}$ ⇒ Boltzmann eqn:

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla} + \frac{\vec{F}}{m}\cdot\nabla_u\right) f = -\frac{(f - f^{(0)})}{\tau} = -\frac{g}{\tau} \qquad (32)$$

↓ this term is responsible for g when there are strong spatial gradients.

To order of mag

$$\frac{|u| f^{(0)}}{L} \sim \frac{|g|}{\tau} \Rightarrow \frac{|g|}{f_0} \ll 1$$

$$\Rightarrow \frac{|u|\tau}{L} \ll 1$$

where L is gradient scale over which properties change.

mean free path

$$\Rightarrow \frac{|g|}{f^{(0)}} \simeq \frac{|u|\tau}{L} \sim \frac{\lambda_{mfp}}{L} \equiv \alpha << 1$$

$$\Rightarrow f = f^{(0)} + \alpha f_1^{(0)} + \alpha^2 f_2^{(0)} ;$$

Chapman-Enskog expansion.

To compute "corrections" use lowest order in (32)

$\Rightarrow$

$$g = -\tau\left(\partial_t + u_i\partial_i + \frac{F_i}{m}\partial u_i\right)f^{(0)} \qquad (33)$$

From (23), $f^{(0)} = \frac{m^{3/2} n(x,t)}{(2\pi k_B T(x,t))^{3/2}} \text{Exp}\left[-\frac{m(\vec{u}-\vec{V}(\vec{x},t))^2}{2k_B T(\vec{x},t)}\right]$, $f = f(n,T,\vec{V})$

So

chain rule for terms in (33)

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial n}{\partial t}\frac{\partial f^{(0)}}{\partial n} + \frac{\partial T}{\partial t}\frac{\partial f^{(0)}}{\partial T} + \frac{\partial V_i}{\partial t}\frac{\partial f^{(0)}}{\partial V_i}$$

use ㉚

use ㉘

use ㉙

$$\frac{\partial f^{(0)}}{\partial x} = \frac{\partial n}{\partial x}\frac{\partial f^{(0)}}{\partial n} + \frac{\partial T}{\partial x}\frac{\partial f^{(0)}}{\partial T} + \frac{\partial V_i}{\partial x}\frac{\partial f^{(0)}}{\partial V_i}$$

$$\boxed{P = nkT}$$

Using (23) for $f^{(0)}$ in (33) and using the $f^{(0)}$ "moment" equations for continuity (30), momentum (28) and energy density (29); we get (set $F_i = 0$ to simplify)

$$g = -\tau\left(\frac{1}{T}\frac{\partial T}{\partial x_i}V_i\left(\frac{m}{2k_BT}V^2 - \frac{5}{2}\right) + \frac{m}{K_BT}\Lambda_{ij}\left(V_iV_j - \frac{1}{3}\delta_{ij}V^2\right)\right]f^{(0)} \qquad (34)$$

with $\Lambda_{ij} \equiv \partial_i V_j + \partial_j V_i$; $\vec{V} \equiv \vec{u} - \vec{v}$

$\rightarrow$

→ That g depends linearly on velocity and temperature gradients is expected, based on our simple argument before, for deviations from Maxwellian dist. → gradients imply deviation from Maxwellian.

→ Linear dependence on τ implies that the longer the time between collisions, the more the deviation from Maxwellian can be sustained, and thus a larger correction g. (collisions tend to make f closer to $f^{(0)}$)

→ Now we can calculate P_{ij}, $\vec{Q}$, and $\mathcal{E}$ for the non-Maxwellian distribution $f = f^{(0)} + g$ with $\langle A \rangle \equiv \frac{1}{n}\int A f\, d^3u$ as defn for averaging of quantity A,

from before:

only g contributes from $f = f^{(0)} + g$

$$\vec{Q} = \frac{nm}{2}\langle \vec{U} U^2 \rangle = \frac{\rho}{2}\int d^3U\, \vec{U} U^2 g$$

Only even powers contribute to integrand so only 1st term on right of (34) contributes:

$$\vec{Q} = -\underline{K}\nabla T, \quad \left(\text{where } \mathbb{K} = \frac{m\tau}{6T}\int d^3U\, U^4\left(\frac{m}{2k_BT}V^2 - \frac{5}{2}\right)f^{(0)} = \frac{5}{2}n\tau\frac{k_B^2T}{m}\right) \qquad (35)$$

[illegible] erg [illegible] erg [illegible]

That $\vec{Q} = -\mathbb{K}\nabla T$ is a familiar form of heat transport equation (which we have derived from a "bottom up" approach).

Also:

$P_{ij} = nm\langle V_i V_j \rangle$, is no longer diagonal

instead:

$$= p\delta_{ij} + \Pi_{ij}, \qquad (35a)$$

with $\Pi_{ij} \equiv m\int d^3V\, V_i V_j\, g$,

from (34) we then have

$$\Pi_{ij} = -\frac{\tau m^2}{k_B T}\Lambda_{kl}\int d^3V\, V_i V_j \left(V_k V_l - \frac{1}{3}\delta_{kl}V^2\right) f^{(0)}$$

but for this integral, only isotropic contributions survive, since $f^{(0)}$ is isotropic (no dependence on vector $\vec{V}$ only its magnitude).

this means →

$$\langle V_i V_j V_k V_l \rangle = a\,\delta_{ij}\delta_{kl} + b\,\delta_{ik}\delta_{jl} + c\,\delta_{il}\delta_{jk}$$

$$\langle V_i V_j \delta_{kl} V^2 \rangle = d\,\delta_{ij}\delta_{kl}$$

→

to find a, b, c : need 3 equations.
Multiply by each separate δ combination:

$$\langle U^4 \rangle = 9a + 3b + 3c \qquad (35)$$

$$\langle U^4 \rangle = 3a + 9b + 3c \qquad (36)$$

$$\langle U^4 \rangle = 3a + 3b + 9c \qquad (37)$$

$$\Rightarrow \quad 0 = 6a - 6b$$

$$0 = 6a - 6c$$

$$0 = 6b - 6c$$

$$\Rightarrow \quad b = c = a = \frac{\langle U^4 \rangle}{15}$$

also $\langle U_i U_j U^2 \delta_{kl} \rangle = d\, \delta_{ij} \delta_{kl}$

$$\Rightarrow 3\langle U^4 \rangle = 9d \Rightarrow d = \frac{\langle U^4 \rangle}{3}$$

$$\Rightarrow \Lambda_{kl} \left\langle U_i U_j U_k U_l - \frac{U_i U_j U^2 \delta_{kl}}{3} \right\rangle \longleftarrow$$

note the interesting equality

$$= \Lambda_{kl} \langle U^4 \rangle \left(\frac{\delta_{ij}\delta_{kl}}{15} + \frac{\delta_{ik}\delta_{jl}}{15} + \frac{\delta_{il}\delta_{jk}}{15} - \frac{1}{9}\delta_{ij}\delta_{kl} \right\rangle \downarrow$$

$$= \frac{2}{15} \langle U^4 \rangle \Lambda_{ij} - \frac{6}{135} \Lambda \delta_{ij} = \frac{2}{15} \langle U^4 \rangle \left(\Lambda_{ij} - \frac{1}{3} \delta_{ij} \Lambda \right)$$

$$\Lambda_{ij} = \frac{\partial_i V_j + \partial_j V_i}{2}$$ ㊲

thus

$$\Pi_{ij} \propto \left(\Lambda_{ij} - \frac{1}{3}\delta_{ij}\Lambda\right)$$

we can write

$$\Pi_{ij} = -2\mu\Big(\Lambda_{ij} - \underbrace{\frac{1}{3}\delta_{ij}\nabla\cdot V}_{=\frac{1}{3}\Lambda\delta_{ij}}\Big) \qquad (38)$$

to get μ evalute one component of Π_{ij}: (from p 35)

$$\Pi_{12} = \frac{\tau m^2}{k_B T}\Lambda_{ke}\int d^3U\, U_1 U_2\left(U_k U_e - \frac{1}{3}\delta_{ke}U^2\right) f^{(0)}$$

$$= -\boxed{2}\frac{\tau m^2}{k_B T}\Lambda_{12}\int d^3U\; \overset{\swarrow}{U_1^2} U_2^2 f^{(0)}$$ since only even powers contribute.

$\tau\, m\, n\, \frac{kT}{m}$

$\left[\frac{\tau c_s^2 \cancel{\rho}}{\cancel{\rho}}\right] = \frac{\mu}{\rho}$

$= [\nu] = \ell\cdot v$

thus: $$\mu = \frac{m^2\tau}{k_B T}\int d^3U\, U_1^2 U_2^2 f^{(0)} = \tau n k_B T \qquad (38a)$$

from (38) since $\langle U_1^2 U_2^2\rangle = \frac{k^2T^2}{m^2}$, use: $\int_{-\infty}^{\infty} q^2 e^{-aq^2} dq = \frac{\pi^{1/2}}{2a^{3/2}}$

and $\int_{-\infty}^{\infty} e^{-aq^2} dq = \frac{\pi^{1/2}}{a^{1/2}}$

and $\langle Q\rangle = \frac{\int Q\, d^3u}{n}$

The off diagonal component of P_{ij} thus has coefficent μ, this is viscosity! density means momentum transport is possible between different flows moving at different velocities More on this later.

with expressions for $\vec{Q}$ and $\dot{P}_{ij}$

we put them into the moment equations:

using P_{ij} and $\Lambda_{ij} \equiv \frac{1}{2}(\partial_j v_i + \partial_i v_j)$; $\Pi_{ij} = -2\mu(\Lambda_{ij} - \frac{1}{3}\delta_{ij}\vec{\nabla}\cdot\vec{v})$

$$\Rightarrow \frac{\partial P_{ij}}{\partial x_j} = \frac{\partial p}{\partial x_i} - \mu\left[\nabla^2 v_i + \frac{1}{3}\frac{\partial}{\partial x_i}(\vec{\nabla}\cdot\vec{v})\right]$$ for constant μ

then plugging into (19)

$$\rho\left(\frac{\partial v_j}{\partial t} + v_i\frac{\partial v_j}{\partial x_i}\right) = -\frac{\partial p}{\partial x_j} + \mu\left[\nabla^2 v_j + \frac{1}{3}\frac{\partial}{\partial x_j}(\vec{\nabla}\cdot\vec{v})\right] + \frac{\rho}{m}F_j \quad (39)$$

from (38), (35a) & defn of Λ_{ij}, we also have

$$P_{ij}\Lambda_{ij} = p\,\nabla\cdot v - 2\mu\left[\Lambda_{ij}\Lambda_{ij} - \frac{1}{3}(\nabla\cdot v)^2\right], \quad (39a)$$

plugging (39) and (38) for Π_{ij} into energy moment eqn (20)

$$\Rightarrow \rho\left(\frac{\partial\varepsilon}{\partial t} + \vec{v}\cdot\vec{\nabla}\varepsilon\right) - \vec{\nabla}\cdot(\underline{K}\,\vec{\nabla}T) + p\,\vec{\nabla}\cdot\vec{v} - 2\underbrace{\mu\left[\Lambda_{ij}\Lambda_{ij} - \frac{1}{3}(\vec{\nabla}\cdot\vec{v})^2\right]}_{\text{heat production by viscous damping}} = 0 \quad (40)$$

now, μ term in (40) and $(\nabla\cdot v)$ term in (39)

are often small, if we neglect them

→

(38)

=> momentum

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V} = -\frac{1}{\rho}\vec{\nabla}p + \vec{F} + \boxed{\frac{\mu}{\rho}}\nabla^2 V \qquad (41)$$

=> energy

$$\rho\left(\frac{\partial \mathcal{E}}{\partial t} + \vec{V}\cdot\vec{\nabla}\mathcal{E}\right) - \vec{\nabla}\cdot(\mathbb{K}\vec{\nabla}T) + \boxed{p\vec{\nabla}\cdot\vec{V}} = 0 \qquad (42)$$

weakly compressible if kept but $\nabla\cdot v$ dropped in μ term

and mass continuity:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \qquad (43)$$

constant ρ

$$\cancel{\frac{\partial \rho}{\partial t}} + \rho\boxed{(\nabla\cdot\vec{v})} + \cancel{v\cdot\nabla\rho} = \boxed{0}$$

are the fluid equations, and we have now used $\vec{F}$ to represent force density.

$$\partial_j P_{ij} = \partial_i P + \partial_j \Pi_{ij}$$

$$= \partial_i P - 2\partial_j\left[\mu\left(\Lambda_{ij} - \frac{\delta_{ij}\vec{\nabla}\cdot\vec{v}}{3}\right)\right]$$

$$= \partial_i P - 2\partial_j\left[\mu\frac{(\partial_i v_j + \partial_j v_i)}{2} - \mu\frac{\delta_{ij}}{3}\vec{\nabla}\cdot\vec{v}\right]$$

$$= \partial_i P - \mu\nabla^2\vec{v} - \frac{\mu}{3}\partial_i(\vec{\nabla}\cdot\vec{v})$$

$$\Rightarrow \quad -\partial_i P_{ij}$$

Fluid Equations

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \qquad \text{(continuity)} \qquad (44)$$

(for constant μ and dropped $\mu\,\nabla\cdot v$ term)

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + F + \underbrace{\frac{\mu}{\rho}}_{\rightarrow\,\nu}\nabla^2\vec{v} \qquad \text{(momentum)} \qquad (45)$$

$$\rho\left(\frac{\partial \varepsilon}{\partial t} + \vec{v}\cdot\vec{\nabla}\varepsilon\right) - \vec{\nabla}\cdot(K\vec{\nabla}T) + \underline{p\vec{\nabla}\cdot\vec{V}} = 0 \qquad \text{(energy)} \qquad (46)$$

(μ-term dropped)

(45) is called Navier-stokes eqn (for constant μ)

These 5 equations constitute a dynamical theory:

$\vec{V}$: 3 quantities

T, ρ, p, ε : 4 quantities

but $p \propto \varepsilon$, and $P = nkT$ so we eliminate 2/7 quantities and are left with 5 equations and 5 variables.

- can also derive the fluid equations from macroscopic stress consideration. I won't do that here.

Vorticity Equation & incompressible flow

take curl of Nav. Stokes equation:

$\vec{\omega} = \vec{\nabla} \times \vec{v} \Rightarrow$

assume $\vec{F} = -\vec{\nabla}\phi$ (conservative force)

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{v} \cdot \vec{\nabla} \vec{v}) = -\nabla \times \frac{1}{\rho} \nabla p + \cancel{\nabla \times \vec{F}} + \boxed{\frac{\mu}{\rho}} \nabla^2 \vec{\omega}$$

ν = constant (assume)

but $(\vec{v} \cdot \vec{\nabla} \vec{v}) = \frac{1}{2} \vec{\nabla}(\vec{v} \cdot \vec{v}) - \vec{v} \times \underbrace{(\vec{\nabla} \times \vec{v})}_{\vec{\omega}}$

$$\Rightarrow \frac{\partial \vec{\omega}}{\partial t} = \nabla \times (\vec{v} \times \vec{\omega}) + \frac{1}{\rho^2} \vec{\nabla}\rho \times \vec{\nabla} p + \nu \nabla^2 \vec{\omega} + \cancel{\nabla \nu}(\nu) \qquad (47)$$

ignore

now, consider an incompressible flow: in such a flow the density remains constant in space and time. The continuity equation then gives

$$\frac{\partial \rho}{\partial t} \xrightarrow{} 0 + \rho \vec{\nabla} \cdot \vec{v} + \cancel{\vec{v} \cdot \vec{\nabla} \rho} \xrightarrow{} 0 = 0 \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{v} = 0$$

For incompressible flow:

$$\frac{\partial \vec{\omega}}{\partial t} = \nabla \times (\vec{v} \times \vec{\omega}) + \nu \nabla^2 \vec{\omega} + \cancel{\nabla \nu}(\,) \xrightarrow{} 0 \qquad (48)$$

compare to magnetic induction equation in incompressible MHD:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \nu_m \nabla^2 \vec{B} \qquad (49)$$

where ν_m is magnetic diffusivity. Note similarity between (48) and (49)!

The similarity of (48) & (49) implies "deep" connections etween behavior of vorticity, or vortex lines and magnetic field lines in incompressible MHD.

more on incompressibility

when is flow incompressible?

We will later see that disturbances in a fluid propagate at the sound speed, c_s. Thus in general, unless the agent causing the disturbance moves faster than c_s, the density will smooth out on time scales short compared to the evolution of the quanties of interest => systems with subsonic material velocities are largely incompressible.

(Note also that for barotropic flows, defined by $P=P(\rho)$, the third term of (47) also vanishes. these flows can be compressible and still have same form as 47 without that 3rd term. More on these later.)

For incompressible flow, energy equation <u>is redundant</u> :

Since $\vec{\nabla}\cdot\vec{V} = 0$; and using $\vec{\nabla}\times\vec{V} \equiv \vec{\omega}$

$\partial_t \vec{\omega} = \vec{\nabla}\times\vec{V}\times\vec{\omega} + \nu \nabla^2 \vec{\omega}$ we can fully solve for $\vec{v}$. If a vector field's divergence and curl are known, then we can solve for vector field. Thus $\vec{\nabla}\cdot\vec{V}$ and (48) are enough to solve for $\vec{V}$. Once we have $\vec{V}$, we get ρ om (44) and p from (45). Thus (96) is never needed, since p and ε are related.

Thus energy-equation is not needed for incompressible flows.

→ This is not true when radiation is important.

⇒ radiative transfer and energy equation are needed. extra terms in the energy equation corresponding to radiation stress are required.

(There is a formal analogy between $f(\vec{x},\vec{u},t)$ and $I_\nu(\vec{x},\underset{\omega}{\hat{n},\nu},t)$)

Specific Intensity →

Hydrostatic Equilibrium

Consider a fluid at rest, so that $\vec{v}=0$

then momentum equation with $\vec{v}=0$

$$\Rightarrow \quad \vec{F} = +\frac{1}{\rho}\vec{\nabla}p \qquad (50)$$

energy equation

$$\vec{\nabla}\cdot(\underline{K}\vec{\nabla}T) = 0 \qquad (51)$$

Consider fluid in gravitational field in equilibrium

$\vec{F} = -g\hat{e}_z$ where $\hat{e}_z$ is vertical direction

the $\hat{z}$ component of (50) then gives

$$-\rho g = \frac{\partial p}{\partial z} \qquad (52)$$

for incompressible flow this completely describes the system

$\Rightarrow$ $p = p_0 - \rho g z$, where $p_0 = p(z=0)$, $\rho =$ constant

and thus p increases as z decreases below 0

for incompressible flow.

→

Now consider a compressible flow and consider the isothermal soln to (51).

Since $p = \frac{k_B}{m} \rho T$ (which we derived from Boltzmann eqn).

For constant T, (52) gives

$$\frac{k_B T}{m} \frac{d\rho}{dz} = -\rho g$$

$$\Rightarrow \rho = \rho_0 \, \mathrm{Exp}\left[\frac{-mgz}{k_B T} \right], \quad \rho_0 \equiv \rho(0) \tag{53}$$

$\Rightarrow$ density falls off exponentially in an isothermal atmosphere

Note that (50) is a fundamental equation of stellar structure. However (51) would have convection and radiative transport terms in addition to the conduction terms present here.

But consider now the solar corona rather than the solar interior

→

Solar Corona

Coronal temperature is hotter than solar surface by factor $\simeq 1000$.

Assume spherical symmetry as a crude approximation. take boundary condition $T = T_0$ at base of corona $r = r_0$.

(we will later discuss more about heating corona)

Mass of corona is negligible so it is under gravitational influence of the sun.

in spherical geometry, hydrostatic equilib momentum eqn

$$\Rightarrow \frac{dP}{dr} = -\frac{G\overset{\text{mass of sun}}{M}}{r^2}\rho = -\frac{GM}{r^2}\frac{\overset{\text{mass of proton}}{m}P}{k_B T} \qquad (54)$$

energy equation:

$$\frac{d}{dr}\left(\underline{K} r^2 \frac{dT}{dr}\right) = 0 \qquad (55)$$

$\underline{K}$ = thermal conductivity $\propto T^{5/2}$ (derived later)

so (55) $\Rightarrow$

$$r^2 T^{5/2}\frac{dT}{dr} = \text{constant}$$

$$\Rightarrow T = T_0\left(\frac{r_0}{r}\right)^{2/7}, \quad T(\infty) = 0, \; T_0 = T(r_0) \qquad (56)$$

Using this for T in (54)

$$\Rightarrow \quad \frac{dp}{p} = \frac{-GM_{\odot}m \; dr}{r^{12/7} r_0^{2/7} k_B T_0}$$

soln is

$$p = p_0 \, \mathrm{Exp}\left[\frac{7}{5}\frac{GMm}{k_B T_0 r_0}\left\{\left(\frac{r_0}{r}\right)^{5/7} - 1\right)\right. \qquad (57)$$

where $p(r_0) = p_0$.

Note that at $r \to \infty$ $p \neq 0$!

No solution with both $T(\infty)$ AND $p(\infty)$ vanishing.

Significance is that hot solar corona can only be in equilibrium if there is a pressure at infinity to keep it from expanding. But since the pressure available is not enough, Parker (1958) used this argument to predict the solar wind! It was detected several years after the prediction!

Temperature dependence of K, the thermal conduction coefficient

(Quick derivation needed for page 46)

u = typical relative velocity between particles.

r_0/u = time during which particles are close enough to make "collision" by Coulomb interaction. Then

$$\Rightarrow \frac{e^2}{r_0^{\cancel{2}}}\frac{\cancel{r_0}}{u} = \underbrace{\Delta p}_{m_e u} = \text{change in momentum} = F \cdot \Delta t \text{ from interaction}$$

we set $\Delta p = p$ to define a "collision"

$\Rightarrow$ for non-relativistic electrons (at fixed $T_i = T_e$, electrons conduct the heat rather than ions because electrons are more mobile)

$$r_0 \simeq \frac{e^2}{m_e u^2}, \text{ since } p = m_e u$$

collision cross section is then πr_0^2

and collision frequency is

$$(\pi r_0^2)(n)(u) = \frac{\pi n e^4}{m_e^2 u^3} \equiv \nu_c$$

use $u \sim \left(\frac{k_bT}{m_e}\right)^{1/2}$

$$\Rightarrow \nu_c = \frac{\pi n e^4}{m_e^{1/2} k_b^{3/2} T^{3/2}}$$

so collision time

$$\tau \simeq \frac{1}{\nu_c} = \frac{(k_bT)^{3/2} m_e^{1/2}}{\pi n e^4}.$$

Then from eqn (35) (page 34)

$$\boxed{K = \frac{m\tau}{cT}\int d^3v\, v^4\left(\frac{m}{2k_BT}v^2 - \frac{5}{2}\right)f^{(0)} = \frac{5}{2}n\tau\frac{k_B^2T}{m}\;)} \quad (35)$$

$$\Rightarrow \underline{K \propto \tau T \propto \underline{T^{5/2}}}, \text{ since } \tau \propto T^{3/2}$$

Since $\left[\nabla\cdot\tilde{K}\nabla T\right] = \left[\frac{\partial \mathcal{E}}{\partial t}\rho\right]$

$$[\tilde{K}] = \left[\frac{\text{length}^2}{\text{temp}}\cdot\frac{\text{energy}}{\text{time}\cdot\text{volume}}\right]$$

$$= \frac{l^2\cdot m\cdot \frac{l^2}{t^2}}{[T]\,t\cdot l^3} = \frac{m\,l}{[T]\,t^3} \quad ✓$$

Check whether explicit expression for $\tilde{K}$ has correct units:

$$\left[n\tau\frac{k_b^2 T}{m_e}\right] = \frac{t}{l^3}\,\frac{l^4}{t^4}\,\frac{m}{[T]} = \frac{m\,l}{[T]\,t^3} \quad ✓$$

also since $\boxed{\left[\nabla\cdot\frac{\tilde{K}}{nk_b}\nabla(Tk_bn)\right]} = \left[\rho\frac{\partial\mathcal{E}}{\partial t}\right] = \frac{\text{energy}}{\text{volume}\cdot\text{time}}$

and $[nk_bT] = \left[\frac{\text{energy}}{\text{volume}}\right]$

$$\Rightarrow \left[\frac{\tilde{K}}{nk_b}\right] = \frac{\text{length}^2}{\text{time}} = (\text{speed})(\text{length scale})$$

thermal diffusivity

Same units as kinematic viscosity

Note:

dynamic viscosity $\equiv \mu$

kinematic viscosity $\frac{\mu}{\rho} \equiv \nu$

$$\underline{\partial_t C = \nu\nabla^2 C}$$

Bernouillis Principle

moving Beyond Hydrostatics : simple hydrodynamic problems involve steady flows. (time independent)

Define a streamline as the curve tangent to the velocity $\vec{v}$ at every point.

when a flow is steady, the streamlines trace the paths of all fluid parcels.

lets write $\vec{F} = -\nabla\phi$ (for conservative force)

the Euler equation (= eqn(45) without the viscous term) 'steady state', is :

$$\underbrace{\nabla\left(\tfrac{1}{2}v^2\right) - \vec{v}\times(\vec{\nabla}\times\vec{v})}_{\vec{v}\cdot\vec{\nabla}\vec{v}} = -\frac{1}{\rho}\nabla p - \nabla\phi \qquad (58)$$

Integrate along Streamline :

$$\int d\vec{\ell}\cdot\left[\vec{\nabla}\tfrac{1}{2}(v^2) - \cancelto{0 \text{ since } d\vec{\ell}\parallel\vec{v}}{\vec{v}\times(\vec{\nabla}\times\vec{v})} + \frac{1}{\rho}\vec{\nabla}p + \vec{\nabla}\phi\right] = 0$$

$$\Rightarrow \int\frac{dp}{\rho} + \frac{1}{2}v^2 + \phi = \text{constant} \qquad (59)$$

= Bernouillis Principle

where the integral is along a streamline.

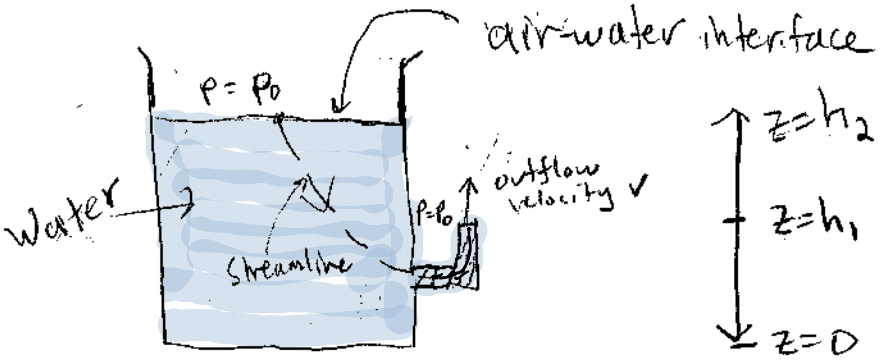
→

incompressible flows, ρ = constant

(59) becomes $\quad \phi = gh$ for gravity

$$\frac{v^2}{2} + \frac{p}{\rho} + gh = \text{constant} \qquad (60)$$

apply this to a tank with outlet:



consider a streamline that extends from air-water interface at top of container to the nozzle.
$p = p_0$ both at top interface and just external to the nozzle

we have from Bernoulli's principle for incompressible flow:

$$\underset{\text{small}}{\cancel{\frac{V_{in}^2}{2}}} + \underbrace{\frac{p_0}{\rho} + gh_2}_{\text{at air water interface}} = \underbrace{\frac{V_{out}^2}{2} + \frac{p_0}{\rho} + gh_1}_{\text{at nozzle}}$$

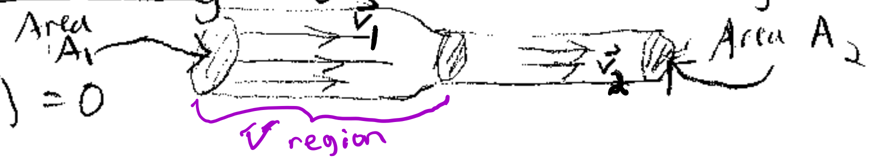
$$\Rightarrow \frac{V_{out}^2}{2} = g(h_2 - h_1) = g\Delta h$$

$$\Rightarrow |V_{out}| = (2g\Delta h)^{1/2} \qquad (61)$$

Note this is independent of the nozzle's direction!
(The subsequent evolution of flow after initial ejection will, however, depend on nozzle direction) ✓

Bernoulli's theorem also implies that pressure drops when velocity of flow increases:

consider continuity equation for steady flow in pipe:



$$\Rightarrow \underbrace{\frac{\partial \rho}{\partial t}}_{0} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

integrating $\Rightarrow$ $$\int \rho\vec{v}\cdot d\vec{S} = 0 = \int_V \vec{\nabla}\cdot(\rho\vec{v})\, d^3x$$

For constant density (incompressible) & constant $\vec{V}$ over the cross sectional area, this implies

$V_1 A_1 = V_2 A_2$. Thus as A decreases V increases:

$$\frac{V_2}{V_1} = \frac{A_1}{A_2} > 1 \quad \Rightarrow \text{velocity increases } V_2 > V_1$$

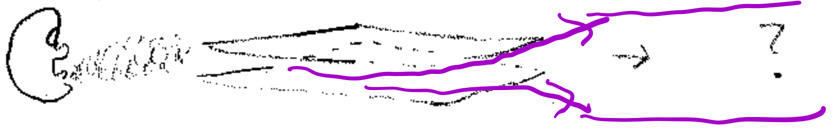
Now from Bernoulli's principle we have:

$$\frac{V_1^2}{2} + \frac{P_1}{\rho} = \frac{V_2^2}{2} + \frac{P_2}{\rho}$$

$$\Rightarrow P_1 = \frac{\rho(V_2^2 - V_1^2)}{2} + P_2 > P_2 \quad \Rightarrow \text{pressure decreases } P_2 < P_1$$

thus as flow is constricted in a pipe its velocity increases and pressure decreases.

By this reasoning, what happens if you hold two pieces of paper parallel and try to separate them by blowing?



Note on Eulerian vs. Lagrangian Descriptions

- Eulerian: describes evolution of flow at fixed coordinates as flow passes through these coords.: e.g. $\rho(\vec{x},t)$
- Lagrangian: describes evolution of flow comoving with fluid element e.g. $\rho(\vec{a},t)$ where $\vec{a}$ labels fluid element

In Lagrangian description: $\vec{x}=\vec{x}(\vec{a},t)$ so that $\vec{x}$ and t are no longer independent variables (!)

parcel of fluid

- Lagrangian derivative:

at time t "particle" is at position $\vec{x},t$ and at time $t+\Delta t$ "particle" is at position $\vec{x}+\Delta x$

=> Lagrangian derivative of function $C(\vec{x},t)$

$$\frac{D\,C(\vec{x},t)}{Dt}=\lim_{\Delta t\to 0}\frac{C(\vec{x}+\Delta\vec{x},t+\Delta t)-C(\vec{x},t)}{\Delta t}$$

the numerator can be written:

$$C(\vec{x},t+\Delta t]+C(\vec{x}+\Delta\vec{x},t+\Delta t)-C(\vec{x},t)-C(\vec{x},t+\Delta t]$$

$$\Rightarrow \frac{DC(\vec{x},t)}{Dt}=\lim_{\Delta t\to 0}\left(\frac{\frac{\partial C}{\partial t}(\vec{x},t)\,\Delta t+\frac{\partial C((\vec{x},t+\Delta t)}{\partial\vec{x}}\Delta\vec{x}}{\Delta t}\right)$$

since $\frac{\partial C}{\partial\vec{x}}(\vec{x},t+\Delta t]=\frac{\partial C}{\partial x}(\vec{x},t)+\Delta t\,\cancel{\frac{\partial^2 C}{\partial\vec{x}\,\partial t}(\vec{x},t)}$

2nd order in small quantities

$$\Rightarrow \underset{\substack{\downarrow\\ \text{Lagrangian}\\ \text{Derivative}}}{\frac{DC(\vec{x},t)}{Dt}} = \underset{\substack{\downarrow\\ \text{Eulerian}\\ \text{Deriviative}}}{\frac{\partial C(\vec{x},t)}{\partial t}} + \underbrace{\underbrace{\vec{v}(x,t)}_{\text{flow velocity}} \cdot \vec{\nabla} C(\vec{x},t)}_{\text{"convective derivative"}}$$

{ in non-cartesian coords, if C is a vector, care must be taken with terms like $\vec{v}\cdot\vec{\nabla}\vec{C}$; because unit vectors are not constant. Can look up components e.g. $(\vec{v}\cdot\vec{\nabla}\vec{C})_r$, $(\vec{v}\cdot\nabla\vec{C})_\phi$, $(\vec{v}\cdot\nabla\vec{C})_\theta$ in sph. coords)

Lagrangian derivative is also called the "material" derivative

Kelvin Circulation Theorem (Helmholtz 1858; Kelvin 1869)

For ideal barotropic or incompressible flow:

$$\partial_t \vec{\omega} = \nabla\times(\vec{v}\times\vec{\omega}) + \cancel{\frac{\nabla\rho\times\nabla p}{\rho^2}} \rightarrow 0 \qquad (62)$$

(relation between S_0 & S_1)

Define flux of vorticity through a surface at time t_0 and t_1 as

$\int_{S_0} \vec{\omega}\cdot d\vec{S}$ and $\int_{S_1} \vec{\omega}\cdot d\vec{S}$. I will show that (62) $\Rightarrow$ Kelvin Circulation Theorem:

$$\frac{D\Phi}{Dt} \equiv \frac{D}{Dt}\int \vec{\omega}\cdot d\vec{S} = 0 \quad \text{or} \quad \int_{S_0}\vec{\omega}\cdot d\vec{S} = \int_{S_1}\vec{\omega}\cdot d\vec{S}. \qquad (63)$$

[The proof below also applies to the magnetic flux, as the induction equation has the same form as (62).]

(64)

→

Derivation of the time evolution of Flux

(Blackman EJP, 34, 489 (2013))

Let $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}$ be the material derivative that follows the time evolution of a quantity as fluid moves with velocity $\vec{v}(\vec{x},t)$. For a vector $\vec{A}$

any vector or pseudovector

$$\frac{D}{Dt}\left(\int_S \vec{A}\cdot d\vec{S}\right) = \frac{D}{Dt}\int_{S_0} \vec{A}\cdot\vec{\vec{J}}\,d\vec{S}_0, \qquad (F1)$$

where $d\vec{S}$ is a surface element on the surface that evolves as the result of the fluid motion and $d\vec{S}_0$ is surface element of a fixed control surface (e.g. at $t=0$.)

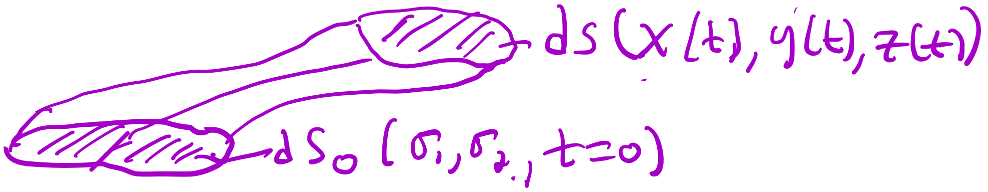
Here $\vec{\vec{J}}$ is the Jacobian for the coordinate transformation. For a surface integral, this is given by

[repeated indices always summed : e.g. $\partial_i V_i = \sum_{i=1}^{3} \partial_i V_i$]

$$J_{k3} \equiv \epsilon_{kij}\frac{\partial X_i}{\partial \sigma_1}\frac{\partial X_j}{\partial \sigma_2} = J_k \qquad (F2)$$

where σ_1 σ_2 are local cartesian coordinates of the fixed surface element dS_0 and X_1, X_2, X_3 are local cartesian coordinates of evolving surface element dS

$$J_{k3}\, dS_3$$
$$\sim$$
$$d\sigma_1 d\sigma_2$$



Because the right side of (1) is an integral over a fixed surface, we can take the $\frac{D}{Dt}$ inside the integral in (1):

$$\frac{D}{Dt}\int_S \vec{A}\cdot d\vec{S} = \frac{D}{Dt}\int_{S_0} \vec{A}\cdot\vec{J}\,dS_0 = \int\left(\vec{J}\cdot\frac{D\vec{A}}{Dt} + \vec{A}\cdot\frac{D\vec{J}}{Dt}\right)dS_0 \qquad (F3)$$

We need an expression for $\frac{D\vec{J}}{Dt}$. On page (F4) I derive this explicitly. For the moment I just take the result:

$$\frac{DJ_q}{Dt} = J_m\left(\delta_{mq}\,(\vec{\nabla}\cdot\vec{v}) - \partial_q V_m\right) \qquad (F4)$$

Using (F4) and

$$\frac{DA_i}{Dt} = \frac{\partial A_i}{\partial t} + (\vec{V}\cdot\vec{\nabla}\vec{A})_i \qquad \left[\text{Note } ((\vec{V}\cdot\nabla\vec{A}))_i = \vec{V}\cdot\vec{\nabla}A_i \text{ in cartesian coords}\right] \qquad (F5)$$

in equation (F3), we obtain

$$\frac{D}{Dt}\int_S \vec{A}\cdot d\vec{S} = \int\Big[\underbrace{\left(\frac{\partial A_i}{\partial t} + (\vec{V}\cdot\vec{\nabla}\vec{A})_i\right)J_i}_{\textcircled{a}} + \underbrace{A_q J_m\big(\delta_{mq}(\vec{\nabla}\cdot\vec{V}) - \partial_q V_m\,dS_0\big)}_{\textcircled{b}} \qquad (F6)$$

the contributions ⓐ and ⓑ can be combined in equation (F6) giving

$$\frac{D}{Dt}\int_S \vec{A}\cdot d\vec{S} = \int_{S_0}\left(\frac{\partial A_i}{\partial t} + \underline{\vec{v}\cdot\nabla A_i} + \underline{A_i\,\vec{\nabla}\cdot\vec{v}} - \underline{\vec{A}\cdot\nabla v_i}\right)\underbrace{J_i\,dS_0}_{dS_i} \qquad (F7)$$

Since $J_i\,dS_0 = dS_i$ we now have an integral over the moving surface in (F7).

In addition, we have the vector identity

$$\vec{\nabla}\times(\vec{v}\times\vec{A}) = \underline{\vec{A}\cdot\nabla\vec{v}} - \underline{\vec{v}\cdot\nabla\vec{A}} - \underline{\vec{A}(\vec{\nabla}\cdot\vec{v})} + \vec{v}(\nabla\cdot\vec{A}) \qquad (F8)$$

Using $J_i\,dS_0 = dS_i$ and (F8) in (F7), we have

$$\frac{D}{Dt}\int_S \vec{A}\cdot d\vec{S} = \int_S\left[\left(\frac{\partial\vec{A}}{\partial t} - \nabla\times(\vec{v}\times\vec{A})\right) + \vec{v}(\nabla\cdot\vec{A})\right]\cdot d\vec{S} \qquad (F9)$$

thus if $\frac{\partial\vec{A}}{\partial t} = \nabla\times(\vec{v}\times\vec{A})$ AND $\nabla\cdot\vec{A} = 0$, then

$$\frac{D}{Dt}\int_S \vec{A}\cdot d\vec{S} = 0$$

//

Derivation of $\frac{D\vec{J}}{Dt}$ (needed in eqn (4))

$$J_k = \varepsilon_{kij} \frac{\partial x_i}{\partial \sigma_1} \frac{\partial x_j}{\partial \sigma_2}$$

$$\frac{DJ_k}{Dt} = \varepsilon_{kij}\left(\left(\frac{\partial}{\partial \sigma_1} \frac{Dx_i}{Dt}\right) \frac{\partial x_j}{\partial \sigma_2} + \frac{\partial x_i}{\partial \sigma_1}\left(\frac{\partial}{\partial \sigma_2} \frac{Dx_j}{Dt}\right)\right) \tag{F10}$$

but $\frac{Dx_j}{Dt} = v_j$ and $\frac{Dx_i}{Dt} = v_i$

$$\frac{DJ_k}{Dt} = \varepsilon_{kij}\left(\frac{\partial v_i}{\partial \sigma_1} \frac{\partial x_j}{\partial \sigma_2} + \frac{\partial x_i}{\partial \sigma_1} \frac{\partial v_j}{\partial \sigma_2}\right) \tag{F11}$$

$$= \varepsilon_{kij}\left(\frac{\partial v_i}{\partial x_m} \frac{\partial x_m}{\partial \sigma_1} \frac{\partial x_j}{\partial \sigma_2} + \frac{\partial x_i}{\partial \sigma_1} \frac{\partial v_j}{\partial x_m} \frac{\partial x_m}{\partial \sigma_2}\right)$$

Use antisymmetry of ε_{kij} in index interchange between $i \leftrightarrow j$ to obtain:

$$= \varepsilon_{kij} \frac{\partial v_i}{\partial x_m}\left(\frac{\partial x_m}{\partial \sigma_1} \frac{\partial x_j}{\partial \sigma_2} - \frac{\partial x_j}{\partial \sigma_1} \frac{\partial x_m}{\partial \sigma_2}\right) \tag{F12}$$

now consider each component of (F12) :

(51e)

the $\hat{X}_1$, $\hat{X}_2$, $\hat{X}_3$ components of (12) obtained by setting $K =$ to 1, 2, and 3 respectively: and using $\varepsilon_{kij} = -\varepsilon_{kji}$ along with the fact that ε_{ijk} vanishes when any two indices are the same. Thus:

$$\frac{DJ_1}{Dt} = \frac{\partial V_2}{\partial X_q}\left(\frac{\partial X_q}{\partial \sigma_1}\frac{\partial X_3}{\partial \sigma_2} - \frac{\partial X_q}{\partial \sigma_2}\frac{\partial X_3}{\partial \sigma_1}\right) - \frac{\partial V_3}{\partial X_q}\left(\frac{\partial X_q}{\partial \sigma_1}\frac{\partial X_2}{\partial \sigma_2} - \frac{\partial X_q}{\partial \sigma_2}\frac{\partial X_2}{\partial \sigma_1}\right) \quad (F13)$$

- though q indices are summed, $q = 3$ does not contribute to first term and $q = 2$ does not contribute to second term.
- recognize that

$$\left.\begin{aligned} \frac{\partial X_1}{\partial \sigma_1}\frac{\partial X_3}{\partial \sigma_2} - \frac{\partial X_1}{\partial \sigma_2}\frac{\partial X_3}{\partial \sigma_1} &= -J_2 \\ \frac{\partial X_2}{\partial \sigma_1}\frac{\partial X_3}{\partial \sigma_2} - \frac{\partial X_2}{\partial \sigma_2}\frac{\partial X_3}{\partial \sigma_1} &= J_1 \\ \frac{\partial X_1}{\partial \sigma_1}\frac{\partial X_2}{\partial \sigma_2} - \frac{\partial X_2}{\partial \sigma_1}\frac{\partial X_1}{\partial \sigma_2} &= J_3 \end{aligned}\right\} \quad (F14)$$

using the equations of (F14), (F13) $\Rightarrow$

$$\frac{DJ_1}{Dt} = -\frac{\partial V_2}{\partial X_1}J_2 + \frac{\partial V_2}{\partial X_2}J_1 - \frac{\partial V_3}{\partial X_1}J_3 + \frac{\partial V_3}{\partial X_3}J_1 + \frac{\partial V_1}{\partial X_1}J_1 - \frac{\partial V_1}{\partial X_1}J_1 \quad (F15)$$

$$= (\vec{\nabla}\cdot\vec{V})J_1 - \vec{J}\cdot\frac{\partial \vec{V}}{\partial X_1}$$

analagously to (13) we have

$$\frac{DJ_2}{Dt} = \frac{\partial V_3}{\partial x_q}\left(\frac{\partial x_q}{\partial s_1}\frac{\partial x_1}{\partial s_2} - \frac{\partial x_q}{\partial s_2}\frac{\partial x_1}{\partial s_1}\right)$$

$$- \frac{\partial V_1}{\partial x_q}\left(\frac{\partial x_q}{\partial s_1}\frac{\partial x_3}{\partial s_2} - \frac{\partial x_q}{\partial s_2}\frac{\partial x_3}{\partial s_1}\right) \qquad (F16)$$

similarly to the treatment below (F13), using (F14) this reduces to:

$$\frac{DJ_2}{Dt} = -\frac{\partial V_3}{\partial x_2} J_3 + \frac{\partial V_3}{\partial x_3} J_2 + \frac{\partial V_1}{\partial x_1} J_2 - \frac{\partial V_1}{\partial x_2} J_1$$

$$= (\nabla \cdot \vec{V}) J_2 - \vec{J} \cdot \frac{\partial \vec{V}}{\partial x_2} \qquad (F17)$$

Finally we have:

$$\frac{DJ_3}{Dt} = \frac{\partial V_1}{\partial x_q}\left(\frac{\partial x_q}{\partial s_1}\frac{\partial x_2}{\partial s_2} - \frac{\partial x_q}{\partial s_2}\frac{\partial x_2}{\partial s_1}\right) - \frac{\partial V_2}{\partial x_q}\left(\frac{\partial x_q}{\partial s_1}\frac{\partial x_1}{\partial s_2} - \frac{\partial x_q}{\partial s_2}\frac{\partial x_1}{\partial s_1}\right)$$

$$= \frac{\partial V_1}{\partial x_1} J_3 - \frac{\partial V_1}{\partial x_3} J_1 - \frac{\partial V_2}{\partial x_2} J_3 - \frac{\partial V_2}{\partial x_3} J_2$$

$$= (\vec{\nabla} \cdot \vec{V}) J_3 - \vec{J} \cdot \frac{\partial \vec{V}}{\partial x_3} \qquad (F18)$$

Combining (F15), (F17) & (F18) :

$$\frac{DJ_q}{Dt} = (\nabla\cdot\vec{V})\,J_q - \vec{J}_m \nabla_q \vec{V}_m$$

$$= J_m\left(\delta_{mq}(\vec{\nabla}\cdot\vec{V}) - \partial_q V_m\right) \qquad (F19)$$

which is eqn (4) //

→

So eqn (F9) shows that only when $\nabla\cdot\vec{A}=0$ AND $\frac{\partial\vec{A}}{\partial t}-\nabla\times(\vec{v}\times\vec{A})=0$ does the material derivative of flux transport hold. when $\vec{A}=\vec{\omega}$ this is the kelvin circulation theorem for vorticity lines. when $\vec{A}=\vec{B}$ this is Alfvén theorem (though not his proof) Note $\vec{\nabla}\cdot\vec{w}=0$ is automatically satisfied

Consider the case when $\vec{\nabla}\cdot\vec{A}\neq 0$. This means a source of monopoles: (eg if $\vec{A}=\vec{B}$) the magnetic monopole density would be proportional to $\nabla\cdot\vec{B}$ (by analogy to $\vec{\nabla}\cdot\vec{E}$ for electric charge). If we set $\vec{A}=\vec{B}$ in F9 and take advantage of the induction equation $\frac{\partial\vec{B}}{\partial t}=\vec{\nabla}\times\vec{v}\times\vec{B}$, then

$$\frac{D}{Dt}\int\vec{B}\cdot d\vec{S}=\int\underbrace{\vec{V}\overset{4\pi\rho_{m,c}}{(\vec{\nabla}\cdot\vec{B})}}\,d\vec{S} \qquad (F20)$$

like an advection of magnetic charge

(there is more to be said about the interpretation of F20, we will discuss...)

Analogy between vorticity & B-field

Maxwells equations:

$$\frac{1}{c}\frac{\partial \vec{B}}{\partial t} + \vec{\nabla}\times\vec{E} = 0 \qquad (70)$$

$$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B} = \frac{4\pi \vec{J}}{c} \qquad (71)$$

$$\nabla\cdot\vec{E} = 4\pi\rho \qquad (72)$$

$$\nabla\cdot\vec{B} = 0 \qquad (73)$$

Ohm's Law:

results from subtracting momentum equations for positive and negative charges. That is one integrates Boltzmann equation for "+" and "−" charges in presence of Electromagnetic force. The result is

$$\boxed{\vec{E} + \frac{\vec{V}\times\vec{B}}{c} = \eta J} + \left(\frac{1}{nec}\vec{J}\times\vec{B} - \frac{1}{ne}\nabla P_e - \frac{m_e}{ne}\left(\frac{\partial J}{\partial t} + \vec{V}_e\cdot\nabla\vec{J}\right) + \eta_2\nabla^2 J\right) \qquad (74)$$

↓ MHD Ohm's Law

(Generalized Ohms Law also includes)

these plasma terms are "usually" small for "colder denser" plasmas of astrophysics; they are important for "hot diffuse" plasmas in the lab

(Using MHD Ohm's law in (70):

$$\Rightarrow \frac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{V}\times\vec{B}) - \eta\nabla\times J \qquad (75)$$

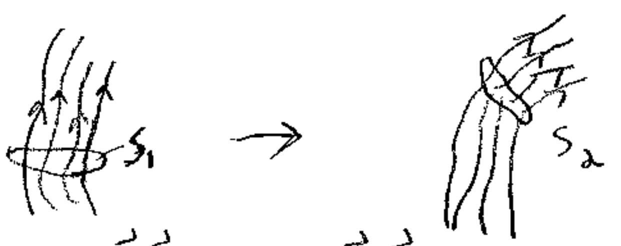
$$\simeq \vec{\nabla}\times(\vec{V}\times\vec{B}) \quad \text{for} \quad \eta = 0$$

thus

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{v}\times\vec{B}) \quad \text{in ideal MHD}$$

So that $\frac{D}{Dt}\int \vec{B}\cdot d\vec{S} = 0$ by analogy to the ideal circulation theorem.

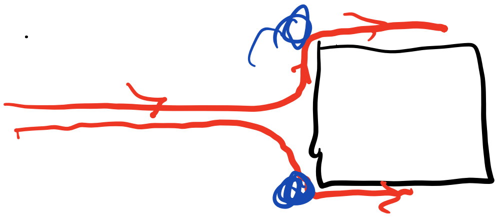
This is flux freezing since $\int \vec{B}\cdot dS = \phi$ is magnetic flux. The interpretation for magnetic flux freezing is identical to vortex line freezing: As we follow fluid elements that consitute a surface S_1 at t_1 as they evolve to surface S_2 at t_2, the normal component of vorticity or magnetic field adjusts to conserve the respective fluxes. E.g.



$$\vec{S}_1\cdot\vec{\omega}_1 = \vec{S}_2\cdot\vec{\omega}_2$$

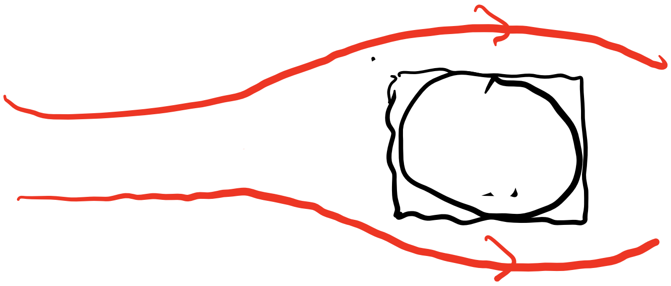
$$\vec{S}_1\cdot\vec{B}_1 = \vec{S}_2\cdot\vec{B}_2$$

How far can one take the analogy between vorticity and magnetic field ? Topic of discussion...



low viscosity

$$\frac{\partial \vec{\omega}}{\partial t} = \nabla \times \vec{v} \times \vec{\omega} + \boxed{\underset{\uparrow}{\nu} \nabla^2 \vec{\omega}}$$



high viscosity

Potential Flows: flow past cylinder

Incompressible ideal flow with no vorticity at any time will remain non-vortical:

$$\vec{\nabla}\times\vec{V}=0 \quad \Rightarrow \quad \vec{V}=-\nabla\phi$$

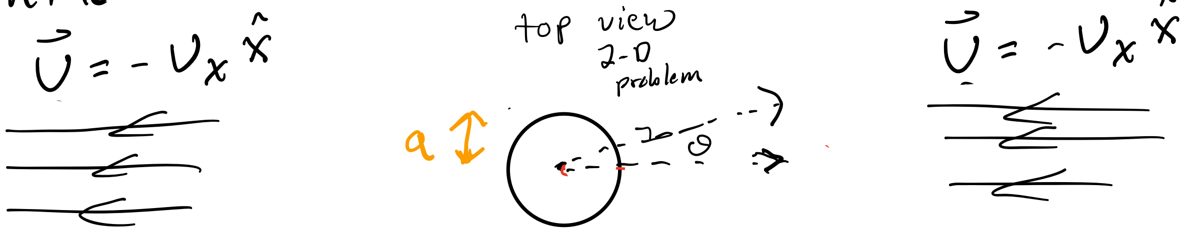
↑ velocity potential

$$\vec{\nabla}\cdot\vec{V}=0 \quad \Rightarrow \quad \nabla^2\phi=0$$

- normal component of $\vec{v}$ at a fixed solid boundary vanishes: ⇒

$$V_{\hat{n}}=-\hat{n}\cdot\nabla\phi=0 \qquad (c1)$$

- Example of flow past a cylinder: Assume flow is uniform far away from cylinder



solve for flow around cylinder assuming ideal (no viscosity)

- Equation C1 at the surface boundary is

$$V_n(a) = \frac{-\partial \phi}{\partial r}\Big|_{r=a} = 0 \quad \text{at} \quad r = a \quad \text{(cylindrical coords)}$$

- at $r \to \infty$ boundary condition is

$$\vec{v}\Big|_{r\to\infty} = -U\hat{x} = -U\cos\theta\,\hat{r} + U\sin\theta\,\hat{\theta}$$

$$\Rightarrow \quad = -\vec{\nabla}\phi \Rightarrow \quad \phi = Ur\sin\theta \quad \text{as } r\to\infty$$

- General solution of Laplaces equation in 2-D cylindrical coords:

$$\phi = (A_0 + B_0 \ln r)(C_0 + D_0\theta) + \sum_{n=1}^{\infty}\left(A_n r^n + \frac{B_n}{r^n}\right)(C_n \cos n\theta + D_n \sin n\theta)$$

A_n, B_n, C_n, D_n are constants

to satisfy above boundary cond at $r \to \infty$:

$\Rightarrow A_0 = B_0 = C_0 = D_0 = 0 = C_{n\geq 1}$ and only A_1 & B_1 contribute

$$\Rightarrow \quad \phi = \left(A_1 r + \frac{B_1}{r}\right) D_1 \cos\theta$$

at $r \to \infty$

must have $\phi = Ur\cos\theta \quad \Rightarrow \quad A_1 D_1 = U$

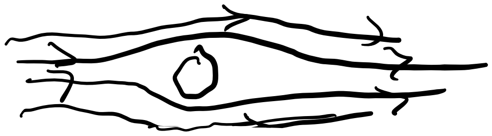
at $r \to a$

$$\partial_r \phi\big|_a = 0 \quad \Rightarrow \quad A_1 - \frac{B_1}{a^2} = 0$$

$$\Rightarrow \quad B_1 = a^2 A_1$$

$$\Rightarrow \quad \phi = \left(A_1 r + \frac{a^2 A_1}{r}\right) D_1 \cos\theta = \boxed{U\cos\theta\left(r + \frac{a^2}{r}\right)}$$

$\Rightarrow$ $\vec{v} = -\vec{\nabla}\phi$



$= -U\hat{x} + U\frac{a^2}{r^2}(\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$

in rest frame of asymptotic flow, the flow around the cylinder is

$$\vec{v}' = U\frac{a^2}{r^2}\left(\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \qquad (C2)$$

- Kinetic energy in a fluid layer of unit thickness parallel to cylinder axis is $K_{fluid} = \frac{1}{2}\rho\cancel{\int_0^1}\int_a^\infty (v')^2\, 2\pi r\, dr\, \cancel{dz}$

$$= \frac{1}{2}\rho\int_a^\infty v'^2\, 2\pi r\, dr$$

from (C2) $= \pi\rho U^2 a^4 \boxed{\int_a^\infty \frac{1}{r^3}\,dr} = -\frac{1}{2r^2}\Big|_a^\infty$

$$E_{fluid} = \underline{\frac{1}{2}\pi\rho U^2 a^2} = \frac{1}{2}M'U^2$$

$$= \frac{1}{2}\rho U^2(\pi a^2)\underset{\substack{\uparrow\\ \text{vertical}\\ \text{thickness}}}{(1)}$$

$M' = \pi a^2 \cdot \underbrace{h}_{=1}\rho$

= mass of fluid displaced per unit length of cylinder

Cylinder mass = M so total kinetic energy of cylinder plus displaced mass is: $\frac{1}{2}(M+M')U^2 = E_{tot}$

$$\Rightarrow \frac{dE_{tot}}{dt} = \underbrace{(M+M')}_{\text{effective mass}}\frac{d\vec{v}}{dt}\cdot\vec{v} = \underbrace{\vec{F}\cdot\vec{U}}_{\text{rate of work done by force acting on cylinder}}$$

$\underbrace{\nu\nabla^2 v}$

$\frac{\nu}{\ell^2}v$

but if $\vec{U}$ = constant then $\vec{F}=0 \Rightarrow$ no force when $\vec{U}$ is uniform; our solution that allowed $\vec{U}$ to be uniform is a system without viscosity; no viscosity means no drag, but drag is very important in real systems, and comes from viscosity in a boundary layer, and for small but finite viscosity this layer will thin but finite

Stream function, ψ, for incompressible flow

$\vec{\nabla}\cdot\vec{v} = 0 \quad \Rightarrow \quad \vec{v} = -\vec{\nabla}\times\left[\psi(x,y)\,\hat{z}\right]$ for 2-D flow

or $v_i = -\varepsilon_{ij3}\,\partial_j \psi(x,y)$

$\Rightarrow \quad v_x = -\partial_y \psi(x,y)$ (1)

$v_y = +\partial_x \psi(x,y)$ (2)

Show ψ is constant on streamlines!

streamline is defined as $\frac{dy}{dx} = \frac{v_y}{v_x}$

$\Rightarrow \quad v_y\,dx - v_x\,dy = 0$

from (1) & (2) $= \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$

$\Rightarrow d\psi = 0$ on streamlines

- If flow is also irrotational then

$\nabla\times\vec{v} = 0$ and $\vec{v} = -\nabla\underset{\text{velocity potential}}{\phi}$; $v_x = -\partial_x \phi(x,y)$, $v_y = -\partial_y \phi(x,y)$

Since $v_x \perp v_y$:

$\underbrace{-\partial_y \psi}_{v_x}$ from (1) is $\perp$ $\underbrace{-\partial_y \phi(x,y)}_{v_y}$

$\underbrace{+\partial_x \psi}_{v_y}$ from (2) is $\perp$ $\underbrace{+\partial_x \phi(x,y)}_{v_x}$

$\Rightarrow \nabla\psi\cdot\nabla\phi = 0$; ϕ and ψ are orthogonal functions

Viscous flows

Ideal fluids are assumed to have no viscosity. This means that $P_{ij} = p\delta_{ij} + \underset{0}{\underset{\downarrow}{\Pi_{ij}}}$ and that the forces on a fluid surface are normal to that surface.

Force density $\propto -\nabla_i P_{ij}$ so jth component of force is $F_j \propto \int \nabla_i P_{ij}\, dV = \int P_{ij}\, dS_i$

but if $\Pi_{ij} = 0$, then $F_j \propto \int p\, dS_j$

so that force points ⊥ to the surface dS and since vector area has direction ⊥ surface.

But this violates our common experience: moving your hand through water you feel a "drag" force which is a force between different layers in a fluid, not a normal force. Thus Π_{ij} cannot in neral be zero. We showed earlier that the fluid momentum equation with viscosity can be written

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = \vec{F} - \frac{1}{\rho}\nabla p + \nu\nabla^2 v \qquad (76)$$

where $\nu \equiv \frac{\mu}{\rho}$ and we have ignored both spatial dependence of ν and $\nabla\cdot v$ term in Π_{ij}.

Recall that the Navier-Stokes equation (reduces to Euler equation with $\nu = 0$) ν is the kinematic viscosity.

The presence of the ν term means that vorticity is no longer conserved:

taking curl of (76) =>

$$\frac{\partial \vec{\omega}}{\partial t} = \nabla\times(\vec{v}\times\hat{\omega}) + \underbrace{\nu\nabla^2\vec{\omega}}_{\text{allows vorticity flux dissipation}}$$

(similarly non-ideal MHD

$$\frac{\partial B}{\partial t} = \nabla\times(\vec{v}\times\vec{B}) + \underbrace{\nu_M\nabla^2\vec{B}}_{\text{allows magnetic dissipation}}$$

)

$V_{in} > V_{out}$

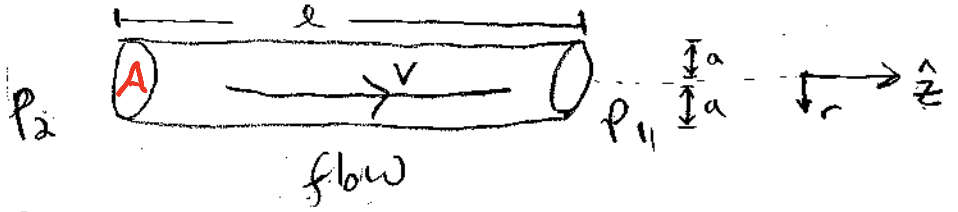
by keplerian formula

$$V = \left(\frac{GM}{r}\right)^{1/2}$$

$$v(r_1) > v(r_2)$$

Viscous flow through pipe

Consider steady flow of incompressible viscous fluid through pipe of circular cross section.



cylindrical coords

$P_2 - P_1 = \Delta p$ pressure drives the flow

$$\cancel{\frac{\partial \rho}{\partial t}} + \cancel{v\cdot\nabla\rho} + \rho\nabla\cdot v = 0$$

$$\rho\nabla\cdot v = 0$$

$$\int \rho\vec\nabla\cdot v\, dV = 0$$

$$= \int \rho v\cdot dS = 0$$

$$\downarrow$$

$$\rho VA = \text{const}$$

For incompressible flow, V should not depend on z: ρVA = constant, where A is cross sectional area. => V = constant for ρ, A constant.

Thus we have $V_z = V_z(r)$

From steady Nav-Stokes eqn:

$$-\nabla p = \cancel{\vec{V}\cdot\vec\nabla\vec{V}} + \mu\nabla^2\vec{V}$$

$(V_z\partial_z V_z = 0)$

$\mu = \rho D$

$$\Rightarrow -\frac{\Delta p}{l} = \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)$$

in cylindirical coords. (77)

2nd order equation so we need 2 boundary conditions →

First boundary condition is that $V_z = 0$ at wall of pipe. For pipe of radius a, $V = 0$ at $r = a$

Second boundary condition is that $v(r)$ profile is symmetric and smooth around $r=0$
$\Rightarrow \frac{dV}{dr} = 0$ at $r=0$.

Integrating (77) with the two bdry conds

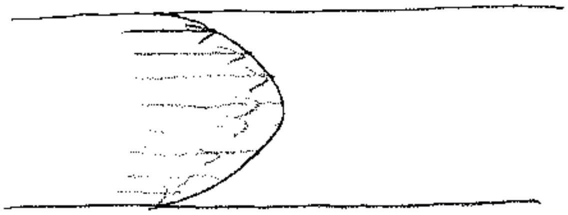
integrate once
$$-\frac{\Delta P r^{\cancel{2}}}{2\ell} = \mu \cancel{r} \frac{dV}{dr} + \cancel{C_1}^{\,0}$$
since $\frac{dV}{dr} = 0$ at $r=0$

integrate again:
$$-\frac{\Delta P r^2}{4\ell\mu} + C_2 = V$$

$C_2 = \frac{a^2 \Delta P}{4\mu\ell} \Rightarrow$

$$\boxed{v(r) = \frac{\Delta P}{4\mu\ell}(a^2 - r^2)} \qquad (78)$$

the velocity profile is parabolic!



Mass flux through pipe is given by

$$Q_M = \int_0^a \rho\, v(r) \cdot 2\pi r\, dr \quad , \quad \text{using (78)}$$

$$= \rho \frac{\pi}{2} \frac{\Delta P}{\mu l} \int_0^a (a^2 r - r^3) \qquad (79)$$

$$= \rho \frac{\pi}{2} \frac{\Delta P}{\mu l} \left(\frac{1}{4} a^4 \right)$$

$$\Rightarrow \boxed{Q_m = \frac{\pi \Delta P}{8 \nu l} a^4}$$

where $\nu = \frac{\mu}{\rho}$

(79) is Poiseuille's formula and can be used to measure viscosity of liquids! :

① measure ΔP ② measure Q ③ l, a are known from pipe shape. ①, ②, ③ imply viscosity can be measured.

The parabolic shape just described is valid for laminar flows which occur at relatively slow velocities but not valid for turbulent flows which occur at larger velocities. The sense of large" and "small" needs to be made precise ⟶

Comment on resistivity vs. viscosity

(rv1)

When Boltzmann equation is integrated separately over electrons and ions for a plasma, one obtains the "two fluid" approximation to plasma physics. When collisions are included, an additional collisional integral must be included for ion-electron collisions:

e.g. $$\frac{Df_e}{Dt} = (\text{electron-electron collision terms}) + (\text{ion-electron collision terms})$$

phase space Lagrange derivative for electrons

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{u}\cdot\nabla + \vec{F}\cdot\vec{\nabla}$$

↑ must include Electromagnetic force per mass $\frac{e(E + v_e \times B)}{m} = \frac{du}{dt}$

The ion-electron collision terms cause a drag on electrons as electrons move with respect to the ions and thus represent the source of resistive damping of the current. Let $\vec{V} = \frac{m_i n_i \vec{V}_i + m_e n_e \vec{V}_e}{n_i + n_e}$ be the bulk flow velocity where $\vec{V}_i, \vec{V}_e$ are bulk ion & electron velocities.

Roughly, in a steady state, subtracting the electron velocity fluid equation from that for the ions gives after some algebra

$$\underbrace{e\left(\vec{E} + \frac{\vec{V}\times\vec{B}}{c}\right)}_{\text{electromagnetic force}} + \underbrace{m_e \nu_{ei} V_e - m_p \nu_{ie} V_i}_{\text{drag force}} = 0 \qquad (\text{rv1})$$

ν_{ei} is frequency with which electrons encounter ions and make a $\frac{\pi}{2}$ deflection

ν_{ie} is frequency with which ions encounter electrons and make a $\frac{\pi}{2}$ deflection →

$$\nu_{ie} = \frac{m_e}{m_p}\nu_{ei}$$

electron thermal speed

outer impact radius Debye length (charge shield scale)

(rv2)

where $\nu_{ei} = n_e \sigma_{ei} U_{th,e}$

and $$\sigma_{ei} \simeq \frac{1}{2\pi}\left(\frac{e^2}{U_{th,e}^2 m_e}\right)^2 \ln\left(\frac{\lambda_D}{b_{\pi/2}}\right) \qquad \text{(rv2)}$$

inner impact radius deflection by $\pi/2$

(see next page for estimate of σ_{ei})

so then (rv1) becomes

$$e\left(E + \frac{v}{c}\times B\right) = + m_e n_e \sigma_{ei}\left(V_i - V_e\right) U_{th,e} \qquad \text{(rv3)}$$

since definition of current density is

$$J = + n_e e\left(V_i - V_e\right)$$

(rv3) =>

$$\left(E + V/c \times B\right) = \underbrace{\frac{m_e \sigma_{ei} U_{th,e}}{e^2}}_{\eta} J$$

resistivity

$$\Rightarrow \eta = \frac{m_e U_{th,e}}{e^2}\frac{1}{2\pi}\left(\frac{e^2}{U_{th,e}^2 m_e}\right)^2 \ln\left(\frac{\lambda_D}{b_{\pi/2}}\right)$$

$$= \frac{e^2}{2\pi U_{th_e}^3 m_e}\ln\left(\frac{\lambda_D}{b_{\pi/2}}\right) \qquad \text{(rv4)}$$

$$\frac{kT}{m_e} = U_{e,th}^2$$

↓

$\propto T_e^{-3/2}$ and independent of density

↑ electron temp.

→ "Spitzer" resistivity

· Estimating σ_{ei}

$$\Delta P_e = m_e \Delta u_e \simeq \underbrace{\frac{Ze^2}{b^2}}_{\text{Force}} \cdot \underbrace{\frac{b}{u_e}}_{\text{time of encounter}} = \text{momentum change per encounter} \quad (m1)$$

$\Delta u_e \simeq u_e$

· The rate of collisions with impact parameters between b & b+db for an electron with ion number density $n_i = n_e$ is

$$n_e u_e \underbrace{2\pi b\, db}_{\text{area of annulus of width db}} \doteq n u_e d\sigma = \frac{d\nu_{ei}}{d\sigma} d\sigma = d\nu_{ei}$$

=> average force: $\Delta P\, n_e u_e 2\pi b\, db \simeq m_e u_e d\nu_{ei} = dF$

Using (m1), setting $\Delta P = m_e u_e$ and integrating:

$$\Rightarrow \int dF = \frac{n_e Z^2 e^4}{m_e u_e^2} \int_{b_{min}}^{b_{max}} \frac{1}{b^2} \cdot 2\pi b\, db \simeq m_e u_e \cdot \nu_{ei} = m_e u_e^2 n \sigma_{ei}$$

$$\Rightarrow \quad = \frac{n_e Z^2 e^4}{m_e u_e^2} \ln\left(\frac{b_{max}}{b_{min}}\right) \simeq m_e u_e^2 n_e \sigma_{ei}$$

$$\Rightarrow \sigma_{ei} \propto \frac{Z^2 e^4}{m_e^2 u_e^4} \ln\left(\frac{b_{max}}{b_{min}}\right)$$

rvB

so Ohm's law is:

$$\vec{E} + \frac{\vec{v}}{c} \times \vec{B} = \eta \vec{J} \qquad (rv5)$$

Induction equation

$$\frac{\partial \vec{B}}{\partial t} = -c \vec{\nabla} \times \vec{E} \quad ; \quad \vec{E} = -\vec{v} \times \vec{B} + \eta \vec{J}$$ from Ohm's law

$\Rightarrow$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}) - c \vec{\nabla} \times \eta \vec{J} \qquad (rv6)$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi \vec{J}}{c}$$ (non relati-)

but $\vec{J} = \frac{c \vec{\nabla} \times \vec{B}}{4\pi}$ so (rv6) $\Rightarrow$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) - \frac{c^2}{4\pi} \vec{\nabla} \times \eta (\vec{\nabla} \times \vec{B})$$

use

$$\left[\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \vec{\nabla}^2 \vec{B} \right] \qquad (rv7)$$

for η = constant, and using $\vec{\nabla} \cdot \vec{B} = 0$, (rv7) $\Rightarrow$

$$\boxed{\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \underbrace{\frac{\eta c^2}{4\pi}}_{\nu_m \,=\, \text{magnetic diffusivity}} \nabla^2 \vec{B}} \qquad (rv8)$$

MHD Induction equation

take curl

$$\Rightarrow \frac{\partial \vec{J}}{\partial t} = \vec{\nabla} \times (\vec{\nabla} \times (\vec{v} \times \vec{B})) + \nu_m \nabla^2 \vec{J} \qquad (rv9):$$

↑ dissipation term for $\vec{J}$

ν_m has units of $\frac{(\text{length})^2}{\text{time}}$ just like viscosity ν

$[\ell^2/t] = [v\ell]$

BUT →

remember that from above, magnetic diffusivity

$$\nu_m = \frac{\eta c^2}{4\pi} \sim \frac{c^2 e^2}{8\pi^2 u_{th_e}^3 m_e} \sim \frac{\sigma_{ei}\, u_{th_e} m_e c^2}{e^2}$$

independent of density

whereas:

$$\frac{\mu}{\rho} = \nu \simeq u_{th,i}\, l_{mfp} \simeq u_{th,i}\frac{1}{n_i \sigma_{ii}}$$

viscosity in fluid momentum equation

thermal velocity of ions (since they carry the inertia)

mean free path of ions

depends on density and inverse dependence on cross section

- viscosity randomizes bulk flow by the action of large random excursions by inertia carrying particles. Thus "less interactions" favor HIGHER VISCOSITY
- magnetic diffusivity dissipates currents by damping relative motions between ions and electrons. Thus "more interactions" (between ions & electrons) favor HIGHER MAGNETIC DIFFUSIVITY

So ν_m and ν rvs

must both have units of $\frac{\ell^2}{t} = [v\ell]$ and thus the product of some velocity and length scale.

$\nu \sim u_{th,i} \lambda_{mfp}$, but what about ν_m?

$$\nu_m = \frac{\eta c^2}{4\pi} \sim \frac{c^2 e^2}{8\pi^2 u_{th,e}^3 m_e} \sim \sqrt{2\pi}\,\frac{c^2 \sigma_{ei}^{1/2}}{4\pi u_{th}}$$

$$\simeq c \cdot \frac{c\,\sigma_{ei}^{1/2}}{u_{th}}$$

speed of light ↓

$$c \cdot \underbrace{c \boxed{\frac{\sigma_{ei}^{1/2}}{u_{th,e}}}}_{\text{distance light travels in time for e- to cross ion-e- cross section}}$$

- time for electron to cross ion electron cross section

Viscosity is dominated by ion motion rvb

BUT thermal diffusivity and resistivity are dominated by electrons; can relate the latter two:

$$K_{Th} \simeq \lambda_{e,mfp} u_{th,e} \propto \frac{u_{th,e}}{n_i \sigma_{ei}} \propto u_{e,th}^{5} \propto T_e^{5/2}$$

$$\nu_m \simeq c \cdot C \frac{(\sigma_{ie})^{1/2}}{u_{th,e}} \propto \frac{1}{u_{th,e}^{3}} \propto T_e^{-3/2}$$

$$\Rightarrow k_{th} \cdot \nu_m \sim T_e$$

↗ Wiedemann - Franz Scaling

Note on Diffusion Equation:

e.g.

$\partial_t \vec{v} = \nu \nabla^2 \vec{v}$ why is this a "diffusion" equation?

Suppose $\vec{v} = \vec{q}(t) e^{i k_x x}$

$$\Rightarrow \frac{\partial \vec{v}}{\partial t} = -\nu k_x^2 \vec{v},$$ consider e.g. $\hat{y}$ component

$$\Rightarrow \ln v_y = -\nu k^2 t + C$$

$$\Rightarrow v_y = \underbrace{v_y(0) e^{-\nu k_x^2 t}}_{\text{exponential decay of initial pattern}} = e^{-t/\tau}$$

$\tau = \frac{1}{k_x^2 \nu}$

$\sim \frac{L^2}{\nu}$

Viscous Flows, Reynolds number, and dimensionless scaling relations

Can a model of a plane or car or astrophysical jet scaled down to "table top" size appropriately model the dynamics of the real thing?

power of dimensionless numbers

consider object of size L velocity U

thus characteristic time is $\sim L/V$

let x', v', t', w' be dimensionless units normalized to these values. Then:

$$x = x'L, \quad v = v'U, \quad t = t'\frac{L}{U}, \quad w = w'\frac{U}{L} \qquad (80)$$

recall that for incompressible flows

$$\frac{\partial \vec{w}}{\partial t} = \vec{\nabla}\times(\vec{v}\times\vec{w}) + \nu\vec{\nabla}^2\vec{w},$$

then, using (80)

$$\left|\frac{\nabla\times(\vec{v}\times w)}{\nu\nabla^2 w}\right| \simeq \frac{1}{\cancel{L}}\frac{V^{\cancel{2}}}{\cancel{L}}\cdot\frac{L^{\cancel{2}}}{\nu\cancel{V}} = \boxed{\frac{VL}{\nu} \equiv R_e}$$

we can write

$$\frac{\partial \vec{w}'}{\partial t} = \vec{\nabla}\times(\vec{v}'\times\vec{w}') + \frac{1}{R_e}\nabla^2 w' \qquad (81)$$

where $R_e = \frac{LU}{\nu}$ is the <u>Reynolds number</u>

note that ν has units of $\frac{\text{length}^2}{\text{time}}$ so R is dimensionless.

This is important: for two systems with the same R_e, the behavior is governed by (81). Thus to properly model astrophysical flows, or planes etc. in the lab, one must do experiments with same R_e

For $R_e \gtrsim 3000$, (using L as radius of pipe and u as velocity of mean flow) flow through pipe is unstable to becoming turbulent. $R_e < 3000$, flow through pipe is laminar.

empirical result from experiments

Note also that R_e appears to indicate the relative importance of the last two terms in (81) but this is not always quite right! → why?? →

For $R_e \ll 1$, (81) becomes

$$\nu \nabla^2 w' \simeq \frac{\partial w}{\partial t}$$

Stokes (1851) showed that a sphere of radius a moving through a viscous fluid with velocity V, density ρ, viscosity ν incurs drag force of $F_D = 6\pi \rho \nu a V$ this is called Stokes Law for viscous flows.

$$\propto \rho a^3 V \frac{\nu}{a^2} \simeq M V \frac{\nu}{a^2}$$

$$\simeq \rho a^2 V c_s \left(\frac{\lambda_{mfp}}{a}\right)$$

$$= \rho a^2 \frac{V^2}{Re}$$

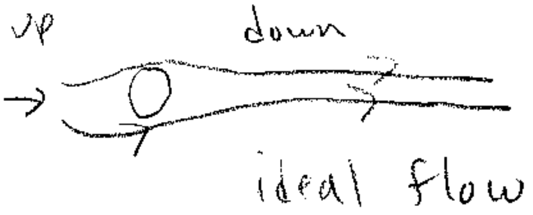
Notice that

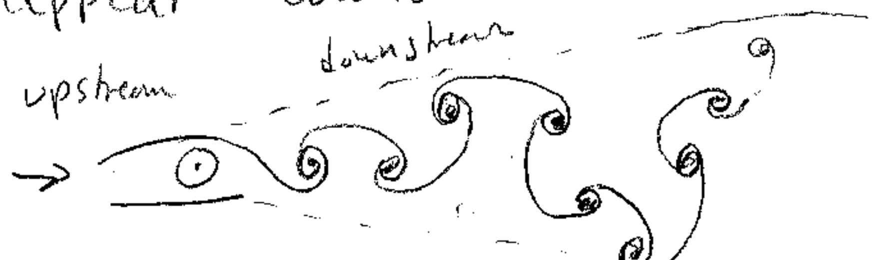
for $R_e \gg 1$, it would appear from (81) that the viscous term (the last term) can be ignored, and one might expect the system to be approximated by an ideal fluid. But it is more complicated in reality, when experiments are performed to test the drag force:

→

Flow past a cylinder for $30 > R_e > 10$

- looks like ideal flow but for $R_e > 30$ vortices begin to appear downstream.





$R_e > 70$

The vortices appear in a "wake" that increases in width farther downstream. (= "karman vortex sheet")

At very large R_e, the wake becomes turbulent, flow has large random velocities → not like ideal flow at all! what is going on to produce this highly non-ideal behavior, despite large R_e?

First, note than when turbulent wake is present, drag on cylinder or sphere much larger than Stokes Law:

In the large R regime:

Reynolds drag

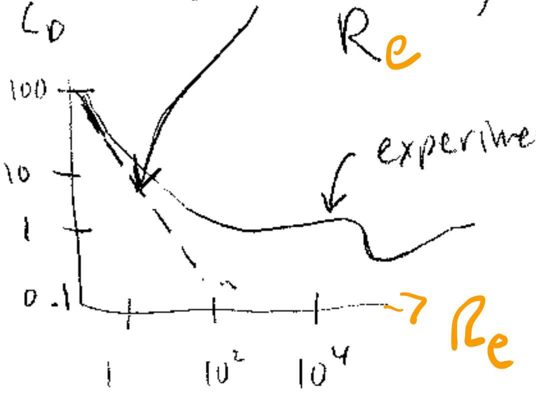
$$F_D \simeq C_D(\pi a^2)\rho\frac{V^2}{2} \quad (82)$$

where C_D can be measured

Stokes' → $= \rho a^2 \frac{v^2}{R_e}$

If the drag force always equaled the Stokes value then setting (82) equal to Stokes drag =>

$C_D = \frac{12}{R_e}$, but experimentally :



the drag coefficient falls off much more slowly at large Reynolds number, R_e

$\left\{ R_e = \frac{va}{\nu} \right\}$

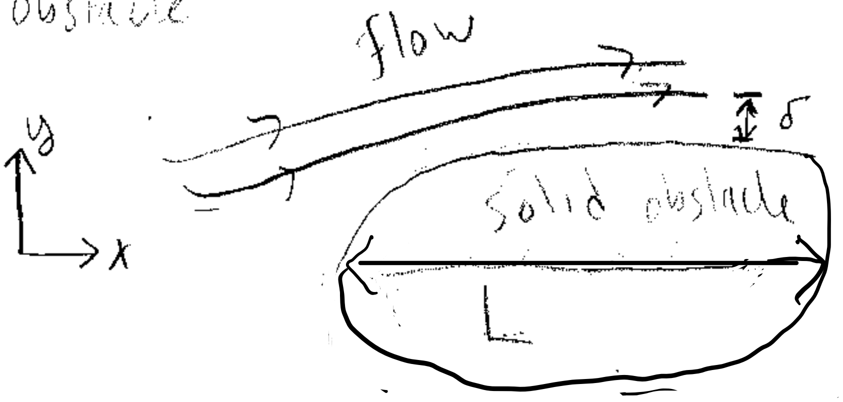
The reason has to do with boundary layers

Near to the surface of the obstacle in the flow, velocity must change from large values to zero. Since this happens over small scales, the effective R_e in that region is not much greater than 1, so near to the obstacle's surface the flow is far from ideal.

∴ the $\nu \nabla^2 v$ term in the Navier-Stokes equation becomes important because v changes on scale $\delta \ll a$, so

$$\underset{\simeq 1}{R_{eff}} \equiv \frac{V\delta}{\nu} \ll R_e \equiv \frac{Va}{\nu} \rightarrow$$

We can see also that the boundary layer grows with distance behind the obstacle:



Navier-Stokes equation for V_x is given by:

$$V_x \frac{\partial V_x}{\partial x} + \cancel{V_y \frac{\partial V_x}{\partial y}} = \cancel{-\frac{1}{\rho} \partial_x P} + \nu \left(\cancel{\frac{\partial^2 V_x}{\partial x^2}} + \frac{\partial^2 V_x}{\partial y^2} \right)$$

Assuming $V_y \lesssim V_x \delta/L$ initially and since P, V_x are not expected to vary much along x, compared to y the dominant terms are

$$\rightarrow \quad V_x \frac{\partial V_x}{\partial x} \simeq \nu \frac{\partial^2 V_x}{\partial y^2}$$

or to order of magnitude

$$\frac{V_x^{\cancel{2}}}{x \cancel{L}} \simeq \nu \frac{\cancel{V_x}}{\delta^2} \Rightarrow \boxed{\delta = \sqrt{\frac{\nu \cancel{L} x}{V_x}}} = L^{1/2} \left(\frac{x^{1/2}}{Re^{1/2}} \right)$$

boundary layer grows as square root of distance downstream!

→

Note that because the viscosity is important in the boundary layer, kelvin's vorticity theorem is violated. Thus flux of vorticity is <u>not conserved</u> there and new vortex lines can form → that explains why vorticres can develop & grow in the turbulent wake.

The reason for the development of a turbulent boundary layer is shear instabilities that develop at the sides of the obstacle from strong velocity gradients (conditions for these instabilities can be derived) the turbulence is then carried downstream. Since the turbulence is a randomization of the bulk velocity, which eventually dissipates as heat, some of the bulk energy of motion of the obstacle is lost ⇒ this is why turbulence produces a drag! Equivalently, one can think of the bulk flow energy being randomized, if object is at rest.

"Order of magnitude" estimates for Stokes, Reynolds & Epstein Drag

---

Stokes Drag: when the flow is lamilar and the object size is much larger than the mean free path of particles, the drag force $\vec{F}_d$ must depend on the flow velocity $\vec{u}$, the object size, a, the flow density ρ, and the viscosity ν. But the only combination of these quantities that produces units of force is

$$\sim \rho u \nu a \sim \underbrace{\left[\frac{\text{mass}}{\ell^3}\,\frac{\ell}{t}\,\frac{\ell^2}{t}\,\ell\right]}_{\text{(units)}} = \left[\frac{m\ell}{t^2}\right] = \underbrace{[F]}_{\text{force units}}$$

More detailed calculations produce

$\vec{F}_d = 6\pi\rho\vec{u}\nu a$ given earlier

Reynolds Drag - when the flow is fast enough that turbulence ensues, the drag no longer depends explicitly on the viscosity. Then one must construct a force with ρ, u, and a only: The combination that works is given by

Replace ν with ν_T

$\nu_T = C_0 u a \pi$

$$F_d \propto \rho u^2 \underbrace{\pi a^2}_{\text{area}} \sim \left[\frac{m}{\ell^3}\,\frac{\ell^2}{t^2}\,\ell^2\right] \sim \left[\frac{m\ell}{t^2}\right] \sim \underbrace{[F]}_{\text{force}}$$

typically a drag constant is empirically measured:

$\Rightarrow F_d \sim \pi C_d \rho u^2 a^2$

RD: $\rho\vec{u}\nu_T a \sim C_0 \rho a^2 u^2 \pi$

Epstein Drag

When the mean free path λ_{mfp} is larger than the object size, then the drag is due to collisions with individual particles

In this case the particles collide with the object at speeds sampled from the particle distribution function $f(p,x,t)$. On average however, for a quasi-Maxwellian distribution, the average particle speed is the sound speed. The drag force must depend on the object speed, the mass of the particles colliding with the object and the frequency at which this occurs. The frequeny of collisions is

$$\sim \underbrace{n}_{\text{number density}} \underbrace{(\pi a^2)}_{\text{area of object}} \underbrace{(C_s - u)}_{\text{relative velocity}} \simeq n \pi a^2 C_s$$

for subsonic $u \ll C_s$ flows.

Combining this frequeny with the particle mass and object speed to form a force requires multiplying by the particle mass and flow speed to obtain

$$F_{d,ep} \propto \underset{\substack{\uparrow \\ \text{e.g. hydrogen}}}{m_H} n \pi a^2 C_s u \sim \underbrace{\rho}_{\text{density of diffuse medium}} \pi a^2 C_s u$$

More detailed calcs give $F_{d,ep} \sim 2 \rho \pi a^2 C_s u$

Another way to think of the drag force is that the time scale for the object to change its speed by an order of magnitude is roughly

$$\tau \sim \frac{V}{\frac{dV}{dt}} \sim \omega_c^{-1} \frac{m_{obj}}{m_H} \quad (*)$$

← H atom for hydrogen dominated system

↑ collision frequency in the case of Epstein drag.

The ratio of masses appears on the right side because, it takes of order 1 collision of an H atom to change the speed of an equivalent mass in the object. Thus we require $N = \frac{m_{obj}}{m_H}$ collisions to change the object speed.

But (*) is the same as the force equation

$$m_{obj} \frac{dV}{dt} = m_H V \omega_c \simeq \rho \pi a^2 c_s V$$

derived on the previous page.

Epstein drag

Planet formation: uses all 3 drag forces.

$$\frac{\partial\vec{\omega}}{\partial t} = \vec{\nabla}\times(\cancel{\vec{v}}\times\vec{\omega}) + \boxed{\frac{1}{\rho^2}\vec{\nabla}\rho\times\vec{\nabla}P} + \nu\cancel{\nabla^2}\vec{\omega}$$

H.

Klahr

Gas Dynamics: the role of compressibility in adiabatic flows

- incompressible limit was considered when using $\vec{\nabla}\cdot\vec{v}=0$
- compressibility is required for sound waves and for shocks
- Basics of compressible flow can be considered for perfect gas.

perfect gas: $P = nk_{B}T = R\rho T =$ (83)

$R = \frac{k_B n}{\rho}$ = mass specific gas constant $= \frac{k_b}{\mu m_H}$

($\mu = 1$ for neutral H)

ε, internal energy per mass

$$\varepsilon = C_v T = \frac{RT}{\gamma-1} = \frac{P/\rho}{\gamma-1} \quad (84)$$

$(C_p - C_v = R)$

$\gamma = \frac{C_p}{C_v}$, ratio of specific heats at constant pressure and volume

for monatomic gas: $\gamma = 5/3$, $C_v = 3/2$, $C_p = 5/2$

Entropy per unit mass:

$$T ds = d\varepsilon + p\,d\left(\frac{1}{\rho}\right) \quad (85)$$

→ from (83), (84) & (85) ⇒

$$ds = C_v \frac{dT}{T} + \frac{P}{T} d\left(\frac{1}{\rho}\right)$$

$$= \frac{C_v}{(P/\rho)} d\left(\frac{P}{\rho}\right) + (\gamma-1)C_v\rho\, d\left(\frac{1}{\rho}\right)$$

$$= C_v\, d\ln P/\rho - C_v\, d\ln(\rho^{\gamma-1})$$

$$= C_v\, d\ln(P/\rho^{\gamma})$$

$$s = C_v \ln\left(\frac{P}{\rho^{\gamma}}\right) + s_0 \quad (86)$$

Note on gas constant

$$-(\gamma-1[c_v \rho]\frac{1}{\rho^\gamma}d\rho$$

$$=1=-c_v d\ln\rho^{\gamma-1}$$

$$P = n k_b T = \underbrace{\frac{N_{mol}}{V}}_{n_{mol}}\ \underbrace{\frac{\text{particles}}{\text{mole}}}_{N_A}\ k_b T$$

$$n_{mol}\, N_A\, k_b T = n_{mol}\tilde{R}\,T$$

$$= \underbrace{n_{mol}\, N_A\, \frac{\text{mass}}{\text{particle}}}_{\rho}\ \frac{k_b T}{\text{mass}/\text{particle}}$$

$$= \rho\frac{k_b T}{\mu m_H} = \rho R T$$

$$R = \frac{\tilde{R}}{N_A \mu m_H}$$

$$\mu m_H \equiv \frac{\sum_i m_i n_i}{\sum_i n_i}$$

We also know that heat dQ satisfies $dQ = dU + pdV$ (87)

and for adiabatic gas $\left(dU = m_H \mu\, d\mathcal{E} \right)$ μ = mean mol weight

$dQ = 0 = dU + pdV$ or

$\frac{ds}{dt} = 0$ along a streamline from (85).
$\left(\text{ie.} \rightarrow \frac{ds}{dt} = \partial_t s + \vec{v} \cdot \vec{\nabla} s = 0 \right)$

so that (86) =>

$$\frac{d(P/\rho^\gamma)}{dt} = 0 \quad \text{for adiabatic gas} \qquad (88)$$

The enthalpy per unit mass, is given by (89)

$$W \equiv \mathcal{E} + \frac{P}{\rho} = \frac{\gamma}{\gamma-1} RT = \frac{P}{\rho} + \frac{P/\rho}{\gamma-1} = \frac{[\gamma-\cancel{1}]\, P/\rho + \cancel{P}/\rho}{\gamma-1} = \frac{\gamma P/\rho}{\gamma-1} = \frac{\gamma}{\gamma-1} RT$$

so that

$$T ds = d\mathcal{E} + p\, d\left(\frac{1}{\rho}\right) = dW - \frac{1}{\rho} dP$$

or $dW = Tds + \frac{1}{\rho} dP$ (90)

for adiabatic flow then ...

$\int \frac{dP}{\rho} = W + \text{constant}_1$ so Bernoulli's principle

$$\frac{1}{2} v^2 + \int \frac{dP}{\rho} + \phi = \text{constant}_2 = \underline{\frac{1}{2} v^2 + \frac{\gamma}{\gamma-1} RT + \phi} \qquad (91)$$

↳ along a streamline, for adiabatic flow. →

Note that adiabatic flows are assumed to have negligible dissipation (negligible heat generation). Thus such flows are intrinsically non-viscous and to focus on effects of compressibility we go back to consideration of ideal fluid (ignoring viscosity and heat conduction) for the moment, under assumption that these transport processes operate on time scales long compared to the compressible effects considered.

Sound Waves

Lets derive sound speed :

Consider homogeneous, initially stationary flow with density ρ_0 and pressure P_0 in absence of external forces

Perturb pressure such that

$$P = P_0 + P_1(\vec{x},t) \tag{92}$$

and density responds with perturbation such that

$$\rho = \rho_0 + \rho_1(\vec{x},t) \tag{93}$$

velocity

$\vec{v} = \vec{V}_1(x,t)$ where subscript 1 indicates perturbed quantities

→

We can write: $\left(\frac{\Delta P}{\Delta \rho} \simeq \frac{P - P_0}{\rho - \rho_0} = \frac{P_1}{\rho_1} \simeq \frac{dP}{d\rho} \right)$ (71)

$$P_1 = \underbrace{\frac{dP}{d\rho}} \rho_1$$

$$\equiv C_s^2 \rho_1 \quad \text{where} \quad \frac{dP}{d\rho} \equiv C_s^2 \qquad (94)$$

(definition)

assuming perturbations evolve on times short compared to viscous or conductive transport the flow is adiabatic, then (88) & (94) imply:

$$\boxed{C_s = \sqrt{\frac{\gamma P_0}{\rho_0}}} \qquad P = K\rho^\gamma \qquad (95)$$

which is the adiabatic sound speed.

$$\frac{dP}{d\rho} = C_s^2 = \frac{\gamma P}{\rho} \simeq \frac{\gamma P_0}{\rho_0}$$

perturbed quantities satisfy continuity eqn:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{V}_1 = 0, \qquad (96)$$

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

$$\partial_t \rho_1 + \underset{0}{\cancel{\vec{V}_0 \cdot \nabla \rho}} + \rho \vec{\nabla} \cdot \vec{v}_1 = 0$$

(where we assume $\rho_1 \ll \rho_0$, $P_1 \ll P_0$, $v_0 = 0$ and neglect quadratic terms in perturbed quantities.)

and recall ρ_0, P_0 are constant in space

Momentum (Euler) eqn:

$$(\rho_0 + \underbrace{\rho_1}_{\text{small}}) \left[\frac{\partial \vec{V}_1}{\partial t} + \overset{\text{small}}{\cancel{(\vec{V}_1 \cdot \vec{\nabla})\vec{V}_1}} \right] = -\vec{\nabla} P_1 \qquad (96a)$$

after linearizing: $\Rightarrow$

$$\rho_0 \frac{\partial \vec{V}_1}{\partial t} = -\vec{\nabla} P_1$$

$$= -C_s^2 \vec{\nabla} \rho_1, \qquad (97)$$

using (94).

time derivative of → $\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \vec{\nabla}\cdot\partial_t \vec{v}_1 = 0$

divergence of → $\rho_0 \partial_t \vec{\nabla}\cdot\vec{v}_1 = -c_s^2 \nabla^2 \rho_1$

combining time derivative of (96) and divergence of (97)

$$\Rightarrow \quad \frac{\partial^2 \rho_1}{\partial t^2} - c_s^2 \nabla^2 \rho_1 = 0 \qquad (98)$$

which is a wave equation for acoustic waves propagating at speed $\underline{c_s}$.

Note: isothermal case, $P_0 =$ constant ρ_0

$P = \underline{\rho} kT \Rightarrow P \propto \rho$; $\frac{}{m_H}$; $\gamma = 1$

$$\Rightarrow c_s = \sqrt{\frac{dP}{d\rho}} = \left(\frac{P_0}{\rho_0}\right)^{1/2}$$

1 atm
≃ 1 bar
≃ 10^5 pascals
1 dyn ≃ 0.1 pascals

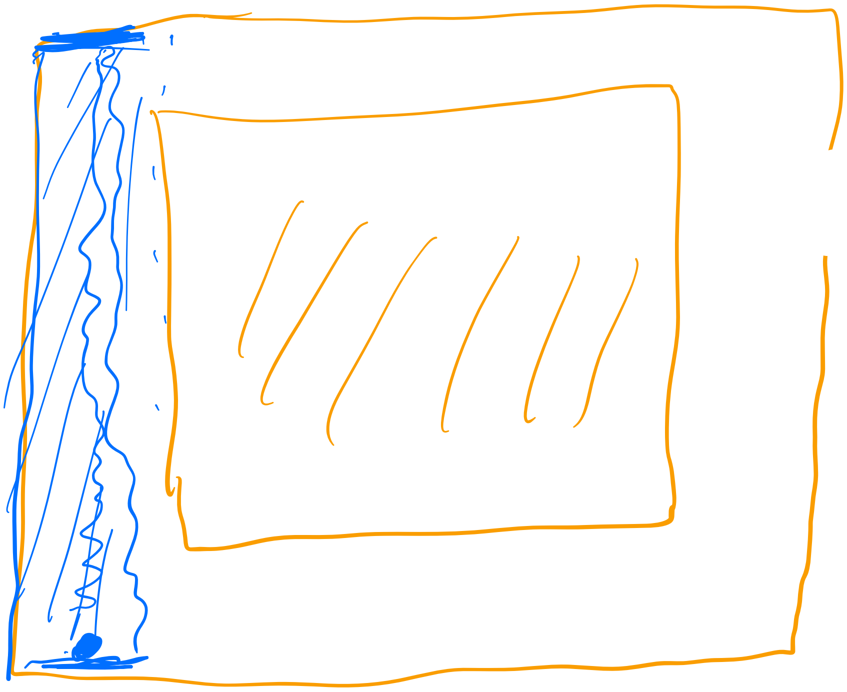
1atm ≃ 10^6 dyn
= erg/cm^3

in air, at 0°C; $P_0 = 1\text{atm} = 10^6$ dyn; $\rho_0 \simeq 10^{-3} \frac{g}{cm^3}$ and atmospheric pressure, $c_s = 2.8 \times 10^4 \frac{cm}{s}$ but this is lower than what is measured experimentally (Newton 1689)

Laplace (1816) was first to take adiabatic case, $\gamma = 1.4 \Rightarrow c_s = 3.32 \times 10^4$ cm/s which agreed with experiment $\left(c_s = \sqrt{\frac{\gamma P_0}{\rho_0}} \right)$

- In liquids do you expect sound speeds to be higher or lower than in gasses?

Higher since they are harder to compress so for given dP, $d\rho$ is smaller and $c_s^2 = \frac{dP}{d\rho}$ is then larger.



73

For linear perturbation analysis, superposition holds and we can decompose perturbation into Fourier components

$$\rho_1 = \tilde{\rho}_1 \exp[i(\vec{k}\cdot\vec{x}-\omega t)] \qquad (99)$$

plugging into (98) => dispersion relation

$$\omega^2 = c_s^2 k^2 \qquad \Rightarrow \left[\omega \frac{\lambda}{2\pi}\right] \simeq c_s \qquad (100)$$

which applies only for simple, initially homogeneous medium (remember we ignored quadratic terms and $\vec{v}_0$)

note that acoustic waves of all frequencies travel with same speed, and group & phase velocities are equal (=> non-dispersive waves)

For stratified atmosphere, group & phase velocities are not equal, and sound waves are dispersive.

Note that $\vec{k}$ gives direction of wave propagation.

(97) => $\vec{v}_1 \parallel \vec{k}$ so sound waves are longitudinal. Since $\vec{\nabla}\cdot\vec{v}_1 \propto \frac{k^2}{\omega}\rho \propto e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

$\vec{\nabla}\cdot\vec{v}$ represents alternating compressions & rarefactions

//

Shocks

For large amplitude waves, quadratic perturbation terms cannot be neglected, in particular the term $\vec{v}_1 \cdot \vec{\nabla} v_1$ [in fluid momentum eqn. (96a)]

More specifically, perturbation approach does not really apply for large amplitude waves.

Consider the 1-D Euler equation; and consider x-direction as both wave propagation and direction of fluid velocities: ($\underline{v = v_x}$ here)

$$\frac{\partial v}{\partial t} + \underbrace{v\frac{\partial v}{\partial x}}_{v^2/\Delta x} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\sim c_s^2/\Delta x} \qquad (101)$$

To assess the influence of the second term we drop the last term and solve the simpler equation for $v(x,t)$:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0 \qquad (102)$$

Consider the curves $dx/dt = v$ in the x,t plane.

note that $\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\frac{dx}{dt}$ so (102) $\Rightarrow$

$\frac{dv}{dt} = 0$ along any curve for which $v = \frac{dx}{dt}$.

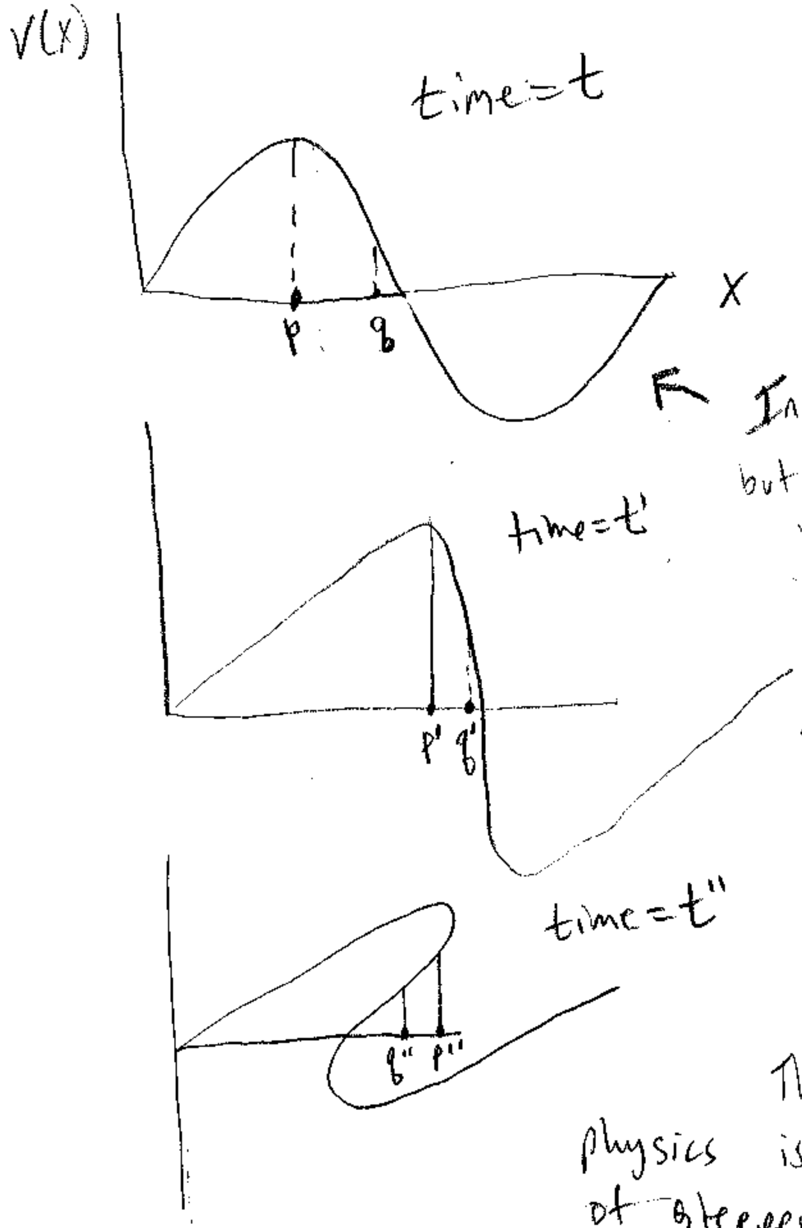
→

We started with partial dif eqn but ended with ordinary diff eqn:

$$\frac{d\vec{v}}{dt} = 0 \qquad (103)$$

(Curves given by (103) are called "characteristic curves" of eqn (102), and $\vec{v}$ is the Riemann invariant)

Consider an initial velocity profile:



under the action of (102), profile evolves as sequence on the left.

Initially p moves faster than q but comoving velocity is constant so

$p-q > p'-q'$:

points become closer together along their trajectories.

eventually p'' overtakes q'' suggesting velocity profile becomes multivalued.

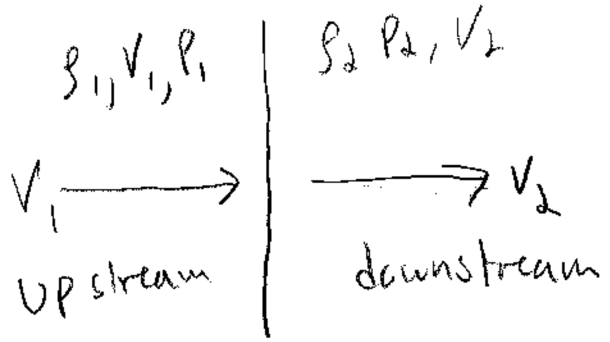
This is unphysical and more physics is need, but represents the onset of steepening of waves to form shocks

→

Shock structure

note how even initially smooth wave can steepen to form a shock. Shock is small region over which fluid variables change dramatically. To see what the change is like we can think of it as discontinuity and solve the "jump conditions"

consider shock propagating in undisturbed medium of density ρ_1, P_1 and lets move into frame at which shock is at rest.



Now we can appeal to <u>flux conservation</u> equations to understand how quantities change across the shock:

Continuity equation: flux of particles

$$\partial_t \rho + \nabla\cdot(\rho \vec{V}) = 0 \qquad (103)$$

Momentum can also be written as conservation flux equations :

$$\rho \partial_t \vec{v} = -\rho \vec{v}\cdot \nabla \vec{v} - \vec{\nabla} P \quad \text{or}$$

$$\partial_t (\rho \vec{v}) = \vec{v}\partial_t \rho - \rho \vec{v}\cdot\vec{\nabla}\vec{v} - \vec{\nabla} P$$

use (103) $\Rightarrow$ $$= -\vec{v}\,\vec{\nabla}\cdot(\rho\vec{v}) - \rho\vec{v}\cdot\nabla\vec{v} - \vec{\nabla}P$$

$$\Rightarrow \partial_t(\rho v_j) = -\partial_i(\rho v_i v_j) + \cancel{\rho\vec{v}\cdot\nabla\vec{v}} - \cancel{\rho\vec{v}\cdot\nabla\vec{v}} - \partial_j P$$

$$\Rightarrow \underbrace{\partial_t(\rho v_j) + \partial_i(\rho v_i v_j + P\delta_{ij}) = 0}_{\text{momentum equation in form of conservation law}}$$

with use of (103) the Euler momentum equation can be written as flux of momentum eqn:

$$\partial_t(\rho V_j) + \partial_i(\rho V_i V_j + P\delta_{ij}) = 0 \qquad (104)$$

Energy equation can be manipulated by use of (103) & (104) to also take form of flux conservation eqn:

$$\partial_t\left(\rho\varepsilon + \frac{1}{2}\rho v^2\right) + \vec{\nabla}\cdot\left[\rho\vec{v}\left(\frac{1}{2}v^2 + \overbrace{\frac{P\gamma}{(\gamma-1)\rho}}^{\text{enthalpy}}\right)\right] = 0 \qquad (105)$$

In (103), (104) (105) we have ignored viscous terms and thermal conduction. In steady state

$$(103) \Rightarrow \quad \underline{\int \rho\vec{v}\cdot d\vec{S}} = 0 = \int \vec{\nabla}\cdot(\rho\vec{v})\, d\vec{V} \qquad (106)$$

Consider a small box which spans across the discontinuity:

$$-\rho V_1 \cancel{S}_1 + \rho V_2 \cancel{S}_2 = 0$$

then $(106) \Rightarrow \boxed{\rho_1 V_1 = \rho_2 V_2}$ for $S_1 = S_2$ (107)

similarly, for steady state (104) and (105) imply that for 1-D flow:

$$\boxed{P_1 + \rho_1 V_1^2 = P_2 + \rho_2 V_2^2} \qquad (108)$$

$$\boxed{\frac{1}{2} V_1^2 + \frac{\gamma P_1}{(\gamma-1)\rho_1} = \frac{1}{2} V_2^2 + \frac{\gamma P_2}{(\gamma-1)\rho_2}} \qquad (109)$$

Now we have 3 equations (107, 108, 109) for 6 variables ($\rho_1 V_1 P_1$; $\rho_2, V_2 P_2$)

Eliminating P_2 & V_2, + algebra =>

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}, \quad \text{where} \qquad (110)$$

$$M \equiv \frac{V_1}{\sqrt{\gamma P_1/\rho_1}} = \frac{V_1}{C_{s_1}}$$ is the Mach Number

and measures the speed of the upstream flow, or the speed at which shock is propagating into upstream flow as measured in the frame where upstream is at rest. Note we expect $M > 1$ for shock: if $M < 1$, then shock could produce acoustic waves moving with sound speed, and there would be no pile up of wave fronts, so any discontinuity would not survive.

We can also write (110)

as $\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{(\gamma-1) + 2/M^2}$ thus, for $M > 1$ (110)

$P = \rho \cancel{kT}$ -- $P \propto \rho^{\cancel{\gamma}}$

$\rho_2/\rho_1 > 1$ and the ratio increases with M. $\gamma = 1$

Thus a faster shock ↔ more compression

As $M \to 1$ $\rho_2/\rho_1 \to 1$: no shock.

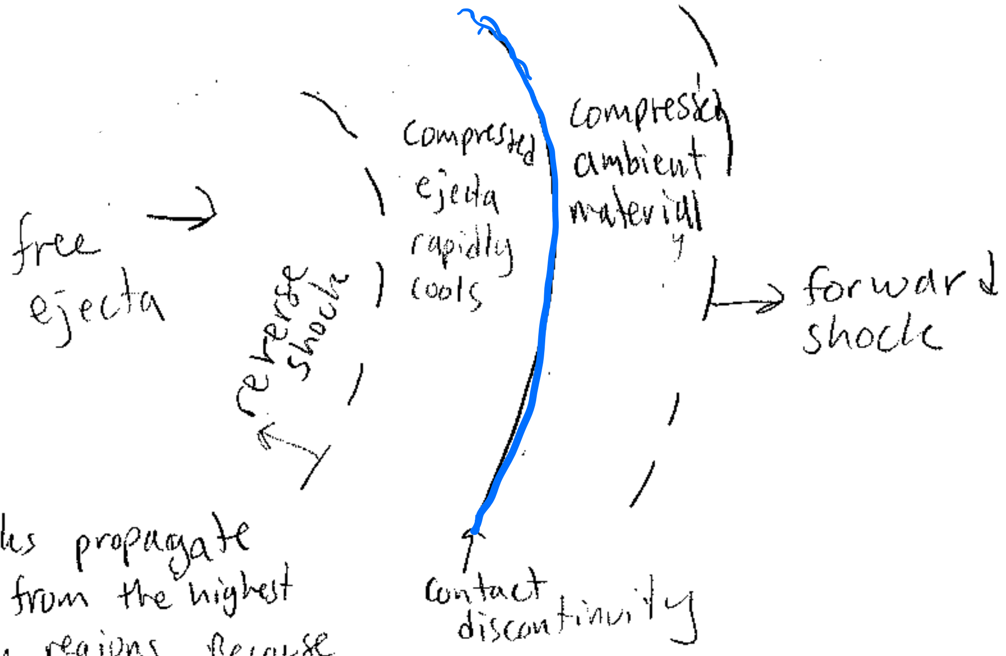
The above calculation was done in the shock frame. For non-relativistic flows the compression ratio (111) is unchanged in the lab frame. In the lab frame, the shock advances into the undisturbed medium, compressing the flow behind it. Note for a monatomic gas $\gamma = 5/3$ and $\frac{\rho_2}{\rho_1} = 4$ as $M \to \infty$. This is maximum compression for a non-relativistic non-radiating gas.

We have neglected viscosity, assuming shock is infinitely thin, but indeed it is the viscosity and transport that determine the shock thickness. Also radiative shocks can have larger compression ratios. Why?

Some aspects of shock propagation through Supernova envelope and ambient Interstellar Medium

S1

- deep in star where energy from outward propagating material comes from radioactivity, thermalization occurs with temp in opt.-UV range.
- when outflow becomes optically thin, effective "temperature" <u>goes up</u> [that is, γ and X-ray photons are not down scattered efficiently so we see high energy non-thermal emission
- Source of energy eventually changes from radioactive decay, to conversion of bulk flow energy at shock (remember shocks are sites of bulk flow dissipation)
- Forward shock <u>and</u> Reverse shocks are present:



[Shocks propagate away from the highest density regions. Because of rapid cooling by Bremsstrahlung in the compressed regions, the high density region also supersonically migrates "backward" into the free ejecta in rest frame away from contact discontinuity →

- note that the ejecta, contact discontinuity and reverse shock are all moving outward in the lab frame, but in the frame of the contact discontinuity there are shocks propagating both outward = forward and inward toward the explosion point = reverse shock.
- forward & reverse shocks are important concepts throughout supersonic astrophysics (jets, GRB, etc..)
- The supernova Remnant SNR (scales $\gtrsim 1000$ AU)

emits by conversion of bulk flow energy at shock: ejecta has kinetic energy

$$\simeq 10^{51}\,\mathrm{erg} = \frac{1}{2} M_{ej} V_{ej}^2$$

$$M_{ej} \simeq 2 M_\odot \Rightarrow V_{ej}^2 \gtrsim 10^{18} \frac{\mathrm{cm}^2}{\mathrm{s}} \Rightarrow V_{ej} \simeq 10^9 \frac{\mathrm{cm}}{\mathrm{s}}$$

$\Rightarrow$ "temperatures" as high as 10^8-10^9 K (using $v = \left(\frac{kT}{m_p}\right)^{1/2}$).

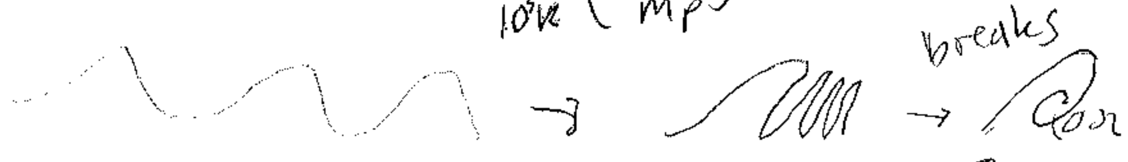
- but there is an important subtlety as the shock reaches these scales $\gtrsim 1000$ AU

Lets look a bit at the shock physics

$\rightarrow$

- Recall from our brief discussion, shocks form as waves steepen non-linearly

waves are calculated as linear perturbations of the hydro equations. They move at speed $\simeq c_s$ for un-magneized plasma. Because pressure disturbance from ejecta moves at $V_{eject} \gg \left(\frac{\underset{10^3 K}{T_{ISM}} k_b}{m_p}\right)^{1/2}$ waves pile up:



randomization of ordered motion → heating. entropy increase

The role of "nonlinearity" arises in the Navier-Stokes equation (fluid momentum)

$$\frac{\partial \vec V}{\partial t} = -\underbrace{\vec V\cdot\vec\nabla\vec V}_{\text{non-linear term}} - \frac{\vec\nabla P}{\rho} + \underbrace{\nu \nabla^2 \vec V}_{\text{Important when there are large gradients, even for small viscosity } \nu} \tag{91}$$

dissipation term, increases entropy

Viscosity is always approximately $\backsim$ speed × length: Typically, for ambient ISM into which shock propagates $\nu \simeq c_s l_{mfp}$.

$\uparrow$ sound speed $\quad \uparrow$ mean free path

Because "non-linear" effects induce dissipation we know shocks are important where $|V\cdot\nabla V| \simeq \nu\nabla^2 V$ (92)

→

Eqn (92) $\Rightarrow$ $|\vec{v}\cdot\vec{\nabla}v| = |\nu_{eff}\vec{\nabla}^2\vec{v}|$ (we'll see why $\nu \to \nu_{eff}$ later)

$$v^{\cancel{2}}/\cancel{\ell} = \nu_{eff}\cancel{v}/\ell^{\cancel{2}}$$

or $\ell = \frac{\nu_{eff}}{V_1} \simeq \frac{c_{s_2}\ell_{eff}}{V_1}$ (93) ← Some "effective" mean free path

In vicinity of shock, the velocity transits from $V_1 \gg c_{s_1}$ to $V_2 \lesssim c_{s_2}$. But $V_2 \simeq \frac{V_1}{4}$ (when no cooling)

So eqn (93) $\Rightarrow$ $\ell \simeq \ell_{eff}$ should be the scale over which the flow changes from "upstream" to "downstream" Typically, therefore we expect the shock thickness to be $\simeq \ell_{eff}$. (In reality, instabilities broaden the shock somewhat, but put that aside for the moment). Now let us estimate ℓ_{eff} for supernova remnants : At ejection velocity

$V_{ej} \simeq 10^9 \frac{cm}{s}$ Kinetic energy per proton in the ejecta is about 2 MeV. As these protons hit (largely neutral) H atoms of the ISM, the latter will ionize.

Cross section of interaction is $\underline{\sigma_{ion}} \simeq 10^{-17} cm^2 \left(\sim \frac{\hbar^2}{m_p^2 V_1^2}\right)$

Energy lost per ionization is $\simeq$ 50 eV (which represents the inelastic part of the collision) The stopping distance of the impinging protons is therefore

$$\ell_{eff} = \frac{E}{\frac{dE}{d\ell}} \simeq \frac{E}{\Delta E}\ell_{mfp,\underline{im}} = \frac{2\,MeV}{50\,eV}\frac{1}{n\sigma_{ion}} \quad (94)$$

for $n \simeq 1/cm^3$

Stopping length $\Rightarrow$ $\ell_{eff} \simeq 4\times10^4 \cdot 10^{17} \simeq 4\times10^{21}\,cm \simeq 10^3\,pc$! for 2 MeV proton in the ejecta

But shock thicknesses observed are <u>MUCH</u> smaller than 10^3pc. In fact the entire remnants become "invisible" (merged with ambient medium) on scales of 50pc. Thus, how can thin shock form if the scale ℓ_{eff} were actually 10^3pc??

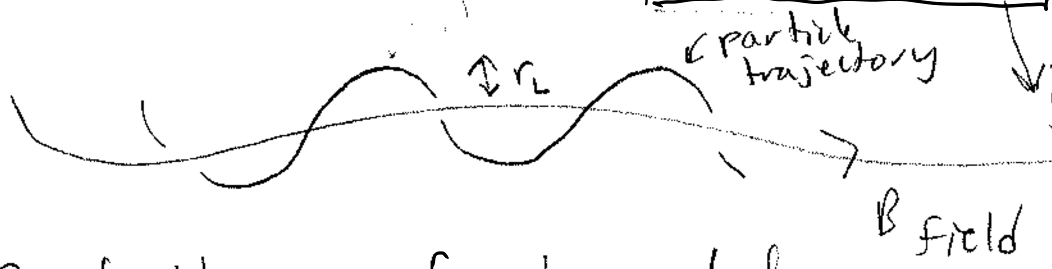
Here the answer is <u>magnetic fields</u>! Calculate the Larmor radius for microgauss field:

$$\ell_L \equiv \frac{m c V_{th}}{eB} \simeq \frac{(10^{-24}\,g)(3\times10^{10}\,\frac{cm}{s})(10^{9}\,\frac{cm}{s})}{(4\times10^{-10})(3\times10^{-6}\,G)}$$

$$\simeq 2.5\times10^{10}\,cm \;\boxed{\simeq 10^{-8}\,pc}$$

note very small strength field of ISM

very small

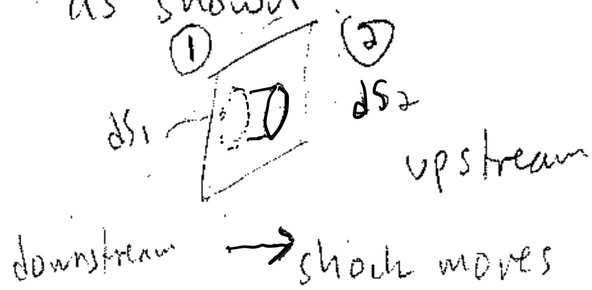


(and replaces the ℓ_{mfp} in (94) with much smaller value so ℓ_{eff} is then also much smaller)

B-fields are fundamental for "collisionless shocks" in astrophysics. They make the "effective mean-free path" equal to the Larmor radius which is much smaller than the collisional mfp even for extremely weak magnetic fields.

More on shock Jump conditions
And Application to Supernova Blast wave

- Assume that the shock represents a thin discontinuity. [This was justified by the role of magnetic fields disussed above]
- Conservation of mass, energy & momentum can all be written $\partial_t Q + \vec{\nabla}\cdot\vec{F}_Q = 0$, as disussed
- If we integrate such a conservation law across the thin discontinuity using the pill box as shown:



In steady state:

$$\cancelto{0}{\partial_t Q} + \vec{\nabla}\cdot\vec{F}_Q = 0 \Rightarrow \vec{\nabla}\cdot\vec{F}_a = 0$$

but volume is arbitrary

so that $$\int \vec{\nabla}\cdot\vec{F}_Q \, d^3x = 0 \underset{\uparrow}{=} \int_{\vec{S}} \vec{F}_Q\cdot d\vec{S} \qquad (95_s)$$

by Gauss' theorem

For mass continuity:

$$\cancelto{0}{\partial_t \rho} + \nabla\cdot(\rho u) = 0 \Rightarrow \oint \rho\vec{u}\cdot d\vec{S} = 0$$

$$\rho_1 u_1 \cancel{dS_1} - \rho_2 u_2 \cancel{dS_2}$$ $dS_1 = dS_2$ for pill box

$$\Rightarrow \boxed{\rho_1 u_1 = \rho_2 u_2} \qquad (96_s)$$

similarly: for flows in which B-field is energetically negligible:

$$w_1 + \frac{1}{2}V_1^2 = w_2 + \frac{1}{2}V_2^2$$ energy conservation (97s)

$$P_1 + \rho_1 V_1^2 = P_2 + \rho_2 V_2^2$$ momentum flux conservation (98s)

(w = enthalpy density $= \frac{\gamma}{\gamma-1}\frac{P}{\rho} = \frac{c_s^2}{\gamma-1}$)

96s–98s are the Rankine-Hugoniot jump conditions

Again we define $M_1^2 \equiv V_1^2/c_{s_1}^2$

Solving (96s–98s) (I leave as exercise)

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2 + M_1^2(\gamma-1)} = \frac{V_1}{V_2} \quad (99s)$$

(100s)

$$\frac{P_2}{P_1} = \frac{(\gamma+1) + 2\,(M_1^2-1)}{\gamma+1}$$

$$\frac{c_{s_2}^2}{c_{s_1}^2} = \frac{T_2}{T_1} = \frac{\left[(\gamma+1) + 2\,(M_1^2-1)\right]\left[(\gamma+1) + (\gamma-1)(M_1^2-1)\right]}{\left[(\gamma+1)^2 M_1^2\right]} \quad (101s)$$

$$P_2 = P_1\,\frac{2M_1^2}{\gamma+1} = \frac{2P_1V_1^2}{(\gamma+1)c_{s_1}^2} = \frac{2}{\gamma(\gamma+1)}\rho_1 V_1^2 \quad (106a)$$

$M_1^2 \gg 1$.

Assume flow is supersonic on side 1

so $M_1 = \frac{v_1}{c_{1s}} > 1$.

Then $$\frac{P_2}{P_1} > 1, \quad \frac{\rho_2}{\rho_1} > 1, \quad \frac{v_2}{v_1} < 1, \quad \frac{T_2}{T_1} > 1.$$

strongest shock $\Rightarrow M_1^2 >> 1$

$$\Rightarrow \underbrace{\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}}_{\text{(limiting relation as } M_1^2 \to \infty!)} \; ; \quad \frac{P_2}{P_1} >> 1, \quad \frac{T_2}{T_1} >> 1 \qquad (102s)$$

$\Rightarrow$ for $\gamma = 5/3 \Rightarrow \frac{\rho_2}{\rho_1} = 4$

(Note: momentum conservation and mass conservation are usually satisfied as in 96s & 98s, but energy conservation can have important radiative terms, chemical reaction terms, thermal conduction..., we ignore these for the moment.)

the above treatment assumes that the viscous terms operate only in the thin layer of the shock itself; this gets back to our notion from the earlier disussion that the shock thickness can be estimated by comparing dissipative & bulk velocity terms:

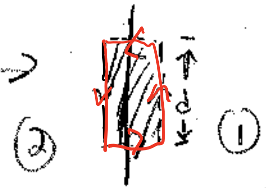
In momentum equation, compare $\vec{v}\cdot\vec{\nabla}\vec{v}$ term to $\nu\nabla^2\vec{v}$ term: (see page 93)

$$\Rightarrow \frac{V^{\cancel{2}}}{\cancel{\ell}} \sim \frac{V_{eff}}{\ell^{\cancel{2}}}\cancel{V} \Rightarrow V = \frac{V_{eff}}{\ell_{eff}}$$

, where V_{eff} is the effective 'viscosity' at the shock.

Now across the shock, the bulk energy of the flow in V_1 gets converted to random thermal energy such that $C_{s_2} \sim V_1$. As discussed earlier the scale l_{eff} is determined by multiples of Larmor radius rather than collisional mean free path

The shock is actually a "current" sheet" when B-field included in jump conditions. This is because maxwells equation require that tangential component of E is conserved across the shock: again consider pill surface across the shock →



from Maxwells equations:

$$\underbrace{\cancel{\frac{1}{c}\frac{\partial \vec{B}}{\partial t}}}_{0 \text{ in steady state}} + \vec{\nabla}\times\vec{E} = 0 \Rightarrow \int(\vec{\nabla}\times\vec{E})\cdot d\vec{S} = 0$$

stokes thm

surface is arbitrary

$$\Rightarrow \oint \vec{E}\cdot d\vec{l} = 0$$

for arbitrarily thin pill surface only sides contribute: (T = "tangential")

$$\Rightarrow \oint \vec{E}\cdot d\vec{l} = 0 = E_{T,1}d - E_{T,2}d = 0$$

$$\Rightarrow \boxed{E_{1,T} = E_{2,T}} \longrightarrow$$

Since Ohms law implies

$$\vec{E} = -\frac{\vec{V}}{c}\times\vec{B} + \eta\vec{J}$$ then

$$E_{1,T} = E_{2,T}$$

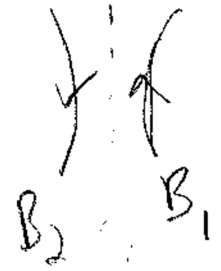
$$\Rightarrow \left(-\frac{\vec{V}}{c}\times\vec{B} + \eta\vec{J}\right)_{1,T} = \left(-\frac{\vec{V}}{c}\times\vec{B} + \eta\vec{J}\right)_{2,T}$$

small away from shock BUT where $\nabla\times B$ increases it can become large

but $\vec{J} = \frac{c}{4\pi}\vec{\nabla}\times\vec{B}$ and away from shock, $\frac{\eta c}{4\pi}\vec{\nabla}\times\vec{B}$ can be considered small;

η is the resistivity and most astro-plasmas have low resistivity. However, near the shock $|\vec{\nabla}\times\vec{B}| \simeq \left|\frac{B}{\ell_{eff}}\right| \simeq \frac{B}{\ell_L}$ ← larmor radius

The gradient scale is small and near the shock $\eta\vec{J}$ is important. This is why a shock is a "current sheet." Magnetic Reconnection provides another example of a current sheet based on same principle: magnetic field annihilation at dotted interface: **Exercise**: show that interface is a current sheet if interface is thin!



(reconnection event)

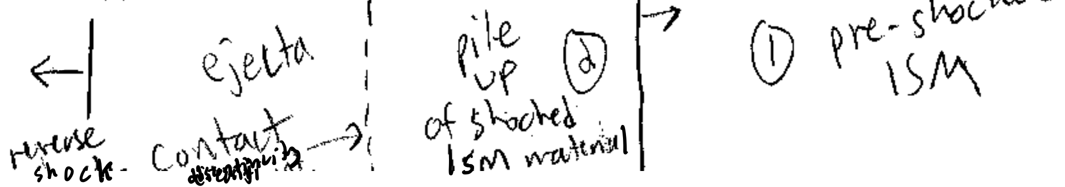
Now back to the evolution of the expanding SN shock: Transition to Sedov phase

During the early stages of the propagation of the optically thin phase of the shock's progress through the envelope and into ISM, the ejecta material has much more inertia than the ISM with which it interacts. The ejecta speed V_1, is thus constant $\Rightarrow r \propto t$

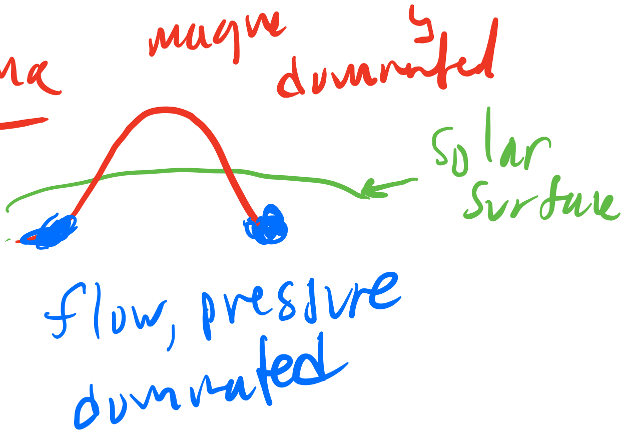
BUT: There exists a critical radius r_c at which the ejecta mass $M_{ejecta} = \frac{4}{3}\pi \rho_{ism} r_c^3$.

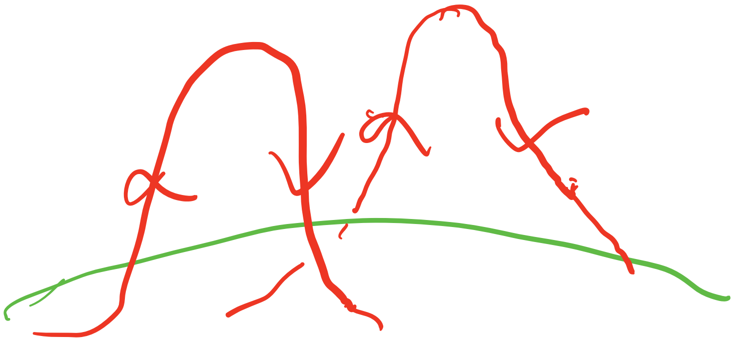
At this point the blast enters the Sedov phase. Now the mass is piling up behind the shock and this mass starts to dominate the total mass of the ejecta.

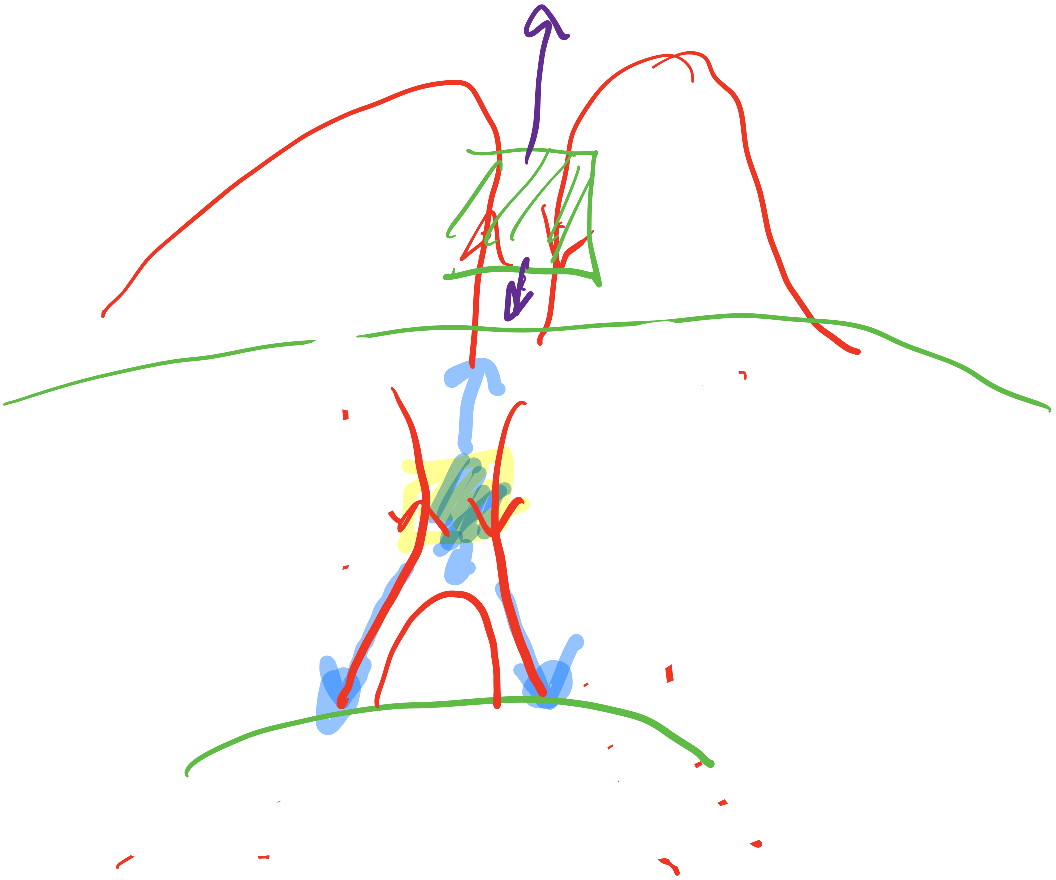
The mass piles up behind the shock, but ahead of the contact discontinuity:



Corona







Once the Sedov phase is underway, the speed of the blast wave is no longer constant: In the Sedov phase mass is dominated by that accumulated from ISM so the Energy is

$$E \simeq \frac{1}{2}\frac{4\pi}{3}\left(\rho_{ISM}\underbrace{r}_{\text{radius of expanding shell}}{}^{3}\right)v_1^2 = \underbrace{\text{constant}}_{\uparrow\ \text{just the initial explosion energy}} \qquad (103_s)$$

constant $\rho_{ism} \Rightarrow$

$$E \propto r^3 v_1^2 \Rightarrow r^3\left(\frac{dr}{dt}\right)^2 = \text{constant} = C$$

$$\Rightarrow r^{3/2}dr = C\,dt \Rightarrow \frac{2}{5}r^{5/2} = t + \cancelto{0}{t_0}$$

$$\Rightarrow \underline{r = (\text{constant})\,t^{2/5}}. \qquad (104_s)$$

Another way to arrive at this is to note that ρ_{ism} and E are constant and

$$E \simeq \frac{1}{2}M\left(\frac{r}{t}\right)^2 = \text{const} \qquad (105_s)$$

$$\rho_{ism} = \frac{M}{\frac{4\pi}{3}r^3} = \text{const.} \qquad (106_s)$$

eliminate M

$$\Rightarrow \frac{E}{\rho_{ism}} = \text{const} = \frac{2\pi}{3}\frac{r^5}{t^2} \Rightarrow r = \left(\frac{Et^2}{\rho_{ism}}\right)^{1/5} \qquad (107_s)$$

→

1 au = 1.5×10^{13} cm   1pc = 3×10^{18} cm

$$\Rightarrow \quad r = \left(\frac{E}{\rho_{ISM}}\right)^{1/5} t^{2/5} = 3\,\mathrm{pc} \left(\frac{E}{10^{51}\,\mathrm{erg}}\right)^{1/5} n_{ISM}^{-1/5} \left(\frac{t}{300\,\mathrm{yr}}\right)^{2/5} \qquad (108_s)$$

(applies only for $r > r_{crit} \simeq \left(\frac{3 M_{ejecta}}{4\pi\rho}\right)^{1/3}$)

$$\Rightarrow \quad V_1 \simeq \frac{r}{t} \simeq \underline{3 \times 10^3 \,\frac{\mathrm{km}}{\mathrm{s}}} \left(\frac{E}{10^{51}\,\mathrm{erg}}\right)^{1/5} n_{ISM}^{-1/5} \left(\frac{t}{300\,\mathrm{yr}}\right)^{-3/5} \qquad (109_s)$$

number density

using $V_1 \simeq C_{s_2} \Rightarrow$

$$T \simeq \frac{m_p}{K_b} C_{s_2}^2 \simeq \frac{m_p}{K_b} V_1^2 \simeq 9 \times 10^8\,\mathrm{K} \left(\frac{E}{10^{51}\,\mathrm{erg}}\right)^{2/5} n_{ISM}^{-2/5} \left(\frac{t}{300\,\mathrm{yr}}\right)^{-6/5} \qquad (110_s)$$

$\Rightarrow$ at $\underline{t \simeq 3.5 \times 10^4\,\mathrm{yr},\ T \simeq 3 \times 10^6\,\mathrm{K}}$

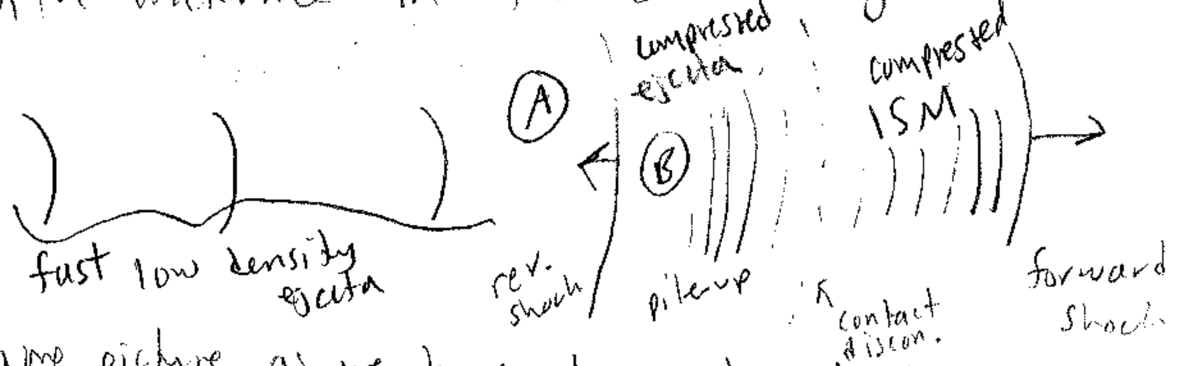
Thus if SNR is observed with $T \simeq 3 \times 10^6$ K

→ then the time in sedov phase to reach that stage is, from (110)

$$t_{sedov} = 3.5 \times 10^4\,\mathrm{yr} \left(\frac{T}{3 \times 10^6}\right)^{-5/6} \left(\frac{E}{10^{51}\,\mathrm{erg}}\right)^{1/3} n_{ISM}^{-1/3} \qquad (111_s)$$

$\Rightarrow$ for given V or T and r observed (to determine if $r > r_{crit}$) age can be determined →

Now, as deceleration becomes significant, the outer shells of expanding sphere decelerate first => material in the inner region catches up with material in the outer regions



(same picture as we have discussed earlier)

Region Ⓐ is supersonic with respect to Ⓑ => reverse shock moves "backward" in frame of contact discontinuity. In lab frame everything is moving outward.

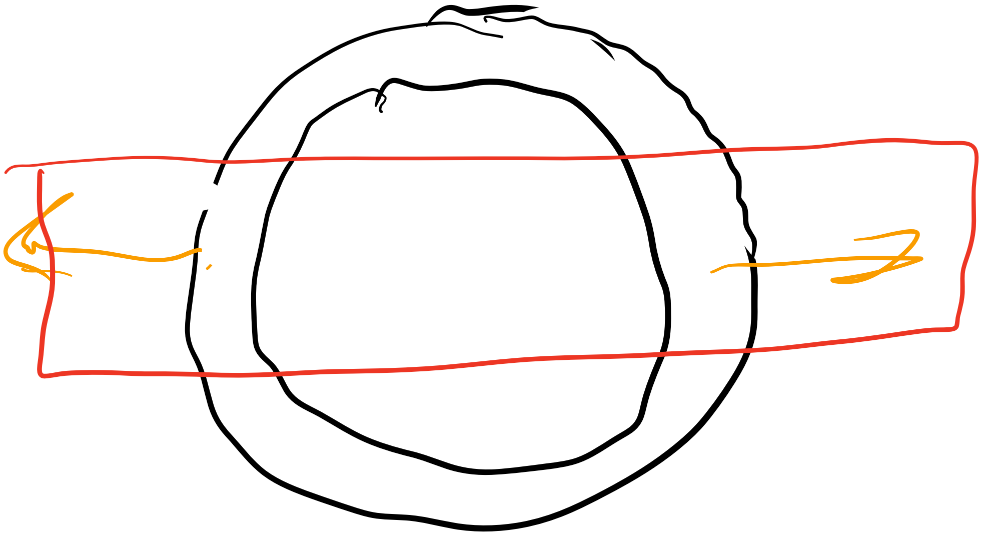
At the reverse shock, kinetic energy of ejecta is re-heated by reverse shock dissipation as it passes through, => implies some of the bulk energy of the ejecta goes back into heat of ejected material. The forward shock converts some of the bulk energy into heating ambient ISM material. (x-ray emission is visible from both shocked regions)

$$E \propto r^3 \dot{r}^2 \underline{\propto t^0}$$

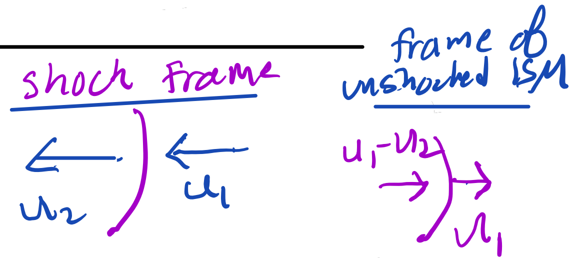
$$P \propto r^3 \dot{r} \propto E/\dot{r}$$

$$\dot{r} \propto t^{-3/5}$$

$$\Rightarrow \underline{P \propto t^{3/5}}$$



Why momentum increases in energy conserving phase



thin shell:

$$\frac{4\pi\rho_1 R^3}{3} = 4\pi\rho_2 R^2 D \qquad (M.1)$$

$$D = \frac{1}{3}R\left(\frac{\gamma-1}{\gamma+1}\right) \simeq 0.1\,R$$ for $\gamma = 5/3$

- $\rho_1 u_1 = \rho_2 u_2$

$$u_2 = \frac{\rho_1}{\rho_2}u_1 = \frac{\gamma-1}{\gamma+1}u_1 \simeq 4u_1$$

relative to unshocked gas, speed of shocked gas is:

$$V = u_1 - u_2 = \frac{2u_1}{\gamma+1} \qquad (M.2)$$

$\simeq$ for $\gamma = 5/3$ $\Rightarrow = \frac{3}{4}u_1$

→ Radial momentum charge per time of shocked gas in shell:

$$\frac{d}{dt}\left[\overbrace{\frac{4\pi\rho_1}{3}R^3}^{\text{shell mass}}\;\overbrace{\frac{2u_1}{\gamma+1}}^{\text{shell speed}}\right] = \underbrace{4\pi R^2 P_{in}}_{\text{pressure × area}} \qquad (M.3)$$

Any gain has to come from pressure on inside of shell P_{in}

Now suppose that this pressure scales with the pressure within the shell, that is:

$$\alpha = \gamma\tilde{\alpha} = 5/3 \cdot 1/2 \sim 5/6$$

$$P_{in} = \overset{\text{constant}}{\alpha}\,\underset{\text{post-shock pressure}}{P_2}$$

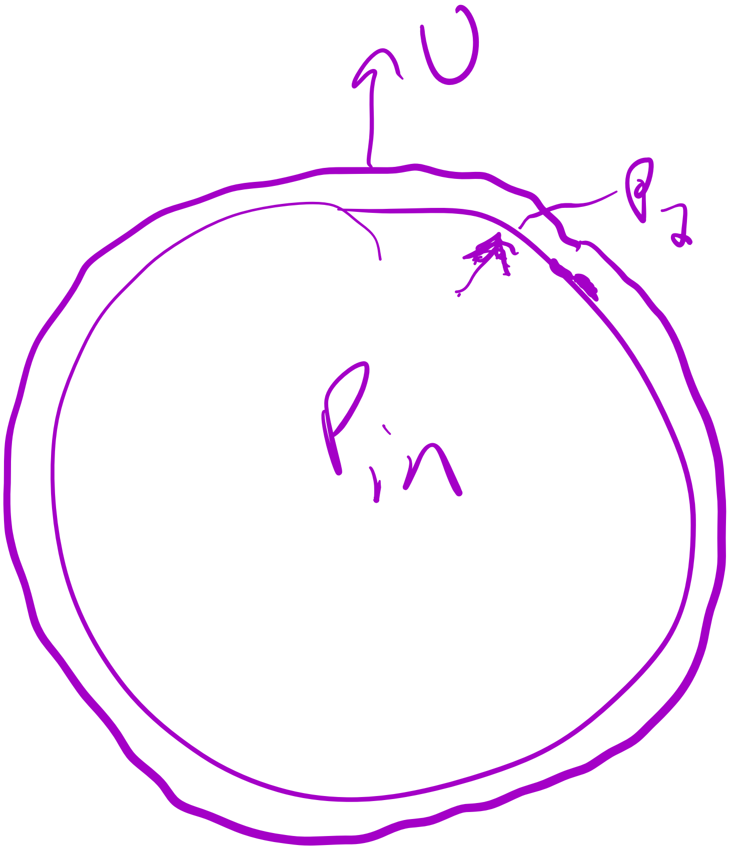
(crude assumption but lets proceed.)

∴ For strong shock ($M_1 \gg 1$):

$$P_2 \simeq P_1 \frac{2M_1^2}{\gamma+1} = \frac{2P_1u_1^2}{(\gamma+1)c_{s_1}^2} = \frac{2}{\gamma(\gamma+1)}\rho_1 u_1^2 \qquad (M.4)$$

$$\frac{\gamma P}{\rho} = c_s^2$$

$\rho_1 = \text{const} = \rho_{ISM}$


U
ρ_2
ρ_1, n

so M.3 then gives:

$$\frac{d}{dt}\left[\frac{\cancel{4\pi\rho_1}}{3}R\,\frac{\cancel{2}u_1}{\cancel{\gamma+1}}\right]=\frac{\overbrace{\cancel{4\pi}R^2}^{\text{area}}\alpha}{\gamma}\,\overbrace{\frac{\cancel{2}}{\cancel{\gamma+1}}\cancel{\rho_1}u_1^2}^{\text{pressure = Force/area}}$$

$$\Downarrow$$

$$\frac{d}{dt}\left[R^3u_1\right]=3\underbrace{\frac{\alpha}{\gamma}}_{\equiv 3\tilde{\alpha}}R^2u_1^2$$

$$=3\tilde{\alpha}R^2u_1^2 \qquad (M.5)$$

(R = radius of blast wave = r from before)

but $u_1=\frac{dR}{dt}$ => M.5 can be written

$$\frac{d}{dt}\left[R^3\dot{R}\right]=3\tilde{\alpha}R^2\dot{R}^2 \qquad (M.6)$$

"guess" that $R\propto t^b$, then M.6

$$\Rightarrow \frac{d}{dt}\left[t^{3b}\,b\,t^{b-1}\right]=3\tilde{\alpha}\,t^{2b}b^2t^{2(b-1)}$$

$$\Rightarrow (4b-1)\cancel{b}\,t^{4b-2}=3\tilde{\alpha}\,b^{\cancel{2}}\,t^{4b-2}$$

$$\Rightarrow \quad b = \frac{1}{4-3\check{\alpha}} \qquad (M.7a)$$

$$\Rightarrow \quad R \propto t^{b} = t^{\frac{1}{4-3\check{\alpha}}}$$

$$\Rightarrow \quad \frac{dR}{dt} \propto t^{\frac{-3+3\check{\alpha}}{4-3\check{\alpha}}} \propto R^{-3+3\check{\alpha}} = R^{3(\check{\alpha}-1)} \qquad (M.7b)$$

So we need to determine $\check{\alpha}$.

For adiabatic blast wave, energy E is conserved and it is distributed into the shell kinetic energy:

$$E_{kin} = \frac{1}{2}\frac{4\pi}{3}\rho_1 R^3 v^2$$

and internal energy, which mostly comes from inner cavity interior to the thin shell:

$$E_{int} = \frac{P_{in}}{\gamma-1}\cdot\frac{4}{3}\pi R^3 \doteq \frac{4}{3}\pi R^3 \frac{\check{\alpha} P_2 \gamma}{\gamma-1} \qquad (M.8)$$

$$\Rightarrow E_{tot} = \frac{4\pi R^3}{3}\left[\frac{1}{2}\rho_1 v^2 + \frac{\tilde{\alpha} P_2 \gamma}{\gamma - 1}\right];$$

using M.2 & M.4:

M.2 M.4

$$\Rightarrow \quad = \frac{4\pi R^3}{3}\left[\frac{1}{2}\rho_1\left(\frac{2u_1}{\gamma+1}\right)^2 + \frac{\tilde{\alpha}}{\gamma-1}\frac{2}{\gamma+1}\rho_1 u_1^2\right]$$

now use again $u_1 = \frac{dR}{dt}$

and M.7a,b $\Rightarrow$

$$R \propto t^b = t^{\frac{1}{4-3\tilde{\alpha}}}$$

$$\frac{dR}{dt} \propto t^{-\frac{3+3\tilde{\alpha}}{4-3\tilde{\alpha}}} \propto t^{b-1}$$

$$E_{tot} \propto R^3 \dot{R}^2 \propto t^{3b}\, t^{2b-2} \propto t^0$$

But energy conservation $\Rightarrow$

$$E_{tot} \propto t^0 \Rightarrow 5b - 2 = 0$$

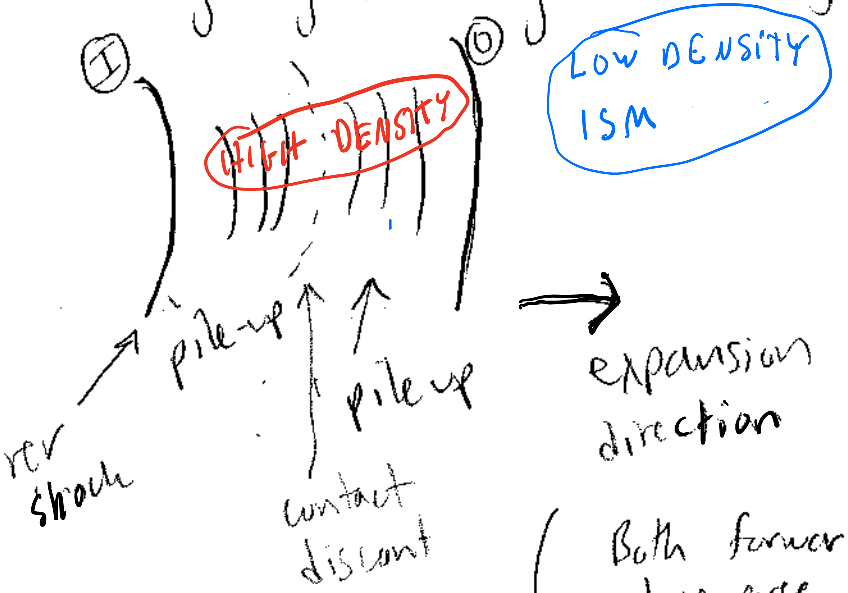
$$\Rightarrow b = 2/5 \Rightarrow 4 - 3\tilde{\alpha} = 5/2$$

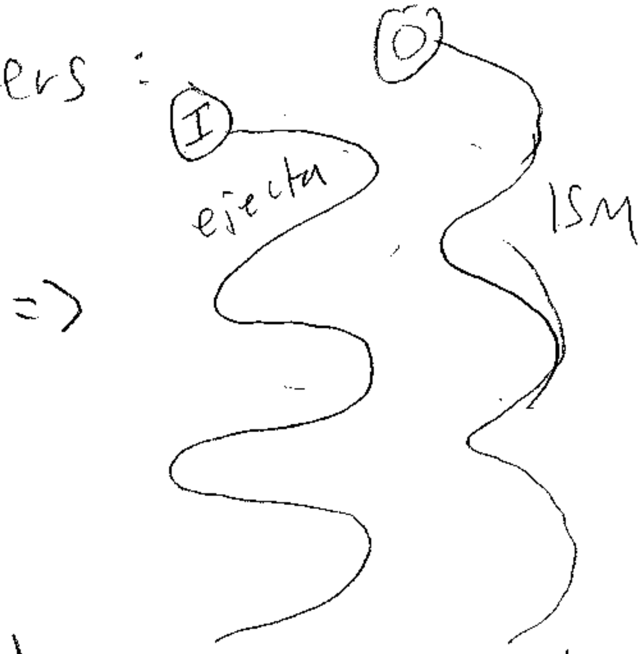
$$\Rightarrow \tilde{\alpha} = \frac{1}{2}$$

$\Rightarrow \quad R \propto t^{2/5} \;; \quad \dot{R} = u_1 \propto t^{-3/5}$

$P_1 \propto u_1^2 \propto t^{-6/5}$

I mentioned, and will discuss later, the Rayleigh Taylor instability, which takes place during the Sedov phase. The Rayleigh-Taylor fingers:





(Both forward and reverse directions have equivalent of "heavy" fluid falling on "light" fluid)

Radiative phase of SNR blast wave

once radiative cooling time becomes short compared to Sedov age we have radiative phase. Sedov age is given by (111s).

For cooling time, note that for $T < 10^6 K$ C, N, O gain e^- and become atomic; cooling by atomic cascade of e^- falling to lower levels dominates:

$$n_H^2 \Lambda(T) = 10^{-22} \, erg/cm^3\text{-}s \cdot n_H^2 \left(\frac{T}{10^6 K}\right)^{-1/2} \qquad (112s)$$

$$t_{cool} \simeq \frac{nkT}{n^2 \Lambda(T)} \simeq 2\times10^5 \left(\frac{T}{3\times10^6}\right)^{3/2} n_H^{-1} \text{ yr} \qquad (113s)$$

number density for compressed region

$t_{cool} < t_{sedov}$ when

from (111s) and (113s)

$$T^{7/3} < \frac{2\times10^5}{3.5\times10^4} \left(3\times10^6\right)^{7/3} \frac{n_H}{n_{ism}^{1/3}} \left(\frac{E}{10^{51}\text{erg}}\right)^{1/3}$$

or $$T < \left(\frac{3.5\times10^4}{2\times10^5}\right)^{3/7} \left(3\times10^6\right)\left(4\right)^{1/3}\left(n_H^{2/3}\right)^{3/7}\left(\frac{E}{10^{51}}\right)^{1/7}$$

compression ratio across shock just before cooling becomes important (see eqn 10ds)

$$\boxed{T < 5.7\times10^6 \text{ K} \left(n_H^{2/7}\right)\left(\frac{E}{10^{51}\text{erg}}\right)^{1/7}} \qquad (114s)$$

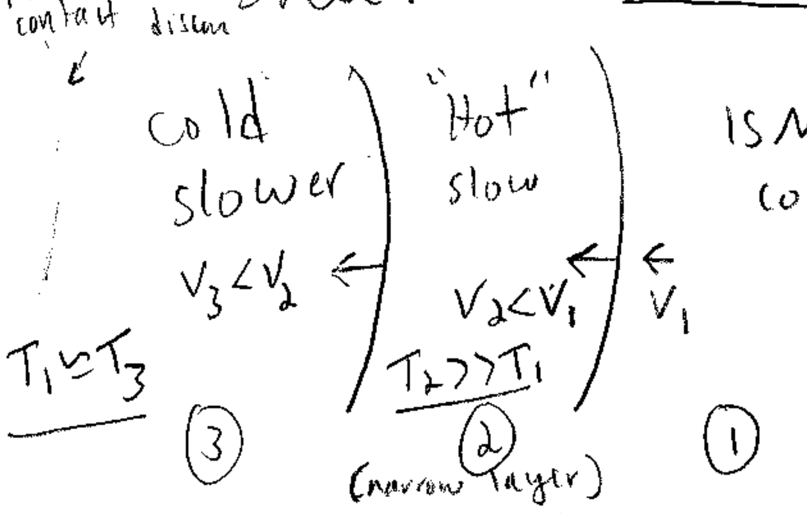
or $$\boxed{v \simeq \left(\frac{kT}{m}\right)^{1/2} \lesssim 240 \frac{\text{km}}{\text{s}} \left(E_{51}\, n_H^2\right)^{1/14}}$$

notice the weak dependence on E and n_H!

In radiative phase

Shock becomes isothermal as it evolves.

Hot interior region but a cooled, isothermal interior shell: In <u>frame of contact discontinuity</u>:



note: $[\rho_1 V_1 = \rho_2 V_2 = \rho_3 V_3]$

=>

$T_3 << 10^6 K$ $\quad$ $T_2 > 10^6 K$ $\quad$ $T_1 << 10^6 K$

cooling takes away most of the shock energy but <u>momentum is conserved</u> because radiation is essentially isotropic. Thus

$$\frac{d}{dt}\underbrace{\left(\overbrace{\frac{4\pi \rho_{ISM}}{3} r^3}^{M} \dot{r}\right)}_{momentum} \simeq 0 \quad \text{in radiative phase} \quad \ldots (115s)$$

$$\Rightarrow r^3 \dot{r} = \text{constant for } \frac{d\rho_{ISM}}{dt} \simeq 0.$$

$$\Rightarrow r^3 dr \simeq dt$$

$$\Rightarrow r \propto t^{1/4}$$

(116s)

and $\dot{r} \propto t^{-3/4}$, $\boxed{\dot{r} \simeq 240 \frac{km}{s} (E_{51} n_H^2)^{1/14} \left(\frac{t}{5.2 \times 10^4 yr}\right)^{-3/4}}$

using (114) & (113)

from Padmanabhan, "Theoretical Astrophysics" vol 2.

S18

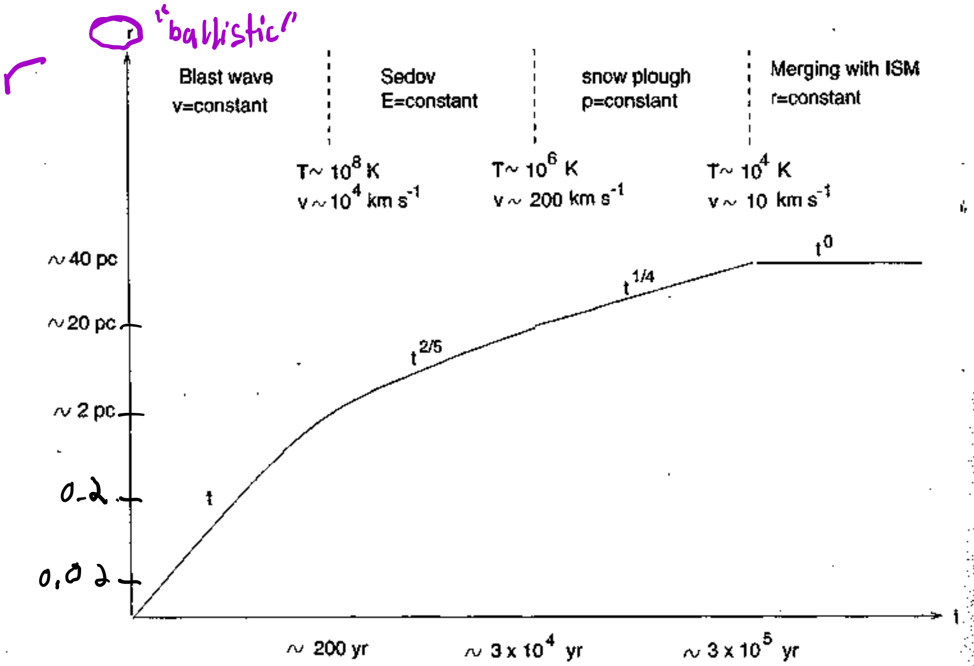


Fig. 4.6. The radius of the supernova shell as a function of time during the different phases.

This integrates to give

$$R = R_0\left[1 + 4\frac{v_0}{R_0}(t - t_0)\right]^{1/4}, \quad \dot{R} = v_0\left[1 + 4\frac{v_0}{R_0}(t - t_0)\right]^{-3/4}. \qquad (4.107)$$

For large t, $R \propto t^{1/4}$ and

$$\dot{R} \propto t^{-3/4} \simeq 200 \text{ km s}^{-1} \, (t/3 \times 10^4 \text{ yr})^{-3/4}. \qquad (4.108)$$

The time constant in relation (4.108) is fixed by equating the Sedov phase velocity of Eq. (4.101) to 200 km s^{-1}.

In the final phase, the speed of the shell drops below the sound velocity of the ISM, which is approximately (10–100) km s^{-1} in a time scale of $t \approx (1\text{–}5) \times 10^5$ yr. Around this time scale, the remnant loses its identity, and it is dispersed by random motions in the ISM. The evolution is shown schematically in Fig. 4.6.

It should be noted that supernova explosions and their eventual dispersion of ejected material have the effect of enriching the ISM with the material processed in stellar interiors. In particular, the heavy elements synthesised inside a star reach the ISM through this process. Because massive stars evolve at shorter time scales and also are more likely to end up as supernovas, the evolution of the first generation of massive stars changes the character of the ISM. Second and later generations of stars condense out of this enriched ISM and will have a higher proportion of heavier elements.

A supernova explosion c
from the surrounding ISM.
a gaseous nebulalike regio
the star in the presupernova
heat and ionise such a regio
an expanding luminous rin
from OIII was detected aro
from the centre of the exp
Supernovas also lead to lig
phenomena discussed in Vo
two light echos were detecte
approximately 1 yr after the
A supernova emits x rays
material behind the shock. T
from the plasma at a tempera
are formed during phase 3, a
in the material with a temper
of the radiating atoms. In add
remnants are also strong so
spiraling in the magnetic fi
Vol. I, Chap. 6, Section 6.11,
electrons per unit volume is t

then the total flux of an optic
be expressed as

$$S_\nu = \frac{G}{d^2} V K B^{(1+p)/2} \nu^-$$

where V is the volume of the
is a numerical factor. In the c
is strongly ionised during the
frozen to the plasma fluid. It f

the energy of individual rela
xpansion of the volume, the e
he pressure of relativistic elect
give $\epsilon \propto r^{-4}$. The total ener

Linear Theory of instabilities and example of convection

- Equilibrium vs. stable equilibrium: Consider simple system:

ball rolling down hill.

though both positions are equilibria, only second is stable. Thus Ⓐ equilibrium is unstable to formation of Ⓑ

- ball may incur oscillations about the stable equilibrium Ⓑ position (corresponds to waves in a fluid system)
- To find equilibria of fluid set ∂_t of all quantities to zero, and solve. But to find stable equilibria and instabilities one must perturb around the equilibrium and see how the perturbations evolve
- when we looked at sound waves we ignored non-linear terms $\vec{v}\cdot\vec{\nabla}\vec{v}$ and we found waves. Arbitrary perturbations can be constructed from superposition of Fourier modes for linear problems.
- One common example of instability is convection when you heat water in pot, conduction transports heat first, then changing over to convection.

→

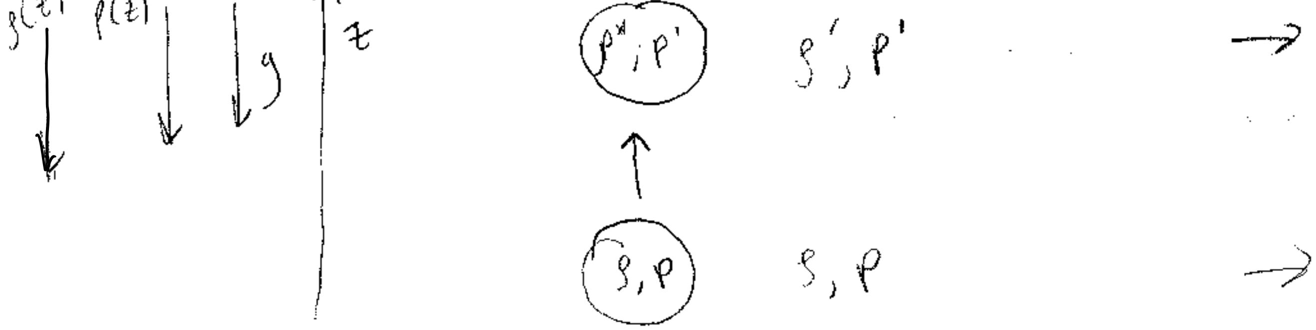
→ system becomes convectively unstable when the temperature gradient from top to bottom exceeds a certain value. Transition to instability also known as bifurcation

→ as perturbations grow, non-linearities ensue in fluids → turbulence. Linear theory not valid for turbulence, but can be useful at least to determine which configurations can be expected to incur transition to turbulence.

→ by considering simple systems one can gain intuition about which systems tend to be unstable.

Convective Instability

Consider perfect gas in hydrostatic equilibrium in uniform gravity. If z axis is chosen such that gravity is in negative z direction then $\rho(z)$ & $p(z)$ decrease with z. Consider vertical displacement of blob as shown:



where initially p and ρ have same density as surroundings. External density and pressure at new position are ρ' & p'. Pressure balance inside and outside is maintained swiftly by acoustic waves, but heat imbalance/exchange takes longer when mediated by conduction. assumption

We can consider the blob to be displaced adiabatically, then let ρ^* be its new density. If $\rho^* < \rho'$, the blob will be buoyant and continue upward, implying instability. If $\rho^* > \rho'$ then the blob will tend to return, making the system stable. So we need to determine ρ^*/ρ':

For adiabatic flow, $\rho^* = \rho \left(\frac{\boxed{p'}}{p}\right)^{1/\gamma}$ (122)

If $\frac{dp}{dz}$ is pressure gradient, we can substitute

$$p' = p + \frac{dp}{dz}\Delta z \qquad (123)$$

and using $\rho^* = \rho \frac{\boxed{\left(p + \frac{dp}{dz}\Delta z\right)^{1/\gamma}}}{p^{1/\gamma}}$ expanding to lowest order in Δz:

$$\Rightarrow \quad \rho^* = \rho + \frac{\rho}{\gamma p}\frac{dp}{dz}\Delta z \qquad (124)$$

since: $\left\{ \left(p + \frac{dp}{dz}\Delta z\right)^{1/\gamma} = p^{1/\gamma} + \frac{1}{\gamma} p^{\frac{1}{\gamma}-1} \frac{dp}{dz}\Delta z \right\}$

but for ambient medium:

$$\rho' = \rho + \frac{d\rho}{dz}\Delta z \qquad (125)$$

then using $\rho = P/RT$

$$\Rightarrow \rho' = \rho + \frac{\rho}{P}\frac{dP}{dz}\Delta z - \frac{\rho}{T}\frac{dT}{dz}\Delta z \qquad (126)$$

where $d\rho/dz$ & dT/dz are density and temp gradients. dP/dz is background pressure gradient

(124) minus (126) $\Rightarrow$

$$\rho^* - \rho' = \left[\overbrace{-\left(1-\frac{1}{\gamma}\right)\frac{\rho}{P}\frac{dP}{dz}}^{\text{Ⓐ}} + \overbrace{\frac{\rho}{T}\frac{dT}{dz}}^{\text{Ⓑ}}\right]\Delta z \qquad (127)$$

$\frac{dT}{dz}$ and $\frac{dP}{dz}$ are <u>both</u> <u>negative</u>;

stable atmosphere ($\rho^* > \rho'$) requires

$$\frac{|\text{Ⓑ}|T}{\rho} = \left|\frac{dT}{dz}\right| < \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\left|\frac{dP}{dz}\right| = \frac{|\text{Ⓐ}|T}{\rho} \qquad (128)$$

This is <u>Schwarzschild stability condition</u>

Important for stellar modeling.

→

Note (127) can also be written:

$$\frac{\rho^* - \rho'}{\rho} = \cancel{\rho}\left[\frac{d\ln T}{dz} - \left(1-\frac{1}{\gamma}\right)\frac{d\ln p}{dz}\right] \quad (*)$$

$$= \cancel{\rho}\,\frac{d\ln\left(\frac{T}{p^{1-1/\gamma}}\right)}{dz}$$

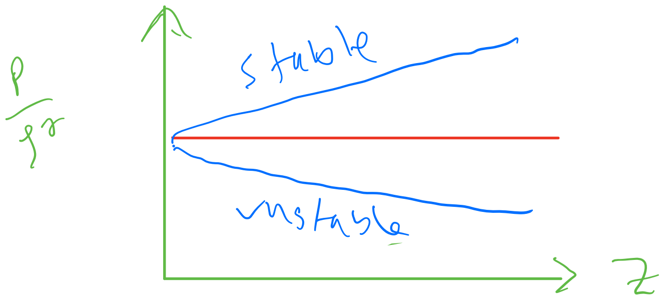
but $T = \frac{p}{\rho R}$ so equation becomes

$$\frac{\rho^* - \rho'}{\rho} = \frac{d\ln}{dz}\left[\frac{p^{1/\gamma}}{\rho R}\right]$$

$$= \underbrace{\frac{1}{\gamma}\frac{d\ln}{dz}\left[\frac{p}{\rho^\gamma}\right]}_{\text{vanishes if adiabatic; } \Rightarrow} + \cancelto{0}{\frac{1}{\gamma}\frac{d\ln}{dz}\left(\frac{1}{R}\right)}$$

stability if $\frac{p}{\rho^\gamma}$ increases with z

instability if p/ρ^γ decreases with z



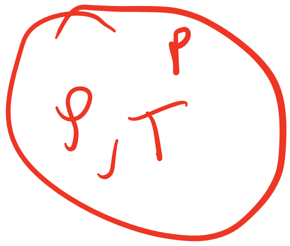
$\Rightarrow$ in (*) if $\left|\frac{dT}{dz}\right|$ is "superadiabatic" $\Rightarrow$ unstable

blob pressure = $\rho^* T^*$

p', T', ρ'

ambient pressure = $\rho' T'$

if "superadiabatic" background $\Rightarrow T' < T^*$



p, T, ρ

$\Rightarrow \rho^* < \rho'$

$\Rightarrow$ instability

$$\rho^* T^* = \rho' T'$$

so if $T' > T^*$ ("super adiabatic")

$\Rightarrow \rho^* < \rho'$

$\Rightarrow$ instability

Since Force per unit volume acting inside displaced blob is $(\rho^*-\rho)(-g)$

quation of motion is approximately

$$\rho^* \frac{d^2}{dt^2}\Delta z = -(\rho^*-\rho')g \qquad (129)$$

Substituting from (127)

$$\Rightarrow \rho^* \frac{d^2}{dt^2}\Delta z = -g\left(\frac{\rho}{T}\frac{dT}{dz} - \left(1-\frac{1}{\gamma}\right)\frac{\rho}{P}\frac{dP}{dz}\right)\Delta z$$

to lowest order in Δz: we replace ρ^* by ρ and then obtain (from 124)

$$\frac{d^2}{dt^2}\Delta z + N^2 \Delta z = 0$$

where $N \equiv \sqrt{\frac{g}{T}\frac{dT}{dz} - \left(1-\frac{1}{\gamma}\right)\frac{g}{P}\frac{dP}{dz}}$

is the Brunt-Väisälä frequency.

$$\overset{N^2}{\rightsquigarrow}$$

$$g\left(\frac{d\ln T}{dz} - \left(1-\frac{1}{\gamma}\right)\frac{d\ln P}{dz}\right)$$

$$= g^{1/2}\frac{d\ln}{dz}\left(\frac{T}{P^{1-1/\gamma}}\right)$$

$$= \frac{g}{\gamma}\frac{d\ln}{dz}\left(\frac{T^\gamma}{P^{\gamma-1}}\right)$$

$$= \frac{g^{1/2}}{\gamma}\frac{d\ln}{dz}\left(\frac{P}{\rho^\gamma}\right)$$

For stable stratification blob will oscillate. In reality such motions give rise to internal gravity waves by disturbing the surrounding medium. We ignored internal gravity waves by ignoring the effect of blobs' motion on external medium. Full treatments account for ; these waves when full perturbative treatment is developed.

Turbulence (intro)

Once instabilities ensue, linear theory of perturbations fail, non-linear theory is required.

Consider a point in phase space at an unstable equilibrium:

if P is unstable equilibrium, then small perturbation around around P sends the system off into an arbitrary direction of phase space. It then becomes impossible to predict the subsequent evolution phase space exactly. For fluid unstable to Kelvin-Helmholtz, Rayleigh-Taylor etc. The eventual consequence of the instability is turbulence.

Random velocities every which way.

Though instabilities lead to turbulence, it is also possible to produce them with random stirring of a fluid at different locations.

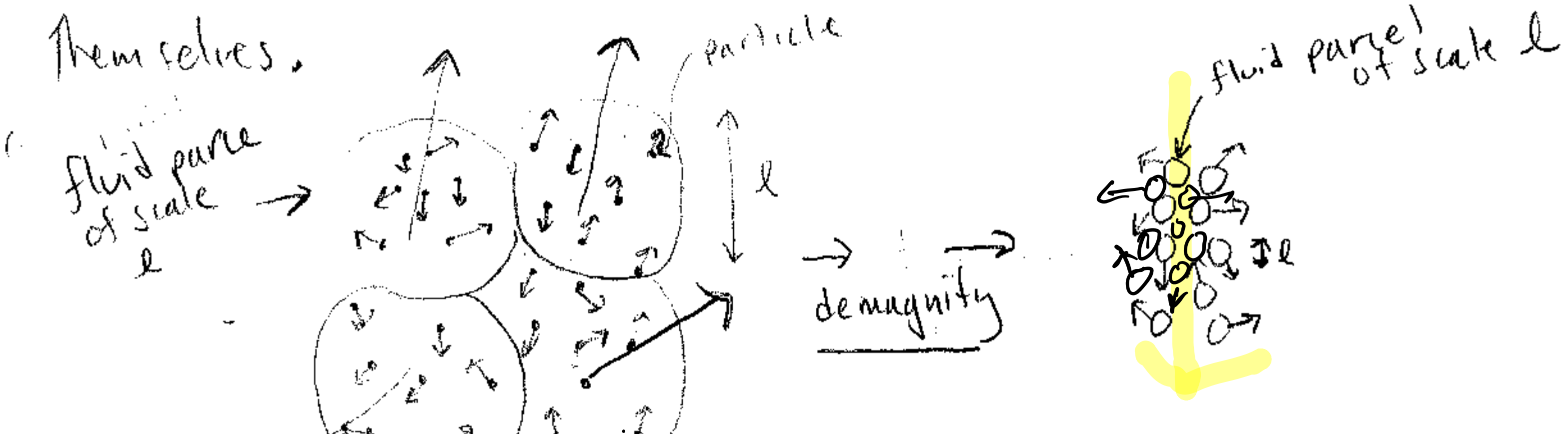
No deterministic theory of turbulence is possible, but one can develop a theory of average properties. What kind of "average"?

→

- Volume average – good for spatially homogeneous or nearly spatially homogeneous
- time average - good for temporally steady or nearly steady
- ensemble average: average over many hypothetical copies of the system having same statistical properties, but which differ in the actual values of quantities like velocity at a given space and given time

When the variation time or spatial gradient scales are large compared to fluctuating scales, then ensemble average and volume or time averages can be thought to be equivalent.

Note the difference between the averaging over fluid velocity fluctuations and the averaging over kinetic theory! to get the fluid eqns themselves.



$$V = V' + \overline{V}$$

$$\overline{V} = \underset{0}{\cancel{\overline{V'}}} + \overline{\overline{V}} = \overline{V}$$

fluid Velocity at given location can be written

$v = \bar{v} + v'$, where $\bar{v}$ is mean & v' is fluctuation. By construction $\overline{v'} = 0$.

Consider the statistical quantity $\overline{v'(x,t) \cdot v'(x+r,t)}$. If $\vec{r}=0$, then this is $\overline{v'^2}$ which is a measure of kinetic energy in the turbulence. But if $\vec{r}$ is large then $\overline{v'(x,t) \cdot v'(x+r,t)} = 0$. Thus such a correlation has sizeable values only within a range Δr.. This is the correlation length. Such correlations contain information about the strength & correlation length of the turbulence. A statistical theory of turbulence is one that develops equations for these correlations. Higher correlations are also often needed, e.g. the 3-point correlation $\overline{v_i(\vec{x})\, v_j(\vec{x}_2)\, v_k(\vec{x}_3)}$.

$$\partial_t \bar{v} = -\overline{v \cdot \nabla v} - \nabla P + \cdots \rightarrow$$

$$= -\overline{v' \nabla v'} - \bar{v} \cdot \nabla \bar{v}$$

$$\partial_t v' = -\underline{(v' \cdot \nabla v')}$$

$$\langle v \cdot \nabla v \rangle = \overline{v \cdot \nabla v}$$

$$= \overline{\bar{v} \cdot \nabla \bar{v}} + \cancel{\overline{v' \cdot \nabla \bar{v}}}$$

$$+ \overline{v' \cdot \nabla v'} + \cancel{\overline{\bar{v} \cdot \nabla v'}}$$

$$\overline{v' \underbrace{\cdot \nabla \bar{v}}} = \cdot \nabla \bar{v} \; \underbrace{\overline{v'}}_{=0}$$

$$=$$

standard averaging procedure employs "Reynolds Rules":

$$\langle v' \rangle = 0$$

$$\bar{\bar{v}} = \langle \bar{v} \rangle = \bar{v}$$

$$\langle \bar{v} v' \rangle = \bar{v} \langle v' \rangle = 0$$

$$\partial_i \langle v_i v_j \rangle = \langle \partial_i v_i v_j \rangle + \langle v_i \partial_i v_j \rangle$$

Even statistical theories of turbulence involve many approximations: Closure problem: differential equations for n-point correlations depend on (n+1)-point correlations and in general an approximation is needed to close the equations.

Even a simple problem like convectively unstable fluid heated from below with top and bottom temperatures given does not have known rigorous solution for $\overline{V_i(\vec{x})\, V_j(\vec{x}+\vec{r})}$.

The statement that "turbulence is an unsolved problem in physics" means that we do not yet understand how to calculate n-point correlations from a fundamental theory.

Kinematics of Homogeneous Isotropic Incompressible Turbulence:

In this simple limit we can derive some properties of turbulence from symmetry:

First, note that a mean flow violates isotropy so we set it to zero and thus $\vec{v} = \overline{\vec{v}} + \vec{v}\,' = \vec{v}\,'$.

Second, homogenity requires $\overline{V_i(x)\, V_j(x+\vec{r})}$ is independent of $\vec{x}$, depending only on $\vec{r}$. Thus we write

$$\overline{V_i(x)\, V_j(\vec{x}+\vec{r})} = R_{ij}(|r|)$$

→

Then $$\frac{\partial R_{ij}}{\partial r_j} = \overline{v_i(x)\frac{\partial v_j(\vec{x}+\vec{r})}{\partial r_j}} = 0$$

(assuming incompressible: $\vec{\nabla}\cdot\vec{v} = 0$). Since R_{ij} depends only on $|\vec{r}|$, $R_{ij} = R_{ji}$

(that is, $\overline{v_i(\vec{x})v_j(\vec{x}+\vec{r})} = \overline{v_j(x)v_i(x+\vec{r})}$),

proof

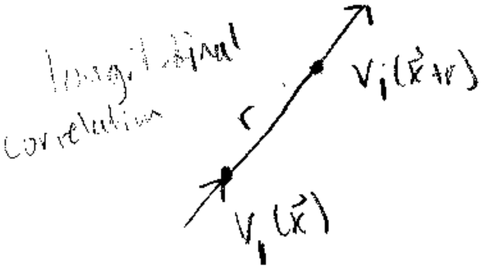
$$= \overline{v_i(\vec{x}-\vec{r})\,v_j(\vec{x})} = \overline{v_i(x')\,v_j(\vec{x}'+\vec{r})}$$
$$= \overline{v_i(x)\,v_j(x+\vec{r})}$$

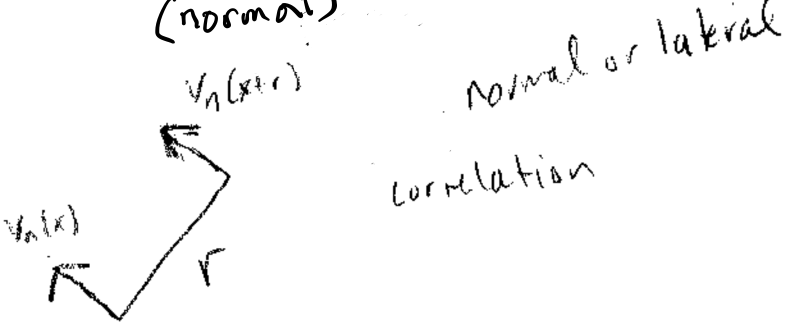
Then $$\frac{\partial R_{ij}}{\partial r_j} = \frac{\partial R_{ji}}{\partial r_j} = 0.$$

Von Kármán & Howarth (1938) showed that the most general tensor function $R_{ij}(r)$ is then

$$R_{ij}(r) = A(r)r_ir_j + B(r)\delta_{ij} \qquad (170)$$

Consider longitudinal and lateral (normal) velocity correlation functions :





Since longitudinal component of $\vec{r}$ is $r_l = r$ and normal component of $\vec{r}$ is $r_n = 0$, we have

$$R_{ll}(r) = A(r)r^2 + B(r) = \frac{1}{3}\overline{v^2}f(r) \qquad (171)$$

$$R_{nn}(r) = B(r) = \frac{1}{3}\overline{v^2}\,g(r) \qquad (172)$$

f(r) & g(r) are defined such that $f(0) = g(0) = 1$. (at $r=0$).

We can then express $A(r)$, $B(r)$ in (170) using (171), (172), in terms of $f(r)$, $g(r)$:

$$R_{ij} = \frac{1}{3}\bar{v}^2\left[\frac{f(r)-g(r)}{r^2} r_i r_j + g(r)\delta_{ij}\right]$$

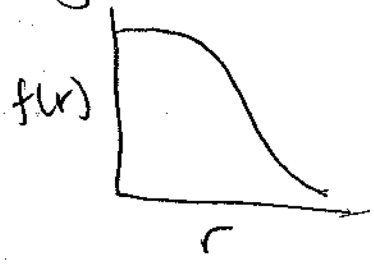
then using $\frac{\partial R_{ij}}{\partial r_i} = \frac{\partial R_{ij}}{\partial r_j} = 0 \Rightarrow$ use

$$\frac{\partial_i(f-g)r_ir_j}{r^2} - \frac{2r\frac{dr}{dr_i}r_ir_j}{r^4}(f-g) + \frac{(f-g)}{r^2}(3r_j+\delta_{ij}r_i) + \partial_i g\delta_{ij} = 0$$

and $\frac{dr}{dr_i} = \frac{r_i}{r}$ and multiply by r_j

$$g(r) = f(r) + \frac{1}{2} r \frac{df}{dr} \qquad (173)$$

Thus if we can determine $f(r)$, we can get all components of the correlation tensor R_{ij}. Since $f(r)$ is the longitudinal correlation function, we expect it to have a decaying form



→

Consider the fourier transform of R_{ij}: $\phi_{ij}(\vec{k}) = \frac{1}{(2\pi)^3}\int R_{ij}(r) e^{-i\vec{k}\vec{r}} d^3r$

Since R_{ij} is spherically symmetric in $\vec{r}$, ϕ_{ij} must be spherically symmetric in $\vec{k}$

so write $\phi(\vec{k}) = \phi(k)$. note "+" sign

Then

$$R_{ij}(r) = \int \phi_{ij}(k) e^{+i\vec{k}\cdot\vec{r}} d^3k \qquad (174)$$

Incompressibility $\frac{\partial R_{ij}}{\partial r_j} = \frac{\partial R_{ji}}{\partial r_j} = 0$ requires, at each k

$$\Rightarrow \quad k_i \phi_{ij} = k_j \phi_{ij} = 0$$

$$\partial_i R_{ij} = i\int \underbrace{k_i \phi_{ij}}_{=0} \underbrace{e^{i\vec{k}\cdot\vec{r}} d^3k}$$

Symmetry considerations then require (e.g. McComb "Physics of Fluid Turbulence").

$$\phi_{ij}(k) = C(k) k_i k_j + D(k)\delta_{ij}, \quad \text{where } k_i \phi_{ij} = 0$$

$$\Rightarrow \quad D(k) = -C(k) k^2, \quad \text{(defines } E(k)\text{)}$$

Then

$$\phi_{ij}(k) = \frac{E(k)}{4\pi k^4}\left(k^2\delta_{ij} - k_i k_j\right) \qquad (175)$$

$$= -C(k)\left(k^2\delta_{ij} - k_i k_j\right)$$

Significance is that

$$\frac{1}{2}\overline{v^2} = \frac{1}{2}R_{ii}(0) = \frac{1}{2}\int \phi_{ii}(k) d^3k \qquad (176)$$

using (174).

$$\phi_{ii} = \frac{E(k)}{4\pi k^4}\left(3k^2 - k^2\right) = \frac{2E(k)}{4\pi k^2}$$

So using (175) in (176) and writing $d^3k = 4\pi k^2 dk$ gives

$$\Rightarrow \frac{1}{2}\overline{v^2} = \int_0^\infty E(k)\,dk = \frac{1}{2}\int_0^\infty \frac{2E(k)}{4\pi k^2}\cdot 4\pi k^2\,dk \qquad (177)$$

$$E(k) = \frac{dEnergy}{dk}$$

Thus $E(k)$ is the energy spectrum of the turbulence. Just as B-body spectrum is composed of contributions at different wavelengths, turbulence can be thought of as being composed of contributions of different Fourier components.

Note that $f(r)$ and $E(k)$ are unspecified and are related such that only one is independent. We have not said anything about the form of $E(k)$, that is where Kolmogorov theory ... fits in

→

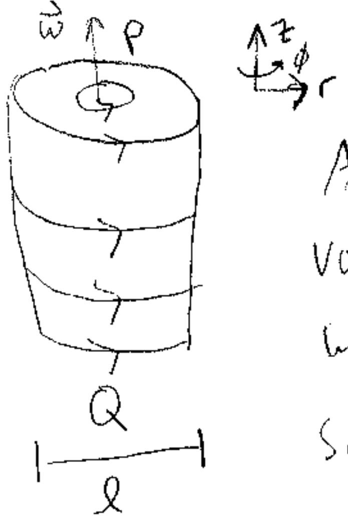
Kolmogorov Equilibrium Theory

A turbulent fluid can be maintained in a steady-state only if energy is continuously fed into system. Reason is viscous dissipation → left alone, turbulent energy will convert to heat. If the fluid is stirred, such that turbulent flow is statistically **homogeneous** & isotropic, then steady state can ensue, and system is in **statistical** equilibrium. Kolmogorov (1941) calculated the energy spectrum for such turbulence.

Imagine the driving (or forcing) to occur on **some scale** ℓ, inducing velocity v. Kolmogorov intuited that the the turbulent parcels (or eddies) of would feed energy to smaller scales, which then feed **energy to** still smaller scales. To see how this **cascade** proceeds, consider incompressible turbulence.

→

Let P & Q be two fluid elements on a vortex tube as shown, with diameter l:



According to Kelvin's theorem vorticity is conserved, or carried with the flow, (e.g. $\frac{d}{dt}\int w\cdot d\vec{s} = 0$ so that $\int \vec{w}\cdot d\vec{s}$ = constant, as derived earlier)

Then, since, statistically speaking, two points in the turbulent flow tend to separate with time, the vortex tube will lengthen as the points separate, still maintaining the coherence of the tube. But incompressibility requires that the the lengthened tube contract: Fixed density means fixed mass for given volume, stretching the tube in length requires decreasing the cross section to maintain the same density for the same mass of material. Thus

constant volume $\Rightarrow \pi (l/2)^2 L$ = constant $\Rightarrow l \sim L^{-1/2}$

thus cross section cascades in scale →

the continuous shrinking of the vortex tubes cannot continue forever because eventually viscous term $\nu \nabla^2 \vec{V}$ becomes important in the Navier-Stokes equation. That is the conditions for

$\uparrow$ viscosity

$$\langle v_d \cdot \nabla v_d \rangle \simeq \nu \nabla^2 v_d$$

Kelvin's theorem are violated for small enough scales (large enough gradients), and vorticity is dissipated. Another way of saying this is that the Reynolds number for the smallest eddies is of order ~ 1 : $R_d \simeq \frac{l_d V_d}{\nu} \simeq 1$ whereas $R_0 \simeq \frac{LV}{\nu} \gg 1$, where subscript d refers to dissipation scale, and R_0 is the Reynolds number for the largest scale (or "outer" scale) of length L & velocity V.

$$\frac{v_d^{\cancel{2}}/\cancel{l_d}}{\nu(1/l_d^{\cancel{2}})\cancel{v_d}} = \frac{v_d l_d}{\nu} = Re_{,d} = 1 \text{ since its } Re(l=l_d)$$

So the idea is that vortex energy is input at scale L with velocity V; it then cascades to scale l_d where it is dissipated into random particle energy (heat).

In steady state energy input rate must equal energy dissipation rate $\rightarrow$

$$\partial_t \vec{V}_j = -\underline{\vec{V}\cdot\nabla\vec{V}_j} - \cancel{\nabla P} + \underline{\nu\nabla^2\vec{V}_j}$$

$$\vec{V}_j\partial_t\vec{V}_j = -\vec{V}_j\cdot\vec{V}\cdot\nabla\vec{V}_j + \nu\vec{V}_j\nabla^2\vec{V}_j + \langle f(t,l_f)\rangle$$

$\langle \ \rangle$

$$\frac{1}{2}\langle\cancel{\partial_t V^2}\rangle = -\underset{Ⓐ}{\langle V_j\vec{V}\cdot\nabla V_j\rangle} + \underset{Ⓑ}{\langle\nu V_j\nabla^2 V_j\rangle} + \langle f(l,t)\rangle$$

0 steady state

for scales $l \gg l_d$

A dominates the cascade

$$\frac{V^3}{l} \simeq \frac{V}{l}\cdot V^2$$

if you force with $f(l_f,t)$
s.t. $l_f \gg l_d \Rightarrow$
term Ⓐ will dominate energy
cascade until you reach
$l = l_d$

$$l_f \geq l \geq l_d$$

and this energy transfer rate will be the same at all scales in a steady state if energy transfer is "local" in k space per unit mass, dimensional analysis leads to

$$\frac{d\mathcal{E}}{dt} \sim \frac{V_\ell^3}{\ell} \quad \text{at scale } \ell \text{ with velocity } V_\ell$$

$$= \frac{V_\ell}{\ell} \cdot V_\ell^2$$

$$\langle \vec{v} \cdot \nabla \times \vec{v} \rangle \neq 0 \quad \text{helical forcing}$$

$$\langle \vec{v} \cdot \nabla \times \vec{v} \rangle = 0 \quad \text{non-helical forcing}$$

(177)

since $\frac{V_\ell^3}{\ell} = \frac{V^3}{L} = \frac{V_d^3}{\ell_d}$

and $V_d \ell_d \sim \nu$ we have, using $R_e = \frac{LV}{\nu}$

$$\frac{L^{\overset{3}{4}} V^3}{\nu^3 \cancel{L}} \sim \frac{\cancel{\nu^3} L^4}{\ell_d^4} \Rightarrow R_e^3 = \frac{L^4}{\ell_d^4} \quad \text{or} \quad \frac{L}{\ell_d} = R_e^{3/4} \quad (178)$$

or $\frac{k_d^{4/3}}{k_L^{4/3}} = R_e$

thus the Reynolds number determines the ratio of largest to smallest scales in the cascade, this range of scales is called the inertial range.

To get the energy spectrum, one just uses $\frac{V_\ell^3}{\ell}$ = constant

$$\Rightarrow \quad V_\ell \propto \ell^{1/3} \Rightarrow V_\ell^2 \propto \ell^{2/3} \quad \text{and} \quad \ell \sim k_\ell^{-1}$$

thus $V_\ell^2 \propto k^{-2/3}$

→

$$\underset{\sim}{\nu}\,\nabla^2 V_d \simeq \langle V_i \nabla V_d \rangle$$

$$\nu \frac{\cancel{V_d}}{l_d^{\cancel{2}}} \simeq \frac{V_d^{\cancel{2}}}{\cancel{l_d}}$$

$$\Rightarrow \underset{\underset{C_{s,i}\lambda_{mfp}}{\parallel}}{\nu} \simeq V_d \, l_d$$

now kinetic energy density per mass v^2 around wavenumber K is $E(k)dk \sim E(k)K$ for $dk \sim k$

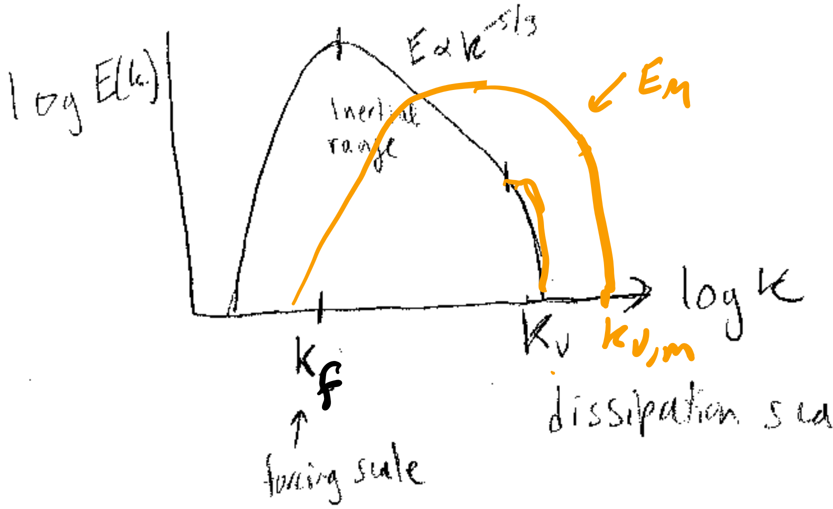
then $V_\ell^2 = E(k)k \propto k^{-2/3}$

$\Rightarrow$ $E(k) \propto k^{-5/3}$ $\Rightarrow$ $\int_{k_{min}}^{k_d} E(k)dk \propto k_{min}^{-2/3} - k_d^{-2/3}$ if $k_d \gg k_{min}$ $\Rightarrow$ $\propto k_{min}^{-2/3}$

$k_{min} \simeq k_f$

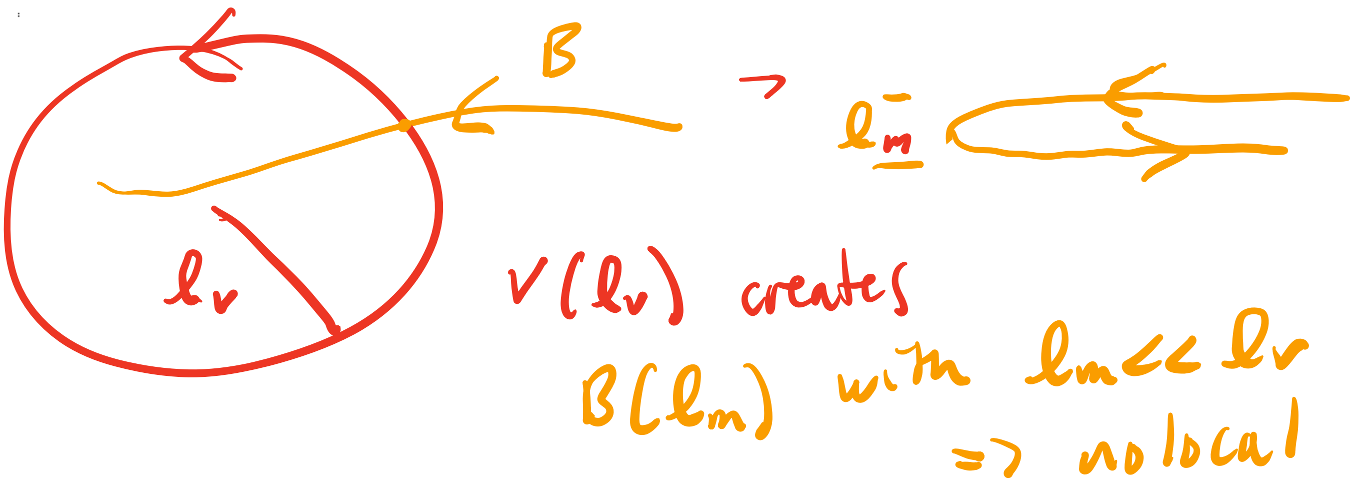
This is the <u>Kolmogorov spectrum</u>.

It applies for the inertial range:



k_ν, k_{ν_m}

$P_m = \left(\frac{R_M}{R_e}\right) = \frac{\nu}{\nu_m}$



$v(\ell_v)$ creates $B(\ell_m)$ with $\ell_m \ll \ell_v$

$\Rightarrow$ no local transfer!

Some nonlocal energy is possible in MHD when $Pm > 1$

Turbulent Diffusion (part 1)

Although homogeneous isotropic turbulence is simplest, real systems have inhomogenities. Turbulence affects transport, and the simplest effect is turbulent diffusion:

If you put sugar in coffee and do not stir, mixing occurs by molecular processes and takes a long time. But if stirred, the coffee becomes turbulent and mixing occurs more quickly

Suppose markers are introduced in fluid at t=0. Displacement of marker after time t=T is

$$\vec{X}(T) = \int_0^T \vec{V}(t)dt \qquad (180)$$

where $\vec{V}(t)$ is fluid velocity at time t.

↓ The mean displacement averaged over all markers must vanish, for a volume fixed in space (e.g. the coffee cup) but the mean squared displacement does not vanish:

$$\overline{X^2(T)} = \int_0^T d\tau \int_0^T dt\ \overline{\vec{V}_i(t)\cdot\vec{V}_i(\tau)} \qquad (181)$$

where $\overline{\vec{V}_i(t)\cdot\vec{V}_i(\tau)}$ is the velocity correlation function for velocities at two different times, but at fixed position. In steady state this must depend only on $t-\tau$, so we write

$$\overline{\vec{V}_i(t)\cdot\vec{V}_i(\tau)} = \overline{V^2} R(t-\tau) \qquad (182)$$

note $R(0) = 1$. We also assume symmetry: (= statistical steady state) $R(t-\tau) = R(\tau-t)$. We expect turbulence to have some correlation time τ_{cor} such that $R(\tau)$ is only substantially finite at $\tau < \tau_{cor}$ (τ_{cor} is typically an eddy turnover time, $\frac{l_{cor}}{V}$)

using (182) in (181) =>

$$\overline{X^2(T)} = \int_0^T dt\, \overline{V^2} \int_0^T d\tau\, R(\tau-t) \qquad (183)$$

Consider $T \ll \tau_{cor}$: then

$R(\tau - t) \simeq 1$, so

$$\overline{X^2(T)} = \overline{V^2}T^2 \tag{184}$$

as expected. But for $T \gg \tau_{cor}$ statistical effects of turbulence emerge:

In this limit, we can change integration bounds to $-\infty$ and $+\infty$ (since away from τ_{cor} there is little contribution)

$$\overline{X^2(T)} \doteq \int_0^T dt\, \overline{V^2} \int_{-\infty}^{\infty} d\tau\, R(\tau - t) \tag{185}$$

← we can choose $t = 0$

Then writing $D_T \equiv \frac{1}{3}\overline{V^2}\int_0^{\infty} R(\tau)\, d\tau = \frac{1}{6}\overline{V^2}\int_{-\infty}^{\infty} R(\tau)\, d\tau$

and assuming $\overline{V^2}$ is independent of time & space

$$\overline{X^2(T)} = 6\, D_T\, T \tag{186}$$

(where $6 = 2 \cdot 3$ and the 2 comes from $\int_{-\infty}^{\infty} \to 2\int_0^{\infty}$)

Now we argue that D_T can be thought of as a diffusion coefficient

⟶

Let $n(\vec{x},t)$ be the density of markers. If the dispersion of markers is diffusive, then $n(\vec{x},t)$ should satisfy

$$\frac{\partial n}{\partial t} = D\nabla^2 n \qquad (187)$$

where D is the diffusion coefficient.

We now prove that $D = D_T$:

If markers are introduced near the origin such that evolution is basically spherically symmetric, then

$$\overline{X^2(t)} = \frac{\int_0^\infty r^2 n(r,t)\, 4\pi r^2 dr}{\int_0^\infty n(r,t)\, 4\pi r^2 dr} = \frac{Ⓐ}{Ⓑ} \qquad (188)$$

Then using (187) $\partial_t n =$

$$\frac{\partial}{\partial t}\int_0^\infty r^2 n\, 4\pi r^2 dr = D\int_0^\infty \underbrace{\frac{\partial}{\partial r}\left(r^2\frac{\partial n}{\partial r}\right)}_{= r^2 \boxed{\nabla^2 n}} 4\pi r^2 dr \qquad (189)$$

$$= -D\int 8\pi r^3 \frac{\partial n}{\partial r} dr$$

$$= D\int 24\pi r^2 n\, dr$$

$$= 6D\int 4\pi r^2 n\, dr$$

integrating right side by parts twice:

$$\partial_t \int_0^\infty r^2 n\, 4\pi r^2 dr = 6D \overbrace{\int n\, 4\pi r^2 dr}^{\propto \text{ mass}} \quad Ⓑ$$

or

$$Ⓐ = \int_0^\infty r^2 n\, 4\pi r^2 dr \underset{\uparrow\text{ (for conserved mass)}}{=} 6Dt \overbrace{\int n\, 4\pi r^2 dr} \qquad (190)$$

thus (188) then implies

$$\overline{X^2(t)} = 6Dt \quad \text{since} \quad \overline{X^2(t)} = \frac{\int r^2 n 4\pi r^2 dr}{\int n 4\pi r^2 dr} \qquad (191)$$

just as in (186) so $D = D_T$ for $t = T$ ✓

and thus $D_T \simeq \frac{\overline{X^2(t)}}{6T} = \frac{1}{3}\overline{v^2}\int_0^\infty R(\tau)\,d\tau$

$v\tau_{cor} \simeq l_{cor}$

$v\tau_{ed} \simeq l_{ed}$

$$\simeq \frac{1}{3}\overline{v^2}\,\tau_{corr} \simeq \overline{v^2\tau_{cor}}$$

$$\simeq \frac{1}{3}|v||l_{corr}| \qquad (192)$$

$v \propto l^{1/3}$

$D_T \sim l^{4/3}$

Note that this coefficient does not depend on e.g. temperature or density <u>unlike</u> the molecular transport coefficients. This is because in the molecular case for example, molecules carrying more thermal energy move faster. For turbulent fluid transport however, the temperature does not have a direct influence on the turbulent velocity when the turbulent velocities are mechanically driven externally (e.g. the rate of sugar transport by turbulence in coffee does not depend on coffee temperature when externally stirred).

→

However, if the quantity transported can backreact on the turbulence, the turbulent diffusion coefficient can change.

Mean Field Equations:

How does turbulent viscosity affect Navier stokes equations?

For an incompressible flow: the NS eqn is

$$\frac{\partial}{\partial t}(\rho v_i) = \rho F_i + \frac{\partial}{\partial x_j}\left(-p\delta_{ij} - \rho v_i v_j + \mu \frac{\partial v_i}{\partial x_j}\right) \quad (193)$$

Now to apply to turbulent flow, follow $\bar{\rho} = \rho$ Reynolds (1895) and break velocity, pressure & force into mean and fluctuating parts: $v_i = \bar{v}_i + v_i'$; $p = \bar{p} + p'$ $F_i = \bar{F}_i + F_i'$. Substituting into (193) and taking average gives: (and using Reynolds rules ued earlier) $\langle \bar{v} \rangle = \bar{v}$; $\langle \bar{v} v' \rangle = \bar{v}\langle v' \rangle = 0$

$$\frac{\partial}{\partial t}(\rho \bar{v}_i) = \rho \bar{F}_i + \frac{\partial}{\partial x_i}\left(-\bar{p}\delta_{ij} - \rho \bar{v}_i \bar{v}_j - \rho \overline{v_i' v_j'} + \mu \frac{\partial \bar{v}_i}{\partial x_j}\right)$$

(194)

(where we used $\rho \overline{v_i v_j} = \rho(\bar{v}_i \bar{v}_j + \overline{v_i' v_j'} + \underbrace{\overline{v_i' \bar{v}_j}}_{0} + \underbrace{\overline{\cancel{v_i'} \bar{v}_j}}_{0})$)

→

(194) is the Reynolds Equation.

Note its similarity to (193) except for replacement of $\vec{v}$ by $\bar{\vec{v}}$ and the extra term $\overline{v_i' v_j'}$. This is an important term however, and is called the Reynolds stress. Note however that it leads to complications: One possibility for dealing with it is to write

$$\partial_t (\overline{v_i' v_j'}) = \overline{v_i' \frac{\partial v_j'}{\partial t} + \frac{\partial v_i'}{\partial t} v_j'} \qquad (194)$$

Then substituting for $\frac{\partial v_j'}{\partial t}$ and $\frac{\partial v_i'}{\partial t}$ by substracting (194) from (193). However, this will then produce triple correlations: $\overline{v_i' v_j' v_k'}$ → trying to deal with this triple in the same way as the double leads to 4th order correlations etc. This is the Closure problem. Since one effect of turbulence is to provide enhanced transport (as discussed in terms of D_T)

The simplest and most naive closure is to write the Reynolds stress as $\overline{v_i' v_j'} = -D_T \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right)$ (194a) this means that we have, for (194)

$$\frac{\partial}{\partial t}(\rho \bar{V}_i) = \rho \bar{F}_i + \frac{\partial}{\partial x_j}\left(-\bar{p}\delta_{ij} - \rho \bar{V}_i \bar{V}_j + \underset{\ell^{4/3}}{\underset{\downarrow}{D_T}}\left(\frac{\partial \bar{V}_i}{\partial x_j} + \frac{\partial \bar{V}_j}{\partial x_i}\right) \cancel{+ \mu \frac{\partial \bar{V}_i}{\partial x_j}}\right) \quad (195)$$

Since μ is the microphysical viscosity we can see that D_T acts in a similar way and is typically <u>much larger</u> than μ since $\mu \simeq \frac{1}{3} V_{th} \ell_{mfp}$ and $D_T = \frac{1}{3} V_T \ell_T$ and $\ell_T \gg \ell_{mfp}$ for wide range of V_{th}.

note also that the diffuse nature of transport is reflected by a diffusion coefficient multiplying two spatial derivatives. For D_T independent of space, its term becomes in (195)

$$D_T \nabla^2 \bar{V}_i + \overset{\text{0 for incompressible flow}}{\cancel{D_T \frac{\partial \vec{\nabla} \cdot \bar{V}}{\partial x_i}}} \quad \text{and}$$

for incompressible $\bar{V}$, the second term vanishes so the closure (194a) is a <u>turbulent diffusion</u> closure to the Navier-Stokes eqn. It works well in many cases.

$$\langle v_i' v_j' \rangle^{(1)} = D_T^{(0)} (\partial_i \bar{v}_j + \partial_j \bar{v}_i)$$

$$\langle v_i'^{(0)} v_j'^{(1)} \rangle + \langle v_i'^{(1)} v_j'^{(0)} \rangle$$

$$D_T^{(0)} = \underline{\langle v_i'^{(0)} \cdot v_i'^{(0)} \rangle} \tau_c$$

$v'^{(0)}$ is fluctuation velocity to 0th order in mean fields. This "base state" of the turbulence is homogeneous and isotropic by assumption

$$\langle v_i' \cdot v_i' \rangle(t) \neq$$
$$\langle v_i'^{(0)} \cdot v_i'^{(0)} \rangle(t)$$

$$\vec{E} = -\vec{v} \times \vec{B} + \eta \bar{J}$$

$$\vec{E} = \langle E \rangle + E'$$

$$\vec{v} = \langle v \rangle + v'$$

$$J = \langle J \rangle + J'$$

$$\Rightarrow$$

$$\langle E \rangle = -\bar{v} \times \bar{B} - \overset{\downarrow}{\langle v' \times B' \rangle} \cancel{+ \eta \langle J \rangle}$$

ignore η for moment

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$$

$$\frac{\partial \langle \vec{B} \rangle}{\partial t} = +\vec{\nabla} \times (\bar{v} \times \bar{B}) + \vec{\nabla} \times \underline{\langle v' \times B' \rangle}$$

$$\langle v' \times B' \rangle^{(i)} = \alpha^{(ij)} \langle B \rangle^{(j)} - D_T^{(ij)} \bar{\nabla} \times \langle B \rangle + \text{higher order derivs of } \langle B \rangle, \bar{v}$$

$\langle \vec{v}\cdot\vec{w} \rangle^{(\omega)} - \langle \vec{j}\cdot\vec{b} \rangle$

$= \langle \vec{v}\cdot\vec{\nabla}\times\vec{v} \rangle^{[\omega]}$

pseudoscalar

Hydrodynamics and Rotation

Need to consider rotating fluids, since most astrophysical objects have ∡ momentum.

Most astrophysical rotators possess differential rotation: two reasons:

(1) viscosity may not be able to act fast enough to smooth out differential rotation

(2) some physical mechanism present to maintain diff rotation.

Consider centrifugal force in rotating body:

Assume axisymmetric steady rotation: $\Rightarrow \partial_t = 0,\ \partial_\phi = 0$ and $V_r = 0$, in the r-component of Navier-Stokes equation in cylindrical coordinates (see Appendix of Shu or Choudhuri)

$$-\frac{V_\phi^2}{r} = g_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \qquad (196)$$

when radial pressure support is unimportant (e.g. thin acc. disks)

$V_\phi \simeq \sqrt{r|g_r|}$; centrifugal force balances grav.

→

In stars, pressure is not negligible, so a balance of rotation & pressure balances gravity. In general the rotation is differential because of ① the gravitational force law ② the ineffectiveness of microphysical viscosity to make the flow uniform and ③ the fact that turbulent "viscosity" in rotating systems, although much stronger than microphysical viscosity may not strong enough and because turbulent viscosity can be highly anisotropic in rotating flows and <u>contribute</u> to sustaining diff rotation as in the solar convection zone.

<u>Rayleigh criterion</u>: not all diff. rot flows are stable:

Consider fluid annulus at distance r_0 from axis rotates with velocity v_0, and this ring is interchanged with ring at $r_1 > r_0$ rotating with v_1, system is stable when displaced ring reacts to return to original position.

conserving ∡ momentum, ring displaced to r_1 acquires velocity

$v_1' = \frac{v_0 r_0}{r_1}$. The centrifugal acceleration at this new position is then $\frac{v_0^2 r_0^2}{r_1^3}$, whereas a ring in equilibrium at r_1 would be ⟶ $\frac{v_1^2}{r_1}$. Thus if

$$\frac{v_0^2 r_0^2}{r_1^{\cancel{3}2}} < \frac{v_1^2}{\cancel{r_1}} \qquad (197)$$

$v_0 = \Omega_0 r_0$
$v_1 = \Omega_1 r_1$

the system is stable and ring will return to original radius. When the inequality is not satisfied, system is unstable to turbulence. Eq (197) can also be written

$$(r_0^2\Omega_0)^2 < (r_1^2\Omega_1)^2 \quad \text{or}$$

$$\frac{d}{dr}\left|(\Omega r^2)^{\cancel{2}}\right| > 0 \qquad (198)$$

Rayleigh's criterion for <u>stability</u>

In astrophysics typically:

$$\Omega r = V \simeq \left(\frac{GM}{r}\right)^{1/2}$$

$\Omega r^2 \propto r^{1/2}$ $\Rightarrow$ $\measuredangle$ mom per unit mass for rotationally supported flow against gravity has

$\measuredangle$ $\frac{\text{mom}}{\text{mass}}$ increasing outward

$\Rightarrow$ Rayleigh Stable

Derivation of Navier Stokes in Rotating frame:

See [Goldstein "Mechanics"]

(e.g. Kageyama & Hyodo 2006)

- Most texts do "Lagrangian" approach which hides details and is not as generally useful
- Lets do Eulerian approach; focus on incompressible flow for now

$$\partial_t \vec{u} + (\vec{u}\cdot\vec{\nabla}\vec{u}) = \vec{f} = -\vec{\nabla}p - \vec{\nabla}\phi + \nu\vec{\nabla}^2\vec{u}$$

Consider the role of both

(using $\vec{u}$ here for flow velocity)

- Galilean transformation
- time dependent rotation of coords.

Start with Galilean transformation of vector between inertial frames L_I and L_I':

$\vec{x}' = \vec{x} - \vec{V}t$ $\qquad \vec{x}' \in L_I'$

$dx' = dx$ $\qquad \vec{x} \in L_I$

For any vector $\vec{a}(\vec{x},t)$ in L_I,

$$\vec{a}'(\vec{x}',t) = G^v\vec{a}(x,t) = \vec{a}(G^v x,t)$$

Where G^v is Galilean operator.

E.g.: $\vec{u}'(x',t) = G^v \vec{u}(\vec{x},t) = \vec{u}(x,t) - \vec{V}$

flow velocity ↗

- For vector function of a vector:

$$\vec{F}'(\vec{a}') = G^v \vec{F}(\vec{a},t) = \vec{F}(G^v \vec{a},t)$$

- So e.g.: if $\vec{F} = \vec{u}\cdot\vec{\nabla}\vec{u}$:

$$\vec{u}'\cdot\vec{\nabla}'\vec{u}' = G^v(\vec{u}\cdot\vec{\nabla}\vec{u}) \equiv (\vec{u}-\vec{v}t)\cdot\vec{\nabla}(\vec{u}-\vec{v}t) \quad (1)$$

(where $\vec{\nabla}' = \vec{\nabla}$ for Galilean transformation

since $\frac{\partial}{\partial x_i'} = \frac{dx_j}{dx_i}\frac{\partial}{\partial x_j} = \delta_{ij}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_i'}$)

Next: rotational transformation

Let $\hat{L}_R$ be rotating frame with constant angular velocity Ω w.r.t. L_I with same origin, and rotation about z-axis:

$$\vec{\Omega} = (0,0,\Omega)$$

- coords of $\vec{x}$ and $\hat{x}$ related by

$$\hat{x} = R^{\Omega t}\vec{x} \quad (1a)$$

where

$$R^{\Omega t} \equiv \begin{pmatrix} \cos\Omega t & \sin\Omega t & 0 \\ -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- A point p fixed in rotating frame $\hat{L}_R$, is seen with circular trajectory in L_I

- consider positions of P in inertial frame L_I at two times $t + \Delta t$

$$\Rightarrow \quad \hat{X}_p = R^{\Omega(\cancel{t+\Delta t})} \vec{X}_{t+\Delta t} = R^{\Omega t} X_t$$

Since inverse to $R^{\Omega t}$ is $R^{-\Omega t}$ $\therefore$

$$\vec{x}_{t+\Delta t} = R^{-\Omega\Delta t} \vec{X}_t \, ,$$ which (1a)

to lowest order in Δt

$$\Rightarrow \quad \vec{X}_{t+\Delta t} = \vec{X}_t + \Delta t \, \vec{\Omega} \times \vec{X}_t$$ (2)

$$\left[R^{-\Omega\Delta t} \vec{X}_t \simeq \begin{pmatrix} 1 & -\Omega\Delta t & 0 \\ +\Omega\Delta t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - \Omega\Delta t \, x_2 \\ x_2 + \Omega\Delta t \, x_1 \\ x_3 \end{pmatrix} \right]$$

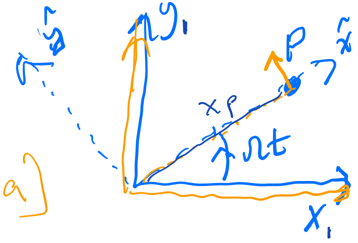
$\Rightarrow$ point P is moving at instantaneous velocity $\boxed{\vec{\Omega}\times\vec{x}} = -\vec{x}\times\vec{\Omega}$

- Consider a vector $\vec{a}(x,t)$ defined in frame L_I. If another frame L_I' moves with velocity $\Omega\times\vec{x}_p$; where $\vec{x}_p$ are coordinates of P in L_I, then

$$\vec{a}'(\vec{x}_p',t) = G^{V}\vec{a}(x_p,t), \quad \text{with } \vec{V}=\Omega\times\vec{x}_p.$$

- Now the components of $\vec{a}'(\vec{x}_p',t)$ and $\hat{a}(\hat{x}_p,t)$ in the rotating frame are related by

$$\hat{a}(\hat{x}_p,t) = R^{\Omega t}\overbrace{G^{\vec{V}}\vec{a}(\vec{x}_p,t)}^{\vec{a}'(x_p',t)} \qquad (2a)$$



at every position $\hat{x}$ in L_R then,

$$\hat{a}(\hat{x},t) = R^{\Omega t}G^{\Omega\times\vec{x}}\vec{a}(\vec{x},t) = R^{\Omega t}\vec{a}'(\vec{x}',t)$$

$$\Rightarrow \hat{u}(\hat{x},t) = R^{\Omega t}\underbrace{\left(\vec{u}(\vec{x},t)-\vec{\Omega}\times\vec{x}\right)}_{u'(\vec{x}',t)} \qquad (3)$$

- Apply to $\underline{(\hat{u}\cdot\hat{\nabla})\hat{u}}$:

$$(\hat{u}\cdot\nabla\hat{u}) = R^{\Omega t} G^{\vec{\Omega}\times\vec{x}} (\vec{u}\cdot\vec{\nabla})\vec{u}$$

$$= R^{\Omega t}\left[(\vec{u}+\vec{x}\times\vec{\Omega})\cdot\vec{\nabla}\,(\vec{u}+\vec{x}\times\vec{\Omega})\right]$$

$$= R^{\Omega t}\left[\{(u+\vec{x}\times\vec{\Omega})\cdot\vec{\nabla}\}\vec{u} + \{(\vec{u}+\vec{x}\times\vec{\Omega})\cdot\vec{\nabla}\}(\vec{x}\times\vec{\Omega})\right]$$

$$= R^{\Omega t}\left[\{(\vec{u}+\vec{x}\times\vec{\Omega})\cdot\vec{\nabla}\}\vec{u} + \{(\vec{u}+\vec{x}\times\vec{\Omega})\overset{\downarrow}{\times}\vec{\Omega}\}\right]$$

follows using identity $(\vec{a}\cdot\vec{\nabla})(x\times\vec{\Omega}) = (\vec{a}\times\vec{\Omega})$ (3a)

$$= R^{\Omega t}\left[\vec{u}\cdot\vec{\nabla}\vec{u} + (\vec{x}\times\vec{\Omega})\cdot\vec{\nabla}u + \vec{u}\times\vec{\Omega} + (\vec{x}\times\vec{\Omega})\times\vec{\Omega}\right]$$

[in General: $\hat{F}(\hat{a},t) = R^{\Omega t} G^{\vec{\Omega}\times\vec{x}}\vec{F}(\vec{a},t) = R^{\Omega t}\vec{F}(G^{\vec{\Omega}\times\vec{x}}\vec{a},t)$]

- Apply to time derivative:

$$\frac{\partial\hat{a}}{\partial t}(\hat{x},t) = \lim_{\Delta t\to 0}\frac{\hat{a}(\hat{x},t+\Delta t)-\hat{a}(\vec{x},t)}{\Delta t} \qquad (3b.)$$

from eqn (3)

$$\hat{a}(\hat{x},t+\Delta t) = R^{\Omega(t+\Delta t)} G^{\vec{\Omega}\times\vec{x}_{t+\Delta t}}\vec{a}(\vec{x}_{t+\Delta t},t+\Delta t) \qquad (4)$$

$$\hat{a}(\hat{x},t) = R^{\Omega t} G^{\vec{\Omega}\times\vec{x}}\vec{a}(\vec{x},t) \qquad (5)$$

use $\vec{X}_{t+\Delta t} = \vec{X}_t + \Delta t\,\vec{\Omega}\times\vec{x}$ in (4)

expand to first order in Δt:

$$G^{\vec{\Omega}\times\vec{x}_{t+\Delta t}}\vec{a}\left(\vec{x}_{t+\Delta t},\, t+\Delta t\right) = G^{\vec{\Omega}\times\vec{x}_t}\vec{a}(\vec{x},t) + \Delta t\left[(\vec{\Omega}\times\vec{x})\cdot\vec{\nabla}\right]G^{\vec{\Omega}\times\vec{x}_t}\vec{a}(x,t)$$
$$\Delta t\, G^{\vec{\Omega}\times\vec{x}_t}\partial_t\vec{a}(\vec{x},t) \qquad (5a)$$

also: from 1a & 2 we see
also that for $\Delta t \to 0$ $R^{\Omega\Delta t}\vec{a} = \vec{a} + \Delta t(\vec{a}\times\vec{\Omega})$

$$\text{or } \lim_{\Delta t\to 0}\frac{R^{\Omega\Delta t}\vec{a} - \vec{a}}{\Delta t} = (\vec{a}\times\vec{\Omega}) \qquad (6)$$

$$\text{or } \lim_{\Delta t\to 0}\frac{R^{\Omega\Delta t}G^{\vec{\Omega}\times\vec{x}}\vec{a} - G^{\vec{\Omega}\times\vec{x}}\vec{a}}{\Delta t} = G^{\vec{\Omega}\times\vec{x}}\vec{a}\times\vec{\Omega} \qquad (7)$$

Lets combine 4, 5a, 7 into 3b systematically:

a

$$\Rightarrow \frac{\partial\hat{\vec{a}}}{\partial t}(\hat{x},t) = \lim_{\Delta t\to 0}\frac{\hat{a}(\hat{x},t+\Delta t) - \hat{a}(\vec{x},t)}{\Delta t} \qquad (3a \text{ again})$$

$$\lim_{\Delta t\to 0}\Big[R^{\Omega(t+\Delta t)}\overset{Ⓐ}{\left[1 + \Delta t\,(\vec{\Omega}\times\vec{x})\cdot\vec{\nabla} + \Delta t\frac{\partial}{\partial t}\right]}G^{\vec{\Omega}\times x_t}\vec{a}(\vec{x},t)$$
$$- R^{\Omega t}G^{(\Omega\times\vec{x})_t}\overset{Ⓑ}{\vec{a}(\vec{x},t)}\Big]/\Delta t \qquad (8)$$

Now use $R^{\Omega(t+\Delta t)} = R^{\Omega t}R^{\Delta\Omega t}$ for small angle $\Omega\Delta t$:
rotations [check using

$$\underbrace{\begin{pmatrix}\cos\Omega t & \sin\Omega t\\ -\sin\Omega t & \cos\Omega t\end{pmatrix}}_{R^{\Omega t}}\underbrace{\begin{pmatrix}1 & \Delta t\Omega\\ -\Delta t\Omega & 1\end{pmatrix}}_{R^{\Delta t\Omega}} = \begin{pmatrix}\cos\Omega(t+\Delta t) & \sin\Omega(t+\Delta t)\\ -\sin\Omega(t+\Delta t) & \cos\Omega(t+\Delta t)\end{pmatrix}$$

for $\Delta t\Omega \ll \Omega t$]

Combine with (7), so that (8) $\Rightarrow$

$$\frac{\partial \hat a(\hat x,t)}{\partial t} = R^{\Omega t}\left[(\vec\Omega\times\vec x)\cdot\vec\nabla + \frac{\partial}{\partial t} \underset{\uparrow}{-\vec\Omega\times}\right] G^{(\vec\Omega\times\vec x)_t}\,\vec a \qquad (9)$$

terms Ⓐ & Ⓑ combined after using (7).

Now apply to $\dfrac{\partial \hat u(\vec x,t)}{\partial t}$:

$$\frac{\partial \hat u}{\partial t}(\hat x,t) = R^{\Omega t}\left[\frac{\partial}{\partial t} + (\Omega\times\vec x)\cdot\vec\nabla - \vec\Omega\times\right] G^{(\vec\Omega\times\vec x)_t}\,\vec u(\vec x,t)$$

use $G^{\vec\Omega\times\vec x}\,\vec u(\vec x,t) = \vec u(\vec x,t) + \vec x\times\vec\Omega$

$\Rightarrow$

$$\frac{\partial \hat u}{\partial t}(\hat x,t) = R^{\Omega t}\left[\partial_t + (\Omega\times\vec x)\cdot\vec\nabla - \vec\Omega\times\right](\vec u + \vec x\times\vec\Omega)$$

$$= R^{\Omega t}\left[\partial_t + (\Omega\times\vec x)\cdot\vec\nabla - \vec\Omega\times\right]\vec u$$

$$+ R^{\Omega t}\left[\cancel{\partial_t} + (\Omega\times\cancel{\vec x})\cdot\vec\nabla - \vec\Omega\cancel{\times}\right]\cancel{(\vec x\times\vec\Omega)}$$

since $\partial_t(\vec x\times\vec\Omega) = 0$

since $(\vec q\cdot\vec\nabla)(\vec x\times\vec\Omega) = (\vec q\times\vec\Omega)$ for any $\vec q$

$$= \boxed{R^{\Omega t}\left[\partial_t + (\Omega\times\vec x)\cdot\vec\nabla - \vec\Omega\times\right]\vec u} \qquad (8)$$

Combining (8) & (3a) =>

$$\overset{\text{use (8)}}{\downarrow}\qquad\qquad \overset{\text{use 3a}}{\downarrow}$$

$$\frac{\partial \hat{u}}{\partial t} + \hat{u}\cdot\nabla\hat{u} = R^{\Omega t}\Big[\overset{a}{\partial_t} + \overset{b}{\cancel{(\Omega\times\vec{x})\cdot\vec{\nabla}}} - \overset{c}{\vec{\Omega}\times}\Big]\vec{u}$$

$$+ R^{\Omega t}\Big[\big\{(\overset{d}{\vec{u}} + \overset{e}{\cancel{\vec{x}\times\vec{\Omega}}})\cdot\vec{\nabla}\big\}\vec{u} + \big\{(\overset{f}{\vec{u}} + \overset{g}{\vec{x}\times\Omega})\times\vec{\Omega}\big\}\Big]$$

$$= R^{\Omega t}\Big[\overset{a}{\partial_t\vec{u}} + \overset{d}{\vec{u}\cdot\nabla\vec{u}} + \overset{c+f}{2\vec{u}\times\vec{\Omega}} + \overset{g}{(\vec{x}\times\vec{\Omega})\times\vec{\Omega}}\Big] \qquad (8a)$$

for the last two terms:

$$R^{\Omega t}\left[2\vec{u}\times\vec{\Omega} + (\vec{x}\times\vec{\Omega})\times\vec{\Omega}\right]$$

$$= \left[2R^{\Omega t}\vec{u}\times\Omega + (R^{\Omega t}\vec{x}\times\vec{\Omega})\times\vec{\Omega}\right]$$

$$= \Big[2\overbrace{(\hat{u} - R^{\Omega t}\vec{x})}^{\text{from (3)}}\times\vec{\Omega} + (R^{\Omega t}\vec{x}\times\vec{\Omega})\times\vec{\Omega}\Big]$$

$$= \Big[2\hat{u}\times\vec{\Omega} - 2\underbrace{R^{\Omega t}\vec{x}}_{=\hat{x}}\times\vec{\Omega} + (\underbrace{R^{\Omega t}\vec{x}}_{=\hat{x}}\times\vec{\Omega})\times\vec{\Omega}\Big]$$

$$= \left[2\hat{u}\times\vec{\Omega} - (\hat{x}\times\vec{\Omega})\times\vec{\Omega}\right] \qquad (9)$$

Combine (8a) & (9) =>

$$\frac{\partial \hat{u}}{\partial t} + \hat{u}\cdot\nabla\hat{u} = \underbrace{R^{\Omega t}\left[\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u}\right]}_{\text{work on this}} + \underbrace{2\hat{u}\times\vec{\Omega} - (\hat{x}\times\vec{\Omega})\times\vec{\Omega}}_{\text{these are coriolis \& centrifugal terms in final form}} \qquad (10)$$

note

$$\partial_t \vec{u} + \vec{u}\cdot\vec{\nabla}\vec{u} = \underbrace{-\vec{\nabla}(p+\phi) + \nu\nabla^2\vec{u}}_{\vec{f}}$$

is lab frame Navier stokes

- First note that $\hat{\nabla} = R^{\Omega t}\vec{\nabla}'$

$$= R^{\Omega t}\vec{\nabla}$$

gradient doesn't change under Galilean transform.

and p, ϕ scalars are invariant so: $-\hat{\nabla}(p+\phi) = -R^{\Omega t}\vec{\nabla}(p+\phi)$ $\qquad (1.1)$

- For the viscous term:

$$\nu\hat{\nabla}^2\hat{u} = \nu\nabla'^2\hat{u} = \nu\nabla^2\hat{u}$$

$$= \nu\nabla^2 R^{\Omega t}\vec{u}'$$

$$= \nu\nabla^2 R^{\Omega t}\vec{u}'$$

$$= \nu R^{\Omega t}\nabla^2(\vec{u} - \vec{\Omega}\times\vec{x})$$

but in our cartesian coords,

$$\nabla^2(\vec{\Omega}\times\vec{x}) = 0$$

$$\Rightarrow \nu\hat{\nabla}^2\hat{u} = \nu R^{\Omega t}\nabla^2\vec{u} \qquad (1.2)$$

Combinig (12), (11), (10) the Navier-Stokes eqn. in the rotating frame is then:

$$\boxed{\frac{\partial \hat{u}}{\partial t} + \hat{u}\cdot\nabla\hat{u} = -\hat{\nabla}(P+\phi) + \nu\hat{\nabla}^2\hat{u} + 2\hat{u}\times\vec{\Omega} - (\hat{x}\times\vec{\Omega})\times\vec{\Omega}}$$

✓

Note:

For invariance of ∇^2 note:

$$\frac{\partial}{\partial x} = \frac{dx'}{dx}\frac{\partial}{\partial x'} + \frac{dy'}{dx}\frac{\partial}{\partial y'}$$

$$\frac{\partial}{\partial y} = \frac{dx'}{dy}\frac{\partial}{\partial x'} + \frac{dy'}{dy}\frac{\partial}{\partial y'}$$

$$R = \begin{pmatrix} \cos\Omega t & \sin\Omega t & 0 \\ -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R\begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow$

$$\frac{\partial}{\partial x} = \cos\Omega t\frac{\partial}{\partial x'} + \sin\Omega t\frac{\partial}{\partial y'}$$

$$\frac{\partial}{\partial y} = -\sin\Omega t\frac{\partial}{\partial x'} + \cos\Omega t\frac{\partial}{\partial y'}$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} = (\cos^2\Omega t)\frac{\partial^2}{\partial x'^2} + \sin^2\Omega t\frac{\partial^2}{\partial y'^2} + 2\sin\Omega t\cos\Omega t\frac{\partial}{\partial x'}\frac{\partial}{\partial y'}$$

$$\frac{\partial^2}{\partial y^2} = (\sin^2\Omega t)\frac{\partial^2}{\partial x'^2} + \cos^2\Omega t\frac{\partial^2}{\partial y'^2} - 2\sin\Omega t\cos\Omega t\frac{\partial}{\partial x'}\frac{\partial}{\partial y'}$$

$$\Rightarrow \nabla^2 = \partial_x^2 + \partial_y^2 = \partial_{x'}^2 + \partial_{y'}^2$$

Lagrangian approach

$$\left.\frac{d\vec{x}}{dt}\right|_{\text{inertial}} = \left.\frac{d\vec{x}}{dt}\right|_{\text{rot}} + \vec{\Omega}\times\vec{x} \qquad \left(\left(\frac{d}{dt}+\vec{\Omega}\times\right)\left(\frac{d}{dt}+\vec{\Omega}\times\right)\right)$$

$$\left.\frac{d^2\vec{x}}{dt^2}\right|_{\text{inertial}} = \left.\frac{d^2\vec{x}}{dt^2}\right|_{\text{rot}} + \vec{\Omega}\times\left.\frac{d\vec{x}}{dt}\right|_{\text{rot}} + \vec{\Omega}\times\left.\frac{d\vec{x}}{dt}\right|_{\text{rot}} + \vec{\Omega}\times\vec{\Omega}\times\vec{x}$$

$$\left.\frac{d\vec{v}}{dt}\right|_{\text{inertial}} = \left.\frac{d\vec{v}}{dt}\right|_{\text{rot}} + 2\Omega\times\vec{v} + \vec{\Omega}\times\vec{\Omega}\times\vec{x}$$

$$-\vec{\nabla}p + \nu\nabla^2\vec{v} = \partial_t\vec{v}\Big|_{\text{rot}} + \vec{v}\cdot\nabla\vec{v} + 2\vec{\Omega}\times\vec{v} + \vec{\Omega}\times\vec{\Omega}\times\vec{x}$$

$$-\frac{1}{2}\vec{\nabla}(\vec{\Omega}\times\vec{x}) = -\frac{1}{2}\nabla\left(\Omega^2\vec{x}^2 - (\vec{\Omega}\cdot\vec{x})(\vec{\Omega}\cdot\vec{x})\right)$$

$$= \Omega^2\vec{x} - (\vec{\Omega}\cdot\vec{x})\vec{\Omega}$$

$$= -\vec{\Omega}\times(\vec{\Omega}\times\vec{x})$$

$$p \to p_{eff} = \left(p + \frac{1}{2}(\Omega\times\vec{x})^2\right)$$

Hydrodynamics in Rotating Frame

Navier-stokes equation in rotating frame

$$\frac{d\vec{u}}{dt} \rightarrow \frac{d\hat{u}}{dt} + 2\vec{\Omega}\times\hat{u} + \vec{\Omega}\times(\vec{\Omega}\times\vec{x}) \qquad (198)$$

(Here, assume that Ω = constant.)

when written in cartesian coords $\vec{r} = (x, y, z)$

lets use $\vec{r}$ for $\hat{u}$:

$$\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \vec{F} + \nu\vec{\nabla}^2\vec{v} \overbrace{- 2\vec{\Omega}\times\vec{V}}^{\text{coriolis force}} \overbrace{- \vec{\Omega}\times(\vec{\Omega}\times\vec{r})}^{\text{centrifugal force}}$$

$\downarrow$ use $\vec{r}$ for $\hat{x}$

note that $-\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = -\frac{1}{2}\nabla(\vec{\Omega}\times\vec{r})^2$ for Ω = constant

$$\Downarrow \qquad\qquad \Downarrow$$

$$\left(-(\vec{\Omega}\cdot\vec{r})\vec{\Omega} + \Omega^2\vec{r} = -\frac{1}{2}\vec{\nabla}(\vec{\Omega}\cdot r)^2 + \frac{1}{2}\vec{\nabla}(\Omega^2 r^2) \right.$$

$$\left. \text{if } \frac{\partial\Omega}{\partial r} = 0 \Rightarrow \quad = -\vec{\Omega}(\vec{\Omega}\cdot\vec{r}) + \Omega^2 r \right)$$

for Ω = constant

thus:

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla}\underbrace{\left(\Phi - \frac{1}{2}|\vec{\Omega}\times\vec{r}|^2\right)}_{\Phi_{eff}} + \nu\nabla^2\vec{v} - 2\vec{\Omega}\times\vec{v} \qquad (199)$$

Since centrifugal force can be written as potential, it is easier to deal with than the coriolis term $-2\vec{\Omega}\times\vec{v}$!

$\rightarrow$

For slowly rotating object like Earth, centrifugal term is small. But large scale flows, hurricanes and ocean currents are influenced by coriolis force. To assess when it's important, compare $\vec{V}\cdot\vec{\nabla}\vec{V}$ to $\vec{\Omega}\times\vec{V}$ for scale L

$$|\vec{V}\cdot\vec{\nabla}\vec{V}| \sim V^2/L$$
$$|\vec{\Omega}\times\vec{V}| \sim \Omega V \quad \Rightarrow \quad R_0 \equiv \frac{V^2}{\Omega V L} = \frac{V}{\Omega L} \leq 1$$

velocity & its structure

e.g. Earth rotation

$\Rightarrow$ coriolis force is important when the Rossby Number $R_0 < 1$, where V, L are typical velocity and length scales of the flow.

For fluid phenomena in the lab, $R_0 \gg 1$, but in atmosphere and oceans $R_0 < 1$. Studying large scale atmospheric or ocean flows is Geophysical Fluid Dynamics. Usually one assumes thin spherical fluid shell with small Rossby number.

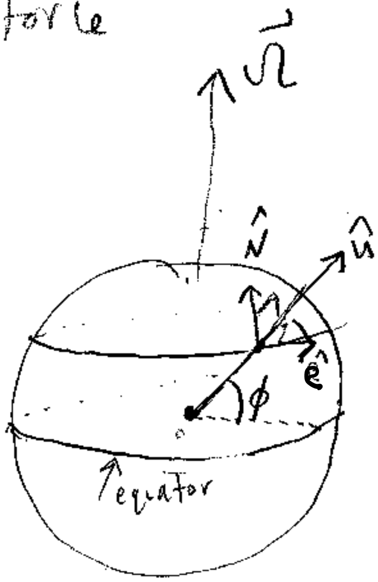
More detailed

Calculation of Rossby number

$R_0 = $ ratio of inertial terms in Navier-Stokes / Coriolis force

$$= \frac{|\vec{v}\cdot\vec{\nabla}\vec{v}|}{|2\vec{\Omega}\times\vec{v}|}$$

Consider Earth:



$\hat{u}$ - "up" normal to plane
$\hat{N}$ north
$\hat{e}$ - east
ϕ = lattitude

coords defined at a point on sphere

in coordinate system $(\hat{e}, \hat{N}, \hat{u})$:

$$\vec{\Omega} = (0, |\Omega|\cos\phi, |\Omega|\sin\phi), \quad \vec{V} = (V_e\, V_N, V_u)$$

$$-2\vec{\Omega}\times\vec{v} = -2\left(\cancel{V_u|\Omega|\cos\phi} - |\Omega| V_N \sin\phi,\ |\Omega| V_e \sin\phi,\ -\cancel{V_e|\Omega|\cos\phi}\right)$$

(V_u usually small) for both $V_u, V_e \ll V_N$ we get:

$\Rightarrow$

$$|2\vec{\Omega}\times\vec{v}| \simeq |2|\Omega| V_N \sin\phi| \Rightarrow$$

$$R_0 = \frac{|\vec{v}\cdot\vec{\nabla}\vec{v}|}{|2|\Omega| V\sin\phi|} \simeq \frac{V^2}{2L\cancel{V}\Omega\sin\phi} = \left|\frac{V}{2L\Omega\sin\phi}\right|$$

Geostrophic approximation

In atmospheric or ocean applications assume flows are nearly "horizontal" (non-radial) in a thin layer (much thinner than Earth's radius).

For low Rossby Number flows, left hand terms of (199) are small compared to leading terms on right side. We can also neglect viscosity and centrifugal terms (slow rotation, nearly inviscid flow). Then (199) =>

$$-\frac{\nabla p}{\rho} - \vec{g}\hat{e}_r - 2\vec{\Omega}\times\vec{v} = 0 \qquad (g=-\nabla\phi) \qquad (200)$$

Usually, coriolis force is small compared to gravity in vertical direction so the $\hat{r}$ component becomes

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = g \qquad (201)$$

but horizontal direction:

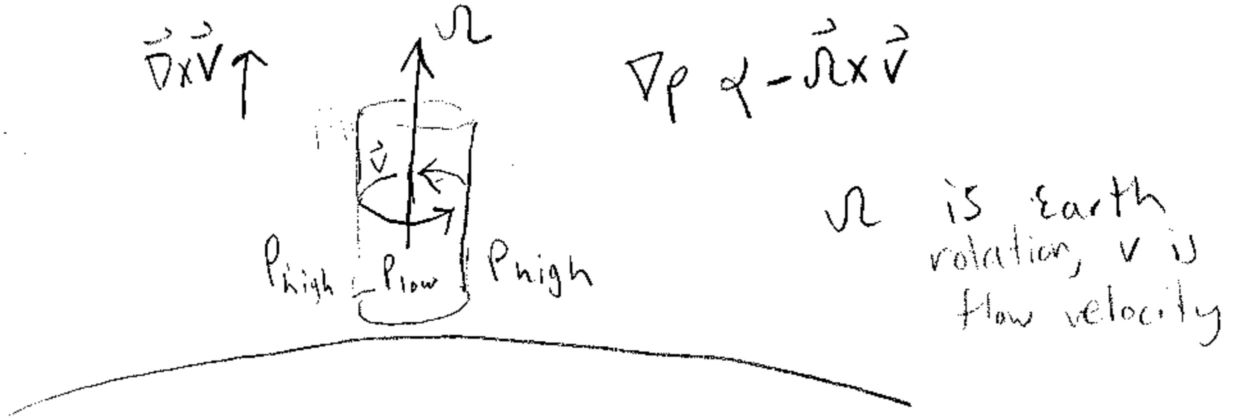
$$\nabla_h p = -2\rho(\vec{\Omega}\times\vec{v})_h$$

Geostrophic Approximation

Horizontal pressure gradient balanced by coriolis force →

Note the interesting fact that the velocity is $\perp$ to the gradient in pressure if cross product with Ω balances $\nabla_h p$.

This means that if there is a low pressure region in the atmosphere, velocity does not flow into low pressure region, but flows around it.:



$\nabla p \propto -\vec{\Omega}\times\vec{v}$

Ω is Earth rotation, v is flow velocity

vortex $\vec{\nabla}\times\vec{v}$ is $\parallel$ to underlying rotation $\vec{\Omega}$

$\Rightarrow$ Cyclonic circulation around low pressure region in a vortex tube

Vorticity in Rotating Frame

Using vector identity for $-v\cdot\nabla v$ we can write (199)

$$\frac{\partial\vec{v}}{\partial t} = \vec{v}\times\vec{\omega} - \nabla\left(\frac{p}{\rho} + \frac{1}{2}v^2 + \phi - \frac{1}{2}(\vec{\Omega}\times\vec{r})^2\right) - 2\vec{\Omega}\times\vec{v} \qquad (202)$$

taking curl $\Rightarrow$ $$\frac{\partial w}{\partial t} = \nabla\times(\vec{v}\times\vec{\omega}) + \nabla\times(\vec{v}\times 2\vec{\Omega}) \qquad (203)$$

assuming Ω = constant we can write (203) as

$$\frac{\partial}{\partial t}(\omega + 2\vec{\Omega}) = \nabla \times [V \times (\omega + 2\Omega)] \qquad (204)$$

This is of the form $\frac{\partial \vec{Q}}{\partial t} = \nabla \times \vec{V} \times \vec{Q}$ with $Q = (\vec{\omega} + 2\vec{\Omega})$ thus we know from derivation of kelvin circulation theorem that

$$\frac{d}{dt}\int (\omega + 2\Omega) \cdot dS = 0 \qquad \text{(Bjerknes Theorem)} \qquad (205)$$

This generalization implies that if Ω is increased, local vorticies must increase OPPOSITELY to underlying rotation to satisfy the theorem.

Self-gravitating masses - Maclaurin & Jacobi ellipsoids:

Consider an initially spherically symmetric, gravitating fluid of uniform density and start it rotating: Flattening near the poles is expected. Assume Ω is constant, then move to frame in which $V = 0$ (rotating frame) and consider the equilibrium configurations. Maclaurin = bi-axial ellipsoid, Jacobi = tri-axial ellipsoid

Accreretion with Angular momentum:
(a context for mean field fluid dynamics and turbulent transport)

- Binaries, (in particular X-ray binaries) are where we have learned a lot about accretion: why binaries?
- orbiting system, tidal forces
  ⇒ shearing of material into a disk; to accrete, ∡ momentum must be shed (or equivalently, transported outward)

two reasons for binary. mass transfer via accretion:

(1) one of the stars may increase in size during evolution: companion can rip off outer layers

(2) ejection of mass by stellar wind and accretion onto the companion

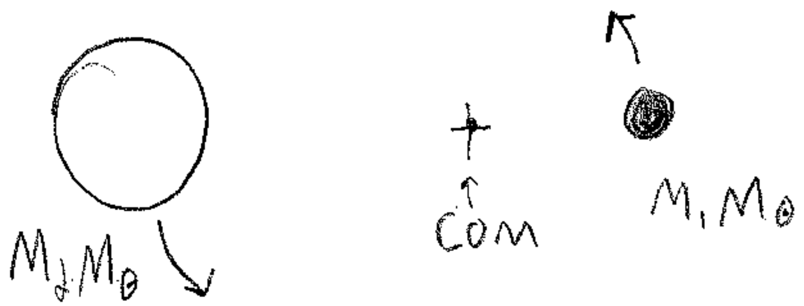
Important concept is Roche lobe overflow

→

- 19th century Edouard Roche studied destruction of planetary satellites (moons, etc.)

- Basic idea was to consider orbit of test particle in grav potential of two orbiting, massive bodies

- Assume two "centrally condensed" stars orbit each other in keplerian, circular orbits, and consider test particle gas motion in the potential (also called "restricted 3-body problem", because gas is assumed not to influence the binary orbit)

- 

- Gas flow between stars governed by Euler equation. In rotating frame:

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} = \underbrace{-\vec{\nabla}\phi_R}_{\frac{\text{Grav + Cent. force}}{\text{mass}}} \underbrace{- 2\vec{\omega}\times\vec{v}}_{\text{coriolis force/mass}} - \frac{1}{\rho}\nabla P$$

$\vec{w}$ from Kepler's law $\vec{w} = w_z\hat{z}$.

$$\vec{w} = \left[\frac{G(M_1+M_2)}{a^3}\right]^{1/2}\hat{z}$$ normal to the orbit plane, a = binary separation

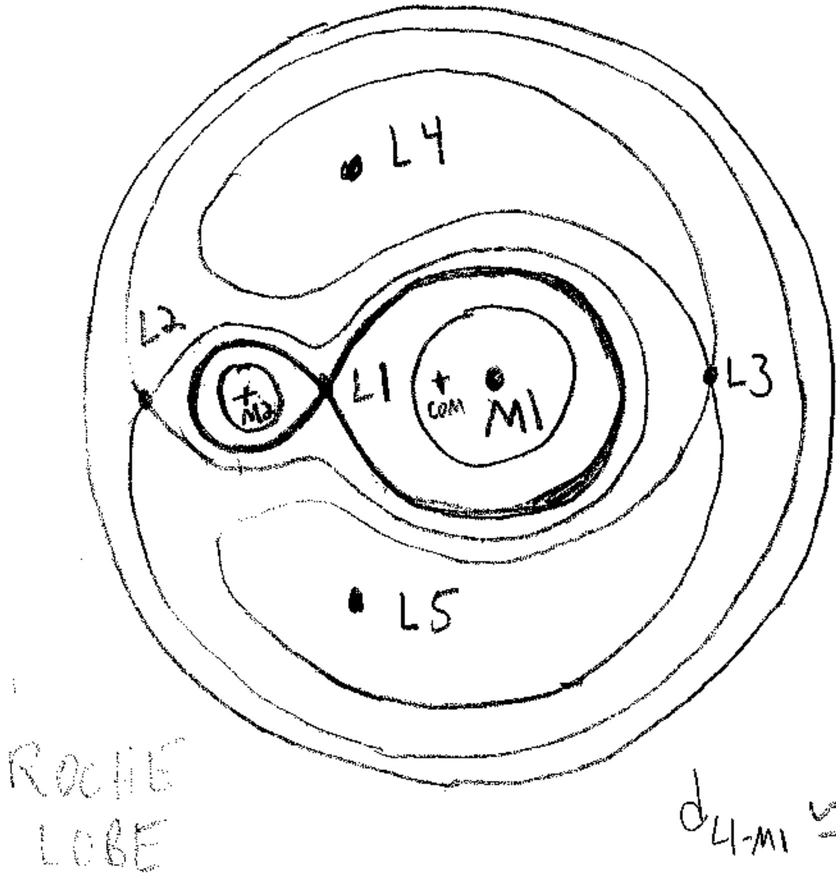
$$\phi_R = -\frac{GM_1}{|\vec{r}-\vec{r}_1|} - \frac{GM_2}{|\vec{r}-\vec{r}_2|} - \frac{1}{2}(\vec{w}\times\vec{r})^2$$

Roche potential

Set ϕ_R = constant and plot:

L4, L5 maxima, (but coriolis force stabilizes)

L1, saddle → material can overflow if e.g. M2 fills lobe, it can accrete onto M1



note: distance L1 —— M1

$$d_{L1-M1} \simeq \frac{1}{2}a - 0.23a\log_{10}\frac{M_2}{M_1}$$

Engine evolution:

① assume both stars are smaller than Roche lobe and are in circular orbit, and tidally locked

=> surface of each star corresponds to circular equipotential surface. (follows from momentum equation with $\vec{v}=0$ and $\nabla p = 0$ on surface of star)

=> Binary is fully <u>detached</u>

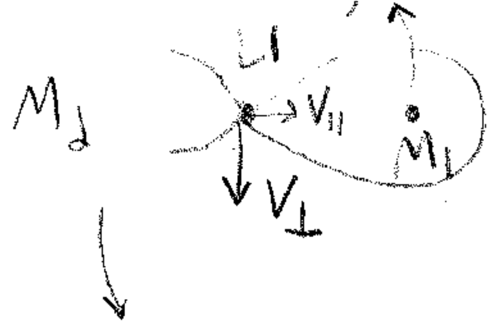
② If one star then swells & fills Roche lobe (usually called <u>secondary</u> star) then primary can accrete this is a <u>semi-detached</u> binary

③ can you guess what a contact binary is?

(both stars fill Roche lobes)

Formation of Disk in Binary

Note that material is pushed through Roche lobe at $\simeq c_s$, but because of orbit, $V_{\parallel} << V_{\perp}$ for typical systems.



→ remember, previous equations are in rotating frame about c.o.m; as soon as material edges over L1, M1 sees it in orbit.

$V_{\parallel} \simeq c_s$; $V_{\perp} \simeq V_K$

typically $V_{\parallel} << V_{\perp}$, $\Big[$ $c_s \sim 10\ \text{km/s}\ (T/10^4)$

$$V_{\perp} \simeq 10^2 M_1^{1/3}\left(1+\frac{M_2}{M_1}\right)^{1/3} P_{day}^{-1/3}\ \text{km/s} \Big]$$

from Kepler's law

⇒ gas has $\measuredangle$ momentum which it needs to shed to accrete

⇒

Gas will first orbit in circle at

at $V_\phi(R_{circ}) = \left(\frac{GM_1}{R_{circ}}\right)^{1/2}$

L1-M1 distance

with $\underbrace{V_\phi(R_{circ})\,R_{circ}}_{\measuredangle\ \text{momentum conservation}} = \left(d_{L1-M1}^{2}\,\omega\right)$ ← orbit velocity

---

Using Kepler: $\left(\begin{array}{l}4\pi^2 a^3 = G(M_1+M_2)P^2 \\ \text{and } P = \frac{2\pi}{\omega}\end{array}\right.$ and $V_\phi(R_{circ}) = \left(\frac{GM_1}{R_{circ}}\right)^{1/2}$)

and formula for d_{L1-M1} on page 134 ⇒

$$\Rightarrow R_{circ}/a = \left(\frac{4\pi^2}{GM_1P^2}\right)a^3\left(d_{L1-M1}/a\right)^4 =$$

$$= R_{circ}/a = \left(1+\frac{M2}{M1}\right)\left(\frac{1}{2} - 0.25\log_{10}\frac{M_2}{M_1}\right)^4$$

(it is possible to have $R_{circ} < R_{*,1}$ ← size of M_1, but never true for NS, BH, or WD systems)

So we have gas orbiting at

$R = R_{circ}$, but that's not

the end of the story!

→

Internal dissipation will lead to radiation → loss of radiation => loss of kinetic energy => material sinks deeper into grav. potential => accretion => loss of ∡ momentum.

→ Now t_{cool} (cooling time) is usually $\ll t_{acc}$ and $t_{dyn} \ll t_{acc}$ so that material spirals in slowly

→ but if material loses ∡ momentum what carries it? : some material actually goes outward, so "initial ring" turns into disk.

→ usually for compact objects disk is not self gravitating ($\rho \ll M/R^3$)

$$\Rightarrow \quad \Omega \simeq \Omega_K = \left(\frac{GM_1}{R^3}\right)^{1/2}$$ Keplerian orbits

⟶

→ kinetic energy of gas element Δm in keplerian orbit is

$\frac{1}{2}\frac{GM\Delta m}{R_*}$ ⇒ luminosity lost during accretion is

$L_{disc} = \frac{1}{2}\frac{GM\dot{M}}{R_*}$, $\underline{but}$ grav. pot. energy is $\frac{GM\Delta m}{R_*}$, so 1/2 is radiated or dissipated in disk, other 1/2 released at surface <u>of star</u>

→ Compare fraction to ∡ momentum: $R^2\Omega(R) \propto R^{1/2}$

now since $R_{circ} \gg R_*$ in general nearly all ∡ momentum must be lost. → dissipation processes [illegible] cause conversion of kinetic energy to heat and transport ∡ momentum

→

Viscous evolution equations for Accretion Disks

Rather than try to "construct" viscous transport from first principles as attempted (and done very incorrectly in some textbooks) lets assume that turbulence acts as a viscosity to then derive the accretion disk transport equations. Note that this "assumption" is really equivalent to what is currently used in disk modeling for direct comparisons with observations but not a fundamentally consistent or complete approach. It is a theoretical frontier to improve the theory.

So for the present we will explicitly assume a "closure" for which the Reynolds stess terms associated with turbulent fluctuations Navier stokes equation take the form:

$$\overline{\vec{u}'\cdot\nabla\vec{u}'} = -\nabla\times(\nu_T \nabla\times\bar{u}) \qquad (1r)$$

where $\underset{\text{fluctuation}}{\vec{u} = \vec{u}'} + \underset{\text{mean}}{\bar{u}}$

(closure to obtain Standard acc. disk theory as mean field theory)

$$\nu_T = \underbrace{\frac{\bar{\Sigma}}{2H}}\underbrace{\eta}_{\text{kinematic viscosity}} \simeq v_T l_T = \text{turbulent viscosity}$$

See nextpage for $\bar{\Sigma}$ and H: $\rightarrow$

The continuity equation is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \qquad \text{for } \rho = \rho(r,\phi,z),\ u = u(r,\phi,z) \qquad (2r),$$

define $\bar{\Sigma} = \underbrace{\dfrac{\int_{-H}^{H}\int_0^{2\pi} \rho \, d\phi \, dz}{2\pi}}_{\text{mean surface density}}$ and $\bar{u} = \dfrac{\int_{-H}^{H}\int_0^{2\pi} \rho u \, d\phi \, dz}{\int_{-H}^{H}\int_0^{2\pi} \rho \, d\phi \, dz}$

$$= \frac{\int_{-H}^{H}\int_0^{2\pi} \rho u \, d\phi \, dz}{2\pi \bar{\Sigma}}$$

(H is 1/2 thickness of disk) = density weighted mean velocity

$\Rightarrow$ $\bar{\Sigma} = \bar{\Sigma}(r)$, $\bar{u} = \bar{u}(r)$ (z, ϕ are averaged out)

then after integrating over $d\phi \, dz$, (1r) $\Rightarrow$

$$\frac{\partial \bar{\Sigma}}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R \bar{\Sigma} U_R\right) = 0 \quad \text{(cylind. coords)} \qquad (3r)$$

Similarly, from the ϕ component of Nav.-Stokes:

$$\bar{\Sigma}\left(\frac{\partial \bar{u}_\phi}{\partial t} + \bar{u}_R \frac{\partial \bar{u}_\phi}{\partial R} + \frac{\bar{u}_R \bar{u}_\phi}{R}\right) = \frac{\partial}{\partial R}\left(\nu_T \bar{\Sigma} \frac{\partial \bar{u}_\phi}{\partial R}\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(\nu_T \bar{\Sigma} \bar{u}_\phi\right) - \frac{\nu_T \bar{\Sigma} \bar{u}_\phi}{R^2} - \frac{2 u_\phi}{R}\frac{\partial \Omega}{\partial R} \qquad (4r)$$

Hereafter for <u>notational simplicity</u> I drop the overbars on $\bar{u}, \bar{\Sigma}$ and write $\nu_T = \nu$. That is $\bar{u} \to u$ and $\bar{\Sigma} \to \Sigma$.

Then multiply eqn 3r by $R u_\phi$:

$\Rightarrow$

$$R u_\phi \frac{\partial \Sigma}{\partial t} + u_\phi \frac{\partial}{\partial R}\left(R \Sigma u_R\right) = 0 \qquad (5r)$$

and multiply Eqn (3r) by R:

$\Rightarrow$

$$R\Sigma \frac{\partial u_\phi}{\partial t} + R\Sigma u_R \frac{\partial u_\phi}{\partial R} + \Sigma u_R u_\phi = R\frac{\partial}{\partial R}\left(\nu\Sigma\frac{\partial u_\phi}{\partial R}\right) + \frac{\partial(\nu\Sigma u_\phi)}{\partial R} - \frac{\nu\Sigma u_\phi}{R} - \frac{\partial u_\phi \partial \eta}{\partial R} \qquad (6r)$$

next page $\rightarrow$

Footnote: the ϕ component of the axisymmetric Navier stokes equation equation that arises if one assumes $\frac{\partial}{\partial\phi} = 0$ of all quantities and assumes $u = \bar{u}$ and $\rho = \bar{\rho}$ and simply replaces $\rho\eta$ with $\bar{\rho}\eta$ is

$$\bar{\rho}\left(\frac{\partial \bar{u}_\phi}{\partial t} + \bar{u}_R\frac{\partial \bar{u}_\phi}{\partial R} + \frac{\bar{u}_R\bar{u}_\phi}{R}\right) = \frac{\partial}{\partial R}\left(\eta\frac{\partial \bar{u}_\phi}{\partial R}\right) + \frac{\partial}{\partial z}\left(\eta\frac{\partial \bar{u}_\phi}{\partial z}\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(\eta\bar{u}_\phi\right) - \frac{\bar{u}_\phi\eta}{R^2} - \frac{\partial \bar{u}_\phi}{R}\frac{\partial \eta}{\partial R}$$

6r can be derived by by integrating this over z. Often the distinction between u and $\bar{u}$ is incorrectly ignored so one should really formally average

Add (5r) + (6r) using $\Omega = \frac{u_\phi}{R}$

@ Hη

$\Rightarrow$

$$R\underbrace{\frac{\partial(\Sigma u_\phi)}{\partial t}}_{\frac{\partial(\Sigma u_\phi R)}{\partial t}} + \overbrace{\frac{\partial}{\partial R}\left(R\Sigma u_R u_\phi\right) + \Sigma u_R u_\phi}^{\frac{1}{R}\frac{\partial}{\partial R}(R^2\Sigma u_R u_\phi)}$$

$$= R\frac{\partial}{\partial R}\left(\nu\Sigma\frac{\partial}{\partial R}(\Omega R)\right)$$

$$+ \frac{\partial}{\partial R}(\nu\Sigma\,\Omega R) - \nu\Sigma\Omega$$

$$+ \frac{2\Omega R}{R}\frac{\partial\Sigma\nu}{\partial R}$$

$$\therefore \frac{\partial(\Sigma u_\phi R)}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R^2\Sigma u_R u_\phi\right) = \frac{1}{R}\frac{\partial}{\partial R}\Big(\nu\Sigma\big(\underset{Ⓐ}{R^3\frac{\partial\Omega}{\partial R}} + \underset{Ⓑ}{\Omega R^2}\big)\Big) \quad (7r)$$

$$Ⓑ + Ⓒ + Ⓓ + Ⓔ = 0$$

$$- 2\nu\Sigma\frac{\partial}{\partial R}(\Omega R) \quad Ⓒ$$

$$+ R\frac{\partial}{\partial R}(\nu\Sigma\Omega) \quad Ⓓ$$

$$- 2\Omega R\frac{\partial(\Sigma\nu)}{\partial R} \quad Ⓔ$$

$$\Downarrow$$

$$= \frac{1}{R}\frac{\partial\left(\nu\Sigma R^3\frac{\partial\Omega}{\partial R}\right)}{\partial R} + 0$$

(143r)

$$\underset{\substack{\uparrow\\ \text{Eulerian change}\\ \text{of } \phi \text{ momentum}\\ \text{per area}}}{\frac{\partial}{\partial t}\left(R\Sigma U_\phi\right)} + \underset{\substack{\uparrow\\ \text{divergence of flux of}\\ \phi \text{ momentum}\\ \text{per area}}}{\frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 U_\phi U_R\right)} = \underbrace{\frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{d\Omega}{dR}\right)}_{\text{viscous } \frac{\text{torque}}{\text{area}}} \quad (8r)$$

$H\eta R^3 \frac{d\Omega}{dR}$

$-\frac{3}{2} H\eta R^{3}$

$-\frac{3}{2} H\eta R^3\Omega$

multiply both sides by $2\pi R dR$ so that equation represents angular momentum evolution of an annulus.

$$dR\frac{\partial}{\partial t}\left(2\pi R^2\Sigma U_R\right) + \frac{dR}{\cancel{R}}\frac{\partial}{\partial R}\left(2\pi \overset{\downarrow}{R^{2}}\Sigma U_\phi U_R\right) = \underbrace{\frac{dR\,\partial}{\partial R}\left(2\pi\nu\Sigma R^3\frac{\partial\Omega}{\partial R}\right)}_{\substack{\text{net viscous torque on}\\ \text{annulus}}} \quad (9r)$$

torque at radius R:

$$dR\frac{\partial G}{\partial R} = dG$$

$\Rightarrow$ $G(R) = 2\pi\nu\Sigma R^3\frac{\partial\Omega}{\partial R}$ ← (Eqn. 10r)

$= R \times (\text{viscous force})$

$= R\left(2\pi\nu\Sigma R^2\frac{\partial\Omega}{\partial R}\right)$

$= R\left(2\pi\nu\, 2\rho H\, R^2\frac{\partial\Omega}{\partial R}\right)$ ← (note: disk thickness $= 2H$)

$= R\Big(4\pi R H \underbrace{\rho\nu R\frac{\partial\Omega}{\partial R}}_{\sigma_{r\phi}}\Big) = R(2\times\text{Area of annulus})(\sigma_{r\phi}) = \text{torque}$

$\sigma_{r\phi}$ = force per unit area in tangential direction acting on surface with radial normal

Check physical consistency:

$G = 0$ for $\frac{d\Omega}{dR} = 0$ ✓

$G < 0$ for $\frac{d\Omega}{dR} < 0$ ✓

↑

total torque on ring of gas between R, $R+dR$:

$$G(R+dR) - G(R) = \frac{\partial G}{\partial R} dR = dG.$$ Now

$$\text{rate of work} = d\vec{F} \cdot \vec{v} \simeq d\vec{F} \cdot (\vec{\Omega} \times \vec{R})$$

$$= \vec{\Omega} \cdot (\vec{R} \times d\vec{F})$$

$$= \vec{\Omega} \cdot d\vec{G} = \pm \Omega |dG|$$

$\Rightarrow$ rate of work (because $d\vec{G} \parallel \pm \vec{\Omega}$)

$$= \Omega \frac{\partial G}{\partial R} dR = \frac{\partial (\Omega G)}{\partial R} dR - G \frac{\partial \Omega}{\partial R} dR$$

integrate: $\Rightarrow$ total work rate

$$= \underbrace{\int_{R_{in}}^{R_{out}} \frac{\partial (\Omega G)}{\partial R} dR}_{\text{boundary term}} - \underbrace{\int_{R_{in}}^{R_{out}} G \frac{\partial \Omega}{\partial R} dR}_{\text{internal dissipation term}}$$

→

dissipation term converts mechanical energy into particle energy → heat → radiation

per area (2 faces of ring) =>

$$\underset{\text{2 faces}}{} \frac{G\frac{\partial \Omega}{\partial R}\cancel{dR}}{4\pi R\cancel{dR}} = \frac{G(R)}{4\pi R}\frac{d\Omega}{dR}$$

$$= +\frac{1}{2}\nu\Sigma R^2\left(\frac{d\Omega}{dR}\right)^2 \left(\text{from } \underline{(1ar)} \atop \text{(page 143)}\right)$$

$= D(R) =$ energy loss rate per unit area from dissipation

Note we need to have $\frac{d\Omega}{dR} \neq 0$

need to know ν, Σ to compare to observations.

→

Viscosity can be estimated by characteristic velocity and length scale associated with particle motions & deflections.

The force density associated with the viscosity of the previous section comes from the $\rho \nu \nabla^2 V$ term in navier stokes equation. Recall that $\nu = \nu_T + \nu_{microphys}$ with $\nu_T \gg \nu_{microphys}$

to recall its importance we can compute the Reynolds number: ratio of $V \cdot \nabla V$ term to $\nu \nabla^2 V$ term

for $V \simeq V_\phi$, $\nabla \sim \frac{1}{R}$, $\nu \simeq \ell_T V_T + \overset{\nearrow \text{small}}{\ell_{mfp} C_s}_{\,microphys}$

$$\Rightarrow \frac{|V \cdot \nabla V|}{|\nu \nabla^2 V|} \simeq \frac{R V_\phi}{\ell_T V_T} = R_{e,eff} \simeq 1 \longrightarrow$$

Note: If turbulence were absent, recall that
ℓ – microphysical deflection scale from coulomb collisions for protons

$C_s \simeq$ average proton speed

$$Re_{micro} = \frac{R V_\phi}{\nu_{micro}} \simeq 10^{14} \left(\frac{n}{10^{15}}\right) \left(M/M_\odot\right)^{1/2} \left(\frac{R}{10^{10} cm}\right)^{1/2} \left(\frac{T}{10^4 K}\right)^{-5/2} \ll R_{e,eff}$$

Thus V_T is associated with Macroscopic, instead of microscopic values.

Shakura & Sunyaev (1973)
parameterized $\nu_T = l V_T = \alpha_{ss} c_s H$
where H is disk height, c_s is sound speed and α_{ss} is parameter.
$\alpha_{ss} < 1$ under assumption that, for disk which is vertically pressure supported, maximum random velocity is c_s, (more on that later). Also, any structure must be < disk height H. Thus $\alpha_{ss} \leq 1$, determining its exact value is an ongoing struggle

leading model is turbulence generated by magneto-rotational instability (e.g. Balbus & Hawley, Rev Mod Phy 1998)

(Note also Blackman et al. 2006 for relation between α and $\beta = \frac{P_{th}}{B^2/8\pi}$: is robust and $\alpha = 0.2/\beta$ for many simulations)

Radial velocity as diffusion velocity

thin disks

$$\Omega \simeq \Omega_K(R) = \left(\frac{GM}{R^3}\right)^{1/2} \Rightarrow V_\phi = R\Omega_K$$

also have V_R, radial drift velocity, **must be second order quantity**. $\left(\frac{H^2}{R^2}\right)$ we'll this more explicitly.

Write conservation equations:
annulus of disk lying between R ~~&~~ R+ΔR
has mass $2\pi R \Delta R \Sigma$, and ∢ momentum $2\pi R \Delta R \Sigma R^2 \Omega$.
Rate of change of mass is (for small changes)

$$\frac{\partial}{\partial t}(2\pi R \Delta R \Sigma) \simeq V_R(R,t)\, 2\pi R\, \Sigma(R,t) - V_R(R+\Delta R,t)\, 2\pi (R+\Delta R,t)\, \Sigma(R+\Delta R,t)$$

$$\simeq -2\pi \Delta R \frac{\partial}{\partial R}(R\Sigma V_R)$$

or

$$R\frac{\partial \Sigma}{\partial t} + \frac{\partial (R\Sigma V_R)}{\partial R} = 0 \quad \text{(mass continuity)} \quad (14.2)$$

(14a) ⟶

$$R^2 \times R^3\Omega \frac{\partial \Sigma}{\partial t} + R^2\Omega \frac{\partial}{\partial R}(R\Sigma V_R) = 0$$

$$(R^3\Omega\Sigma) \ldots = R^2\Omega \frac{\partial}{\partial R}(R\Sigma V_R) \text{ for } \frac{\partial \Omega}{\partial t} = 0$$

for ∡ momentum same idea:

$$\frac{\partial}{\partial t}\left(2\pi R \Delta R \Sigma \; R^2 \Omega\right) = V_R(R,t)\, 2\pi R \Sigma(R,t)\, R^2 \Omega(R,t)$$

$$- V_R(R+\Delta R,t)\, 2\pi R \Sigma(R+\Delta R,t)\,(R+\Delta R)^2\, \Omega(R+\Delta R,t)$$

$$+ \frac{\partial G}{\partial R}\Delta R$$

↖ from before, the torque

$$\simeq -2\pi \Delta R \frac{\partial}{\partial R}\left(\Sigma V_R R^3 \Omega\right) + \frac{\partial G}{\partial R}\Delta R$$

or

$$\therefore \partial_t\left(\Sigma R^3 \Omega\right) + \frac{\partial}{\partial R}\left(\Sigma V_R R^3 \Omega\right) = \frac{1}{2\pi}\frac{\partial G}{\partial R} \quad (143)$$

∡ mom cons.

$$G(R,t) = \pi \nu \Sigma R^3 \frac{\partial \Omega}{\partial R} \quad \text{from } ([illegible]) \qquad (143a)$$

Using (142, 142a) to eliminate 1st term of (143) (using $\frac{\partial R}{\partial t} = \frac{\partial \Omega}{\partial t} = 0$) and $\frac{\partial \Omega}{\partial t} = 0$ (fixed grav potential)

$$\Rightarrow \Sigma R V_R \frac{\partial\left(R^2 \Omega\right)}{\partial R} = \frac{1}{2\pi}\frac{\partial G}{\partial R} \qquad (144)$$

use (142) & (144) to get V_R:

$$R\frac{\partial \Sigma}{\partial t} = -\frac{\partial}{\partial R}(R\Sigma V_R)$$

$$= -\frac{\partial}{\partial R}\left[\frac{1}{2\pi \frac{\partial}{\partial R}(\Omega R^2)}\frac{\partial G}{\partial R}\right]$$

from 143a $\frac{\partial}{\partial R} 2\pi R\Sigma \nu R^3 \frac{d\Omega}{dR}$

Now since $\Omega \propto R^{-3/2}$ keplerian

$\Rightarrow$

$$\boxed{\frac{\partial \Sigma}{\partial t} = \frac{1}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(\nu_T \Sigma R^{1/2})\right]} \quad (145)$$

surface density equation

Given soln of (145)

V_R follows from (142)

$$\boxed{U_R = -\frac{1}{\Sigma R^{1/2}}\frac{\partial}{\partial R}\left[\nu_T \Sigma R^{1/2}\right]} \quad (146)$$

For (artificially) constant ν_T: (145) implies

$$\Rightarrow \frac{\partial}{\partial t}(R^{1/2}\Sigma) = \frac{\nu_T}{R}\left(R^{1/2}\frac{\partial}{\partial R}\right)^2 (R^{1/2}\Sigma)$$

let $s = 2R^{1/2}$ $\Rightarrow$ $\frac{\partial}{\partial R} = \frac{\partial s}{\partial R}\frac{\partial}{\partial s} = R^{-1/2}\frac{\partial}{\partial s}$

Comment on 2

$$\Rightarrow \frac{\partial}{\partial t}(R^{1/2}\Sigma) = \frac{4\nu_T}{s^2}\frac{\partial^2}{\partial s^2}(\Sigma R^{1/2})$$

$$\frac{\partial}{\partial t}(s\Sigma) = \frac{4\nu_T}{s^2}\frac{\partial^2}{\partial s^2}(\Sigma s) \longrightarrow$$

$$\frac{\partial}{\partial t}\left(\Sigma R^{1/2}\right) = \frac{4\nu_T}{s^2}\frac{\partial^2}{\partial s^2}\left(\Sigma R^{1/2}\right)$$

let $Q = R^{1/2}\Sigma = C(t)e^{ik^{1/2}s}$ just as an example

$$\frac{\partial}{\partial t}\underbrace{\left(R^{1/2}\Sigma\right)}_{Q} = -\frac{4\nu_{Ti}K}{s^2}\underbrace{\left(R^{1/2}\Sigma\right)}_{Q}$$

$\Rightarrow$

$$R^{1/2}\Sigma = Q = Q_0\, e^{-4\frac{\nu_T}{s^2}Kt}, \quad Q_0 = R^{1/2}\Sigma(0)$$

$\Rightarrow$ <u>diffusion</u>; effect of constant viscosity is to diffuse mass density. (true for any separable $R^{1/2}\Sigma = f(t)g(s)$)

$$t_{visc} \simeq \frac{s^2}{4\nu_T K} \simeq \frac{R}{\nu_T K}$$

$$\simeq \frac{R^2}{\nu_{Ti}} \text{ for } k \sim \frac{1}{R}$$

effect of viscosity is to spread structure of of radius R on <u>time scale t_{visc}.</u>

→

From (46)

$$V_R = -\frac{1}{\Sigma R^{1/2}} \frac{\partial}{\partial R}\left(\nu_T \Sigma R^{1/2}\right)$$

$\simeq -\frac{\nu_T}{R}$ when $\Sigma \propto R^q$ $q > -\frac{1}{2}$ and
(for ν_T = const)

$\sim +\frac{\nu_T}{R}$ when $q \leq -1/2$;

V_R is in general a diffusion velocity

Shakura-Sunyaev (1979) approximation

$$\frac{V_R}{R} = \alpha_{SS} C_S \frac{H}{R} \quad \text{for} \quad \nu_T = \alpha_{SS} c_s H$$

Note also that $C_s \ll V_\phi$ for $\frac{H}{R} \ll 1$

this comes from hydrostatic equilibrium:

→

hydrostatic equilibrium

vertical disk structure, steady state

$$\frac{1}{\rho}\frac{dP}{dz} = \frac{\partial}{\partial z}\left[\frac{GM}{(R^2+z^2)^{1/2}}\right] \qquad \text{mom eqn}$$

non-self gravitating disk

for thin disk $z \ll R$

$$\Rightarrow \frac{1}{\rho}\frac{dP}{dz} = -\frac{1}{2}\frac{GM \; 2z}{(R^2+\cancel{z^2})^{3/2}}$$

$$\simeq -\frac{GMz}{R^3}$$

typical scale height of disk is H

$$u_\phi = \left(\frac{GM}{R}\right)^{1/2}$$

So $\Rightarrow$

$$\frac{1}{\rho}\frac{P}{H} \simeq -\frac{GMH}{R^3}$$

$$\Rightarrow |c_s^2| \simeq \left|\frac{GM}{R}\right|\frac{H^2}{R^2} \simeq v_\phi^2\frac{H^2}{R^2}$$

$$\Rightarrow \boxed{c_s^2 \ll v_\phi^2 \iff H^2/R^2 \ll 1}$$

Steady Thin Disks

- radial structure in thin disk evolves on viscous time scales $\simeq R^2/\nu = t_{visc}$
- this presents another way turbulent viscosity is motivated: even knowing nothing about disk properties, viscous time must be less than or equal age of system presumed to be an accretor
- star forming disk ages $\lesssim$ few x 10^7 years
- if accretion is to explain observed features then $t_{age} > t_{visc}$ but for molecular viscosities $R \simeq 10^{14}$cm, $c_s \simeq 10^5$cm/s, $l_{mfp} \simeq 10$cm

$$\Rightarrow R^2/c_s l_{mfp} \simeq 3 \times 10^{14} \text{yr} !$$ too long

$\Rightarrow$ at least in YSO systems, accretion models require turbulent diffusion

- In many systems we can assume mass transfer rate changes on timescales longer than t_{visc}
- System will adjust to steady state structure

- In steady state $-u_R \Sigma R = \text{constant} = \frac{\dot{M}}{2\pi}$

from ∡ momentum eqn (143)

$$\frac{\partial}{\partial R}\left(\Sigma u_R R^3 \Omega\right) = \frac{1}{2\pi}\frac{\partial G}{\partial R}$$, now integrate

$$\Rightarrow \quad \Sigma u_R R^3 \Omega = \frac{G}{2\pi} + \tilde{C}$$

$$= \frac{\nu}{\pi}\Sigma R^3 \frac{\partial \Omega}{\partial R} + \tilde{C}$$

$$\Rightarrow \quad \overbrace{-\nu\Sigma}^{\substack{\text{note combo}\\ \downarrow}} \frac{\partial \Omega}{\partial R} = -u_R \Sigma \Omega + \frac{\tilde{C}}{R^3} \qquad (147)$$

now, for $R \simeq R^*$ (radius of star)

Ω, which is $\simeq \Omega_K$ in disk must slow down to match to "star" which is rotating below break-up ($\Omega < \Omega_K$). There is a thin boundary layer where $\partial\Omega/\partial R \to 0$ this allows us to determine $\tilde{C} \rightarrow$

at $R = R_* + b$, $b << R$

$$\frac{\partial \Omega}{\partial R} \Rightarrow 0$$ since

$$\Omega(R+b) \simeq \left(\frac{GM}{R_*^3}\right)^{1/2}\left[1 + O(b/R_*)\right]$$

$$\Rightarrow \breve{C} \simeq R_*^3 \Sigma u_R \Omega \Big|_{R_*+b}$$ from (147)

$$\Rightarrow \breve{C} = -\frac{\dot{M}(GMR_*)^{1/2}}{2\pi},$$ (since $\bar{M} = 2\pi R \Sigma H u_R$
$u_R < 0$)

then plugging into (147)

$$\Rightarrow -R\nu\Sigma \frac{\partial \Omega}{\partial R} = -V_R \Sigma \Omega R - \frac{\bar{M}(GMR_*)^{1/2}}{2\pi R^2}$$

for $\Omega = \Omega_K$, $\frac{\partial \Omega}{\partial R} = -\frac{3}{2}\frac{\Omega_K}{R}$, then divide by Ω :

$$\Rightarrow \frac{3}{2}\nu\Sigma = -\underbrace{V_R \Sigma R}_{\frac{\dot{M}}{2\pi}} - \frac{\dot{M} R_*^{1/2}}{2\pi R^{1/2}} \frac{\cancel{\Omega_K}}{\cancel{\Omega}}$$

$$\Rightarrow \nu\Sigma = \frac{2\dot{M}}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right] \qquad (148)$$

<u>Important</u>

recall

$$D(R) = \frac{1}{2} \underbrace{\nu \Sigma}\left(R \underbrace{\frac{\partial \Omega}{\partial R}}_{-\frac{3}{2}\frac{\Omega}{R}}\right)^2$$ (from page 145)

$$= \frac{1}{2} \nu \Sigma \frac{9\Omega^2}{4} = \frac{9}{8} \nu \Sigma \Omega^2$$

=> using (148)

$$D(R) = \frac{3GM\dot{M}}{4\pi R^3}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right] \qquad (149)$$

= energy loss rate per area from dissipation

notice it does not depend on viscosity except in combination $\nu\Sigma$, why?

$$\dot{M} = -2\pi V_R \Sigma R \simeq -2\pi \frac{\nu}{\cancel{R}} \Sigma \cancel{R}$$

constant accretion rate => $\nu\Sigma$ constant. Since $\dot{M}$ determines energy dissipated a low ν can be compensated for by high Σ. (Amazing result really ...)

luminosity emitted from $R_1 < R < R_2$

$$L(R_1, R_2) = \underset{\text{2 sides of disk}}{2} \int_{R_1}^{R_2} D(R)\, 2\pi R\, dR$$

$$\overbrace{-\frac{1}{R} + \frac{2R_*^{1/2}}{3}\frac{1}{R^{3/2}}\Big|_{R_*}^{\infty}}$$

$$= \frac{3GM\dot{M}}{2} \underbrace{\int_{R_1}^{R_2} \left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right] \frac{dR}{R^2}}$$

let $R_1 \to \infty$, $R_2 \to R^*$ — $\left[-\frac{1}{R} + \frac{2R_*^{1/2}}{3R^{3/2}}\right]_{R_1}^{R_2}$ gravitation energy available

$$\simeq \frac{GM\dot{M}}{2R_*} = \frac{1}{2}\frac{dU_g}{dt}$$

(other half is left for boundary layer dissipation; $\frac{1}{2}$ available energy dissipated in disk
$\frac{1}{2}$ dissipated in boundary layer)

?

Formal 11/13 (160)

Check that $V_R \sim \nu/R$, & $\Omega \simeq \Omega_K$ from momentum eqn

consider radial component of momentum equation

$$V_R \frac{\partial V_R}{\partial R} - \frac{V_\phi^2}{R} + \frac{1}{\rho}\frac{\partial P}{\partial R} + \frac{GM}{R^2} = \nu_t \nabla^2 V_R \qquad (150)$$

- hydro static equilib tells us $c_s^2 = \frac{H^2}{R^2} V_\phi^2$ ✓
- Viscous term is assumed small. then:

$$\Rightarrow V_R \frac{\partial V_R}{\partial R} - \frac{V_\phi^2}{R} + \frac{GM}{R^2} \simeq 0$$

but $V_R = -\frac{\dot M}{2\pi R \Sigma} = -\frac{\cancel{\dot M}\, \nu\, 3\cancel{\pi}}{2\cancel{\pi} R \cancel{\dot M}}\left(1 - \left(\frac{R_*}{R}\right)^{1/2}\right)^{-1}$ from (148) (151)

$= -\frac{\nu}{R}\left(1 - \left(\frac{R_*}{R}\right)^{1/2}\right)^{-1}$ so indeed ✓

→ $-V_R \simeq O\left(\frac{\nu}{R}\right)$, then (150)

$\Rightarrow \nu = \alpha c_s H \Rightarrow \frac{V_R^2}{R} \ll \frac{V_\phi^2}{R}$ so

$V_\phi^2 \simeq \frac{GM}{R}$ = Keplerian ✓

from (150)

Spectrum from acc disk (opt thick)

Assume acc disk is optically thick:

$\tau_{eff} \equiv \Sigma \kappa_{eff} \gg 1$; $\kappa_{eff} \equiv \frac{\sigma_{eff}}{m_p}$ = cross section for photon absorption and scattering per mass.

Σ ↓ surface density

τ_{eff} = optical depth to abs. + scattering

(see Rybicki & Lightman)

then at each radius it radiates as blackbody: $\sigma T^4(R) = D(R)$

(recall $D(R)$ is energy/time·area = flux)

then from (149):

$$T(R) = \left(\frac{D(R)}{\sigma}\right)^{1/4} = \left\{\frac{3GM\dot{M}}{8\pi R^3 \sigma}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]\right\}^{1/4}$$

for $R \gg R_* \Rightarrow T(R) = T_i \left(R/R_*\right)^{-3/4}$

$$T_i = \left(\frac{3GM\dot{M}}{8\pi\sigma R_*^3}\right)^{1/4} = 4\times10^4 \left(\frac{\dot{M}}{10^{16}\,g/s}\right)^{1/4}\left(\frac{M}{M_\odot}\right)^{1/4}\left(\frac{R}{10^9\,cm}\right)^{-3/4} K$$

WD ←

$$= 10^7 \left(\frac{\dot{M}}{10^{17}\,g/s}\right)^{1/4}\left(\frac{M}{M_\odot}\right)^{1/4}\left(\frac{R}{10^6\,cm}\right)^{-3/4} K$$

NS ←

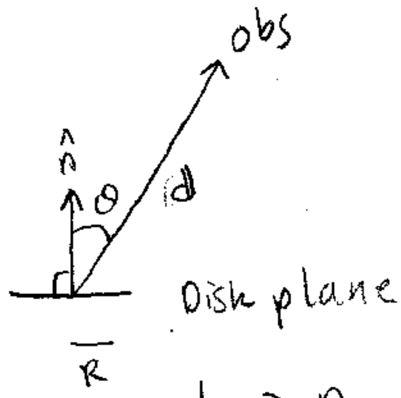
WD should be UV sources ✓

NS should be X-ray Sources ✓

Spectrum emitted by each element of disk is (note: $dE = I_\nu(\hat{k}\cdot\hat{n})\,d\Omega\,dt\,d\nu\,dA$) ← specific intensity $= I_\nu$; $d\Omega \equiv$ solid ∡ element (Rybicki & Lightman or Shu vol 1)

$$I_\nu = B_\nu(T(R)) = \frac{2h\nu^3}{c^2\left(e^{h\nu/kT(R)}-1\right)} \quad \frac{\text{erg}}{\text{cm}^2\text{-s-Hz-ster}}$$

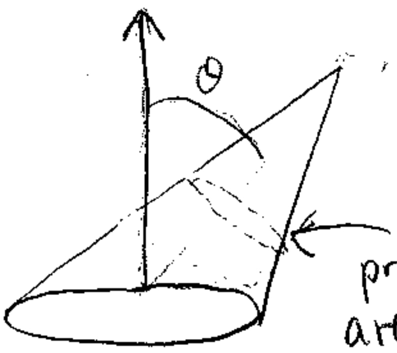
$$F_\nu = \int I_\nu \cos\theta \, d\Omega \quad \text{← solid angle}$$



$d \gg R$

$$\simeq \int_{R_*}^{R_{out}} I_{\nu_f} \cos\theta \, \frac{2\pi R\,dR}{d^2}$$

(using
$d^2\,d\Omega = 2\pi R\,dR$)



projected area normal to observer cosθ factor

$$\simeq \frac{2\pi\cos\theta}{d^2}\int_{R_*}^{R_{out}} I_\nu \, R\,dR$$

→

$$F_{\nu_f} = \frac{2\pi\cos\theta}{D^2}\,\frac{4\pi h}{c^2}\,\nu_f^3 \underbrace{\int_{R_*}^{R_{out}} \frac{R\,dR}{e^{h\nu_f/kT(R)}-1}}$$

$$T(R) = T_i\left(\frac{R}{R_*}\right)^{-3/4}$$

$$x = \frac{h\nu_f}{kT(R)} = \frac{h\nu_f}{kT_{out}}\left(\frac{R}{R_{out}}\right)^{3/4}$$

$$\frac{dx}{dR} = \frac{3}{4}\,\frac{h\nu_f R_{out}}{kT_{out}}\left(\frac{R}{R_{out}}\right)^{-1}$$

$$R = x^{4/3} R_{out}\left(\frac{kT_{out}}{h\nu_f}\right)^{4/3}$$

$$R\,dR = \frac{4}{3}\,\frac{kT_{out}}{h\nu_f R_{out}}\left(\frac{R}{R_{out}}\right)^{1/4} R\,dx$$

$$= \frac{4}{3}\,\frac{kT_{out}\,R^{5/4}}{h\nu_f R_{out}^{5/4}}\,dx$$

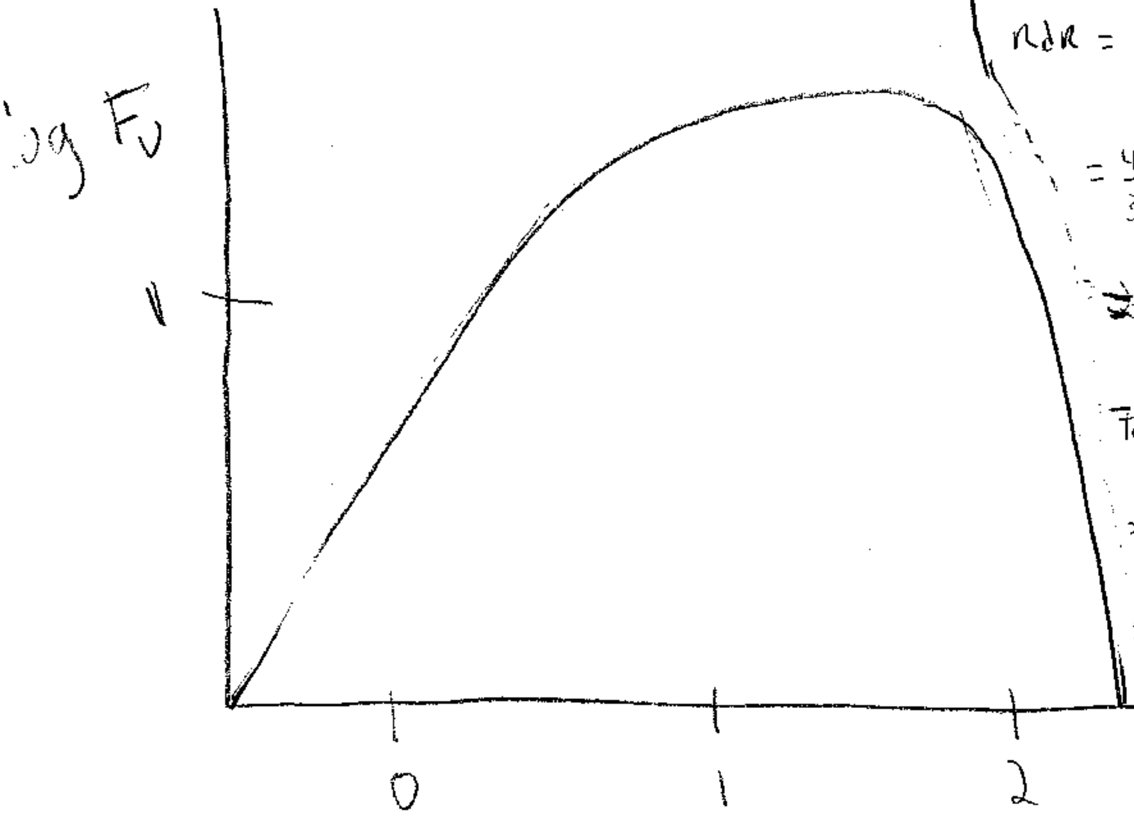
$$= \frac{4}{3}\,\frac{kT_{out}}{h\nu_f}\,x^{5/3}\left(\frac{kT_{out}}{h\nu_f}\right)^{5/3} dx$$

$$= \frac{4}{3}\left(\frac{kT_{out}}{h\nu_f}\right)^{8/3} x^{5/3}\,dx$$

$$\Longrightarrow \nu_f^3 \int_{R_*}^{R} \frac{R\,dR}{e^{h\nu_f/kT}-1}$$

$$\propto \nu_f^{1/3} \int_{h\nu_f/kT_{in}}^{h\nu_f/kT_{out}} \frac{x^{5/3}}{e^x-1}\,dx$$

plot $F_{\nu_f}(\nu_f)$



log $(h\nu / kT_{out})$

(Stretched out black body)

Toward MHD from a two-fluid approach

consider multiparticle phenomena in a plasma on scales much larger than Debye Length and time scales much larger than plasma frequency; then charge separation in plasma can be neglected.
$\lambda_D = \left(\frac{kT}{8\pi n e^2}\right)^{1/2}$, $\omega_{p,e}^2 = \frac{4\pi n e^2}{m_e}$)

But: when considering long time scales collisions cannot be neglected. Here I will address the derivation of collisional MHD starting from two-fluid approach, and also modeling the collision contributions.

In a two fluid approach, we consider a fully ionized plasma of protons and electrons. The protons are treated as one fluid and the electrons as another fluid.

→

Consider e^- fluid first:
Collisions between them do not change ie e^- fluid momentum: only when e^- and ions collide is momentum transferred.

Thus, the equations for the electron fluid, (to first order in V_e) is given by (assuming $n = n_- = n_+$)

$$m_e n \frac{\partial \vec{V}_e}{\partial t} = -\nabla p_e - ne\left(E + \frac{\vec{V}_e}{c} \times B\right) - m_e n \nu_c (\vec{V}_e - \vec{V}_i) \quad (251)$$

where n is number density, ν_c is a collision frequency between electrons and ions. For the moment we neglect the fluid viscosity and will restore it later. Eqn (251) is basically the fluid equation like an Euler eqn for electrons with the extra ν_c term and dropping the $V_e \cdot \nabla V_e$ term on the grounds that it is second order.

Since the current density: $j \equiv ne(\vec{V}_i - \vec{V})$ (251a)
the last term in (251) is proportional to the current density, thus:

$$m_e n \frac{\partial V_e}{\partial t} = -\nabla p_e - ne\left(\vec{E} + \frac{\vec{V}_e}{c} \times \vec{B}\right) + ne\eta\vec{j} \quad (252)$$

with $\eta = \frac{m_e \nu_c}{ne^2}$ (253)

η can be explained as follows:

consider homogeneous (uniform) plasma in steady-state with $\vec{E}$-field driving a current $\vec{J}$. Then (252) becomes

$$-ne\vec{E} + ne\eta\vec{J} = 0 \qquad (254)$$

So that

$$\vec{E} = \eta\vec{J} \qquad (255)$$

$\Rightarrow$. η is the plasma resistivity.

To calculate it, we need expression for ν_c.

To get the collision frequency, consider an approximate approach:

If the impact parameter between e^- and p is large then minimal deflection takes place. Thus define impact parameter r_0 for which e^- deflection is sufficiently large to change its momentum by of order its original momentum.

$\longrightarrow$

to estimate r_0, let u be typical relative velocity between the proton & e^-, so that r_0/u is effective interaction time. Since strongest interaction force is $\frac{e^2}{r_0^2}$, the impulse (= F·Δt)

$$= \frac{e^2}{r_0^2}\frac{r_0}{u} = \frac{e^2}{r_0 u} \simeq \Delta p \qquad (256)$$

by definition of r_0, $\Delta p = p = m_e u$

so that $r_0 \simeq \frac{e^2}{m_e u^2}$ (257)

Moreover, the effective collision cross section is given by πr_0^2, thus the collision frequency $\simeq$

$$\nu_c \simeq n \sigma u = n \pi r_0^2 u = \frac{\pi n e^4}{m_e^2 u^3}, \qquad (258)$$

using (257). For thermal velocities, plug in $u = \left(\frac{k_B T}{m_e}\right)^{1/2}$ and (258) gives:

$$\nu_c = \frac{\pi n e^4}{m_e^{1/2}(k_B T)^{3/2}}, \qquad (259)$$

Using (259) in (253)

$$\rightarrow \quad \eta \simeq \frac{\pi m_e^{1/2} e^2}{(k_B T)^{3/2}} \qquad (260)$$

A rigorous calculation (Spitzer & Härm 1953) gives

$$\boxed{\text{Spitzer resisitivity}} \rightarrow \eta = \underbrace{\left(\frac{Z\pi^{1/2}\ln\Lambda}{g(Z)\,4\sqrt{2}}\right)}_{\text{extra factor}} \underbrace{\left(\frac{\pi m_e^{1/2} e^2}{(k_B T)^{3/2}}\right)}_{(260)} \qquad (261)$$

(Z = charge number for + ions)

The extra factor is typically of order 1-10.

(ie: $\ln\Lambda = \ln\left(\frac{3}{2Ze^2}\frac{k_B^{3/2}T^{3/2}}{\pi^{1/2}n^{1/2}}\right)$, $g(Z=1) \simeq 0.6$).

Thus the rough treatment at least gets the basic scalings reasonably well.

For $Z=1$, proton-electron plasma, the sptizer resistivity is

$$\eta = 7.3 \times 10^{-9} \frac{\ln\Lambda}{T^{3/2}} \text{ sec} \qquad (262)$$

Now, having obtained η, let us consider equation of motion for the ion fluid →

The collision term in the ion equation should be equal and of opposite sign to that in the e⁻ equation, since momentum is exchanged between the two. Thus

$$m_i n \frac{\partial \vec{V}_i}{\partial t} = -\nabla P_i + ne\left(\vec{E} + \frac{\vec{V}_i}{c} \times \vec{B}\right) - ne\eta \vec{J} \quad (263)$$

(where again we worked to first order in $\vec{V}$ so $\vec{V}_i \cdot \vec{\nabla} V_i$ is neglected).

Now we combine (263) with (252) to get a 1 fluid model:

The total density and net fluid velocity are

$$\rho = n(m_i + m_e) \quad (264)$$

$$\vec{V} = \frac{m_i \vec{V}_i + m_e \vec{V}_e}{m_i + m_e} \quad (265)$$

Then adding (263) and (252) =>

$$n \frac{\partial}{\partial t}(m_i V_i + m_e V_e) = \frac{ne(\vec{V}_i - \vec{V}_e)}{c} \times \vec{B} - \nabla(P_i + P_e)$$

→ or, using (264), (265) & $j = ne(\vec{V}_i - \vec{V}_e)$:

$$\rho \frac{\partial \vec{V}}{\partial t} = \frac{\vec{J} \times \vec{B}}{c} - \nabla P \quad (266)$$

where $p \equiv p_i + p_e$.

Now, actually the pressure is really a tensor, as we discussed early in the course in deriving the hydrodynamic equations. the pressure tensor was given by

$$P_{ij} = nm\langle v_i v_j\rangle - \underset{\uparrow}{nm\bar{v}_i\bar{v}_j}$$

this term was neglected

in our present approach. When it is not neglected, (266) becomes

$$\rho\frac{\partial\vec{v}}{\partial t} = -\vec{v}\cdot\nabla\vec{v} - \nabla p + \underbrace{\vec{J}\times\vec{B}}_{\substack{\text{magnetic term}\\ \text{now added to Euler equation}}} \qquad (267)$$

Note $\vec{E}$ force has disappeared. Note also that the Navier-Stokes eqn had is like the Euler eqn but with the added viscosity term that resulted from deviations from Maxwellian. The derivation of that term would proceed the same had we started from the Boltzmann eqn for e- and protons separately in deriving (252) and (263). Thus I will add the term without a rigorous derivation Thus the momentum equation for single fluid MHD is given by

$$\rho \frac{\partial \vec{V}}{\partial t} = - \vec{V} \cdot \nabla \vec{V} - \nabla p + \vec{J} \times \vec{B} + \boxed{\nu \nabla^2 \vec{V}} \qquad (268).$$

(for non-uniform ν), this term should be, in general $(= -\nabla \times (\nu \nabla \times v))$)

Now (268) was derived by taking the sum of e^- & ion fluid eqns. By taking the difference, we get an expression for the electric field in terms of the $\vec{v}$ & $\vec{B}$. Such a relation is needed when the magneto-fluid equations are combined with maxwell's equations.

Multiplying (252) by m_i and subtracting (263) multiplied by m_e gives:

$$m_i m_e n \frac{\partial}{\partial t} (\vec{v}_i - \vec{V}_e) = ne(m_i + m_e) \vec{E} + \frac{ne}{c} (m_e \vec{V}_i + m_i \vec{v}_e) \times \vec{B} - m_e \nabla p_i + m_i \nabla p_e - (m_i + m_e) ne \eta \vec{J} \qquad (269)$$

Using (264), (265), (251a) and

$$\begin{aligned} m_e \vec{V}_i + m_i \vec{V}_e &= m_i \vec{V}_i + m_e \vec{V}_e + (m_e - m_i)(\vec{V}_i - \vec{V}_e) \\ &= \frac{\rho}{n} \vec{V} + \frac{m_e - m_i}{ne} \vec{J} \end{aligned} \qquad (270)$$

we get from (269)

$$\vec{E} + \frac{\vec{v}}{c} \times \vec{B} = \eta \vec{J} + \frac{1}{e\rho}\left[\frac{m_e m_i n}{e} \frac{\partial}{\partial t}\left(\frac{\vec{J}}{n}\right) + (m_i - m_e)\frac{\vec{J} \times \vec{B}}{c} + m_e \nabla p_i - m_i \nabla p_e \right] \quad (270)$$

This is the <u>Generalized Ohm's law</u>.

when system changes on time scales long compared to collision time the $\frac{\partial}{\partial t}\left(\frac{J}{n}\right)$ term is small compared to $\eta \vec{J}$ term.

then (270) =>

$$\vec{E} + \frac{\vec{v}}{c} \times \vec{B} = \eta \vec{J} + (m_i - m_e)\frac{\vec{J} \times \vec{B}}{c} + m_e \nabla p_i - m_i \nabla p_e \quad (271)$$

often in astro the <u>Hall Effect</u> term $(m_i - m_e)\frac{\vec{J} \times \vec{B}}{c}$ and the pressure gradient terms are ignorable compared to $\eta \vec{J}$. Thus in many cases

$$\vec{E} + \frac{\vec{v}}{c} \times \vec{B} = \eta \vec{J} \quad (272)$$

is the appropriate Ohm's law.

(Note that for $\underset{e^-,e^+}{\text{pair}}$ plasma, there is no Hall effect or ∇p terms in (271)).

Collecting the important equations:

(268) →

$$\rho \frac{\partial \vec{v}}{\partial t} = -\vec{v}\cdot\nabla\vec{v} - \nabla p + j \times B + \nu \nabla^2 \vec{v}$$

(272) →

$$\vec{E} + \frac{\vec{v}}{c} \times \vec{B} = \eta j$$

mass continuity (same as for unmagnetized fluids)

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho v) = 0 \quad . \tag{273}$$

Recall that for incompressible flows, $(\nabla\cdot v = 0)$ we dont need to worry about the energy equation. Most of the MHD we will consider will be incompressible

We thus have time evolution eqn for v, ρ and we have eqn that relates E to B, but we need equation for $\partial_t \vec{B}$. This comes from combining (272) with maxwell's equations:

Note that from Maxwell's eqns:

$$\frac{1}{c}\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0 \tag{274}$$

$$\nabla \times \vec{B} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 4\pi \frac{\vec{J}}{c} \tag{275}$$

combining (273) & (274)

$$\Rightarrow \quad \frac{\partial B}{\partial t} = \nabla\times(v\times B) - c\nabla\times\eta\vec{J} \qquad (276)$$

we then use (275) :

$$\nabla\times j = \underbrace{\frac{c}{4\pi}\nabla\times\nabla\times B}_{\mathcal{O}\left(\frac{c}{\ell^2}B\right)\ \textcircled{1}} + \underbrace{\frac{1}{4\pi}\frac{\partial(\nabla\times E)}{\partial t}}_{\mathcal{O}\left(\frac{v}{\ell c t}B\right)\ \textcircled{2}} \qquad (277)$$

this ratio of ①/② $\simeq \frac{c^2 t}{\ell v} \simeq \frac{c^2}{v^2}$

thus we ignore ② =>

=> (277)

$$\simeq \frac{c}{4\pi}\nabla\times\nabla\times B = -\frac{c}{4\pi}\nabla^2\vec{B} \qquad (278)$$

(278) into (276) =>

$$\frac{\partial B}{\partial t} = \nabla\times(v\times B) + \frac{\eta c^2}{4\pi}\nabla^2\vec{B}$$
$$= \nabla\times(v\times B) + \nu_m\nabla^2\vec{B} \qquad (279)$$

for $\nabla\eta = 0$. (279) is Magnetic Induction Eqn

Basic Magnetohydrodynamics (cont)

Apr 4

The momentum equation as derived last time, now has the additional $\frac{\vec{J}\times\vec{B}}{c}$ term. This magnetic force can be re-written using $\nabla\times B = \frac{4\pi\vec{J}}{c}$ (From maxwell's equations for non-relativistic flows.)

Thus:

$$\vec{J}\times\vec{B} = \frac{\cancel{c}}{4\pi}\frac{j\times B}{\cancel{c}} = \frac{1}{4\pi}(\nabla\times B)\times B \qquad (280)$$

$$= \frac{1}{4\pi}\left(\varepsilon_{ijk}\partial_j B_k\right)\varepsilon_{min}B_n$$

$$= \frac{1}{4\pi}\left(\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn}\right)B_n\,\partial_j B_k$$

$$= \frac{1}{4\pi}\vec{B}\cdot\nabla\vec{B} - \frac{1}{8\pi}\nabla B^2 \qquad (281)$$

Thus we can write the MHD momentum equation

$$\rho\frac{\partial\vec{v}}{\partial t} = -\vec{v}\cdot\nabla\vec{v} - \nabla(P + \underbrace{P_{mag}}_{=\frac{B^2}{8\pi}}) + \frac{(\vec{B}\cdot\vec{\nabla})\vec{B}}{4\pi} + \nu\nabla^2\vec{v} + \vec{F} \qquad (282)$$

acts as additional pressure; what about $\frac{\vec{B}\cdot\nabla\vec{B}}{4\pi}$? →

We can show that $\vec{B}\cdot\nabla\vec{B}$ acts as a tension force: consider the tensor M_{ij} defined such that

$$M_{ij} = \frac{B^2}{8\pi}\delta_{ij} - \frac{B_i B_j}{4\pi} \qquad (283)$$

so that

$$(J \times B)_i = -\partial_j M_{ij} \qquad (284)$$

from (281), and $\vec{\nabla}\cdot\vec{B} = \partial_i B_i = 0$.

Suppose we choose the $\hat{z}$ axis as the local direction of the magnetic field. Then from (283):

$$M_{ij} = \begin{pmatrix} B_z^2/8\pi & 0 & 0 \\ 0 & B_z^2/8\pi & 0 \\ 0 & 0 & -\frac{B_z^2}{4\pi} \end{pmatrix} = \qquad (285)$$

This shows that $\perp$ to the field (assumed to be only in z-direction), there is a pressure $\frac{B_z^2}{8\pi}$, so that force in $\hat{x}$ & $\hat{y}$ directions are

$$-\partial_j M_{xj} = -\partial_x \frac{B_z^2}{8\pi} = -\nabla_x P_{mag} \qquad (286)$$

and

$$-\partial_j M_{yj} = -\partial_y \frac{B_z^2}{8\pi} = -\nabla_y P_{mag} \qquad (287)$$

but along the $\hat{z}$ direction

Force is: $$-\partial_j M_{zj} = +\nabla_z \frac{B_z^2}{4\pi} \qquad (288)$$

This corresponds to a force that increases in the direction of increasing B_z. This is a <u>tension force</u> that resists stretching much like a rubber band. Note that the pressure force ⊥ B_z is in the direction of decreasing B_z^2, just like particle pressure force, whereas the tension force is in the direction of increasing B_z along the field line.

Having discussed the physical meaning of the terms in the momentum eqn let us consider some aspects of the magnetic induction equation:

$$\partial_t B = \nabla \times v \times B + \nu_m \nabla^2 \vec{B} \qquad (289)$$

First, note that the order of magnitude ratio of the 1st term on the right, to the second term on the right is given by →

$$R_M \equiv \frac{VB/L}{\nu B/L^2} = \frac{LV}{\underset{\nwarrow \text{magnetic diffusivity}}{\nu_m}} = \text{Magnetic Reynolds Number} \qquad (240)$$

where V, L are characteristic velocities & scale of field variation in problem of interest.

(R_M is reminiscent of the the Reynolds number for hydrodynamic flows $\frac{LV}{\nu}$ ↖ (dynamic viscosity))

From (279) & (267)

$$\nu_m = 5.5 \times \frac{10^{11}}{T^{3/2}} \ln \Lambda \qquad (291)$$

which is $\nu_m \simeq 10^7 \mathrm{cm}^2/\mathrm{s}$ for $T = 10^4$ k, $\ln \Lambda = 10$.

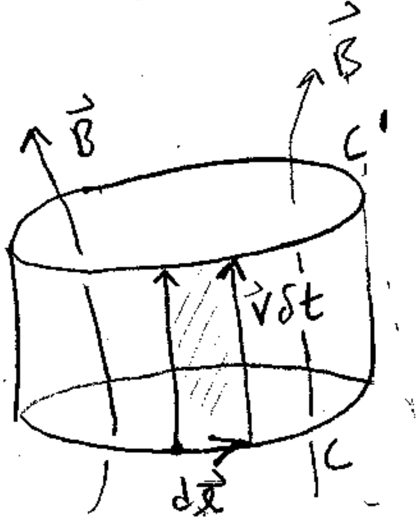
For a laboratory system, $L \simeq 10^2$ cm, $V \simeq 10$ cm/s

$\Rightarrow R_m \simeq 10^{-4}$.

For solar convection zone, $L \simeq 10^8$ cm, $V \simeq 10^5$ cm

$\Rightarrow R_m = 10^6$. Due to smaller scales and velocities involved, but temperatures that need not be hugely different, typically lab R_m is much smaller that astro R_m and usually R_m in astro $\gg 1$. →

For large R_m, the ν_m term can be ignored in the induction equation under most circumstances (but not all!). This leads to concept of <u>flux freezing</u> in astrophysics: (similar to kelvin circulation theorem).



To prove: consider flux $\int \vec{B}\cdot d\vec{S}$ through closed contour C, moving with the fluid. Initial position at time t is closed contour C, and <u>after</u> time δt it has undergone displacement $\vec{v}\delta t$ to new position C'. Let $d\vec{S}_c$ be area element on C and $d\vec{S}'_{c'}$ be area element on C'. The area element with outward normal (shaded) is given by $d\vec{\ell} \times \vec{v}\delta t$.

Now $\nabla\cdot B = 0$ implies that $\int \nabla\cdot B\, dV = \oint \vec{B}\cdot d\vec{S} = 0$. integrated around the closed cylinder. Thus

$$\int_C d\vec{S}_c \cdot B(t+\delta t) - \int_{C'} d\vec{S}'_c \cdot \vec{B}(t+\delta t) - \int_C \vec{B}(t+\delta t)\cdot(d\vec{\ell}\times\vec{v}\delta t) = 0 \qquad (292)$$

Now:

$$\delta\Phi \equiv \int_{C'} d\vec{S}'_{C'}\cdot\vec{B}(t+\delta t) - \int_C d\vec{S}_c\, \vec{B}(t) \qquad (293)$$

which, using (292), becomes

$$\delta\Phi \equiv \int_c d\vec{S}_i\, \vec{B}(t+\delta t) - \int_c \vec{B}(t+\delta t)\cdot(d\vec{l}\times\vec{v}\,\delta t) - \int_c d\vec{S}_i\,\vec{B}(t) \qquad (294)$$

$$= \delta t\left[\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}_c - \underbrace{\int \vec{B}(t+\delta t)\cdot(d\vec{l}\times\vec{v})}_{\simeq \int B(t)\cdot(d\vec{l}\times\vec{v}) \text{ for small } \delta t)}\right]$$

$$\Rightarrow \quad = \delta t\left[\int \underbrace{\left(\vec{\nabla}\times(\vec{v}\times\vec{B}) + \nu_m\nabla^2\vec{B}\right)}_{\text{from } (279)}\cdot d\vec{S}_c - \int B(t)\cdot(d\vec{l}\times\vec{v})\right]$$

$$= \delta t\left[\left[\int(\vec{v}\times\vec{B})\cdot d\vec{l} + \int \nu_m\nabla^2\vec{B}\cdot d\vec{S}_c\right] - \int B(t)\cdot(d\vec{l}\times\vec{v})\right]$$

But $(\vec{v}\times\vec{B})\cdot d\vec{l} = \vec{B}\cdot d\vec{l}\times\vec{v}$ (vector identity)

so

$$\Rightarrow \delta\Phi = \delta t\int \nu_m\nabla^2\vec{B}\cdot dS_c \qquad (295)$$

or $\dfrac{d\Phi}{dt} = \int \nu_m\nabla^2\vec{B}\cdot dS_c \Rightarrow$ for $R_m >> 1$

$\dfrac{d\Phi}{dt} \simeq 0 \equiv$ <u>Flux Freezing</u>

→

Flux freezing is simply the statement that the magnetic field moves with the plasma so as to maintain $\int \vec{B} \cdot d\vec{S}$ = constant with time. If flux freezing were to apply during the collapse of a star like the sun, could it be a simple explanation for the origin of Neutron star magnetic fields? The sun has a mean field of order 2-10 Gauss. Flux freezing from $R_\odot \simeq 10^{11}$ cm to $R_{NS} \simeq 10^6$ cm implies an increase in field strength of order $\frac{R_{NS}^2}{R_\odot^2} \Rightarrow B_{NS} \leq 10^{11}$ Gauss.

Not bad. Many people believe this is possible, but others feel that young NS incur neutrino driven turbulent convection hich can destroy the frozen in field with enhanced diffusion but also generate new field by dynamo action.

Magnetohydrostatics

As simple examples of MHD, consider time independent, velocity free equilibria:

$$f - \nabla p + \frac{1}{4\pi}(\nabla \times B) \times B = 0 \qquad (296)$$

Consider Body forces $= 0 \Rightarrow$

$$\nabla p = \frac{1}{4\pi}(\nabla \times B) \times B \qquad (297)$$

A magnetic field satisfyng (297) is called a <u>pressure balanced</u> field.

An important dimensionless parameter is the plasma beta:

$$\beta \equiv \frac{P}{B^2/8\pi} \qquad (298)$$

often in lab, $\beta \ll 1$. In astrophysics, the definition of "corona" for MHD people is often taken to be the region in stellar atmospheres or above accretion discs above which β drops below 1.

Note that when $\beta << 1$, (297) becomes

$$(\nabla \times B) \times B = 0 = \vec{J} \times \vec{B} \qquad (299)$$

This is called the <u>force-free</u> condition and implies that the magnetic pressure and tension forces conspire to balance. Note also that $J \times B = 0$

$$\Rightarrow J \parallel B, \text{ so that } \vec{\nabla} \times \vec{B} \parallel \vec{B}. \qquad (300)$$

Now consider an example of a pressure balanced column. We work in cylindrical coordinates, assuming cylindrical symmetry (no variation in θ, z).

Then from $\nabla \cdot B = 0$; $\frac{1}{r} \partial_r (B_r r) = 0$

or $B_r = \frac{\text{constant}}{r}$ but in order not to diverge at $r=0$, constant must be zero. Thus $B_r = 0$.

We then write

$$\vec{B} = B_\theta(r) \hat{e}_\theta + B_z(r) \hat{e}_z \qquad (301)$$

→

using 30 in 297 : =>

$$\frac{\partial p}{\partial r} = \frac{1}{4\pi}\left(-\frac{\partial B_z}{\partial r}\hat{e}_\varphi + \frac{1}{r}\frac{\partial}{\partial r}(r B_\varphi)\hat{e}_z\right) \times \vec{B}$$

$$= \frac{1}{4\pi}\left(-\frac{1}{2}\frac{\partial B_z^2}{\partial r} - \frac{1}{2}\frac{\partial (B_\varphi^2)}{\partial r} - \frac{B_\varphi^2}{r}\right)$$

$$\Rightarrow \frac{\partial}{\partial r}\left(p + \frac{B_\varphi^2}{8\pi} + \frac{B_z^2}{8\pi}\right) + \frac{B_\varphi^2}{4\pi r} = 0 \qquad (302)$$

assuming $p = p(r)$. Now consider that the magnetic field in the plasma column is produced by driving a current $j = j(r)\hat{e}_z$ along the axis of the column. This would only produce a field in the toroidal direction since

$$\frac{c}{4\pi}\nabla \times B = J_z \quad \text{and } B_r = 0 \Rightarrow \vec{B} = \vec{B}_\varphi$$

this relation is then

$$\frac{1}{r}\frac{d}{dr}(r B_\varphi) = \frac{4\pi J_z}{c} \qquad (303)$$

→

$\frac{V_A^3}{\lambda_\parallel} = \frac{V_\perp^3}{\lambda_\perp}$

$\frac{\lambda_\parallel}{V_A} =$

$\frac{\lambda}{V_A} =$

$\frac{V_A^3}{\lambda_\parallel} = \frac{V_\perp^3}{\lambda_\perp}$

$\lambda_\perp = \frac{V_\perp^3}{V_A^3}\lambda_\parallel$

$\boxed{\frac{\lambda_\parallel}{V_A} = \frac{\lambda_\perp}{V_\perp}}$

$V_\perp = \frac{\lambda_\perp}{\lambda_\parallel}V_A$

$\frac{V_A^3}{\lambda_\parallel} = \left(\frac{\lambda_\perp}{\lambda_\parallel}\right)^3\frac{V_A^3}{\lambda_\perp}$

$\frac{V_\perp^3}{\lambda_\perp} = \text{const}$

$\frac{V_A}{\lambda_\parallel} = \frac{V_\perp}{\lambda_\perp}$

$\frac{\lambda_\perp^2}{\lambda_\parallel^3}V_A^3 = \text{const}$

$k_\parallel^3 = \frac{\text{const}}{V_A^3}$

If we now assume constant $j(r) = \bar{j}$, so that $\partial_r \bar{j} = 0$, then we can integrate (303):

$$\Rightarrow \quad \boxed{B_\varphi = \frac{2\pi j_z}{c} r} \qquad (304)$$

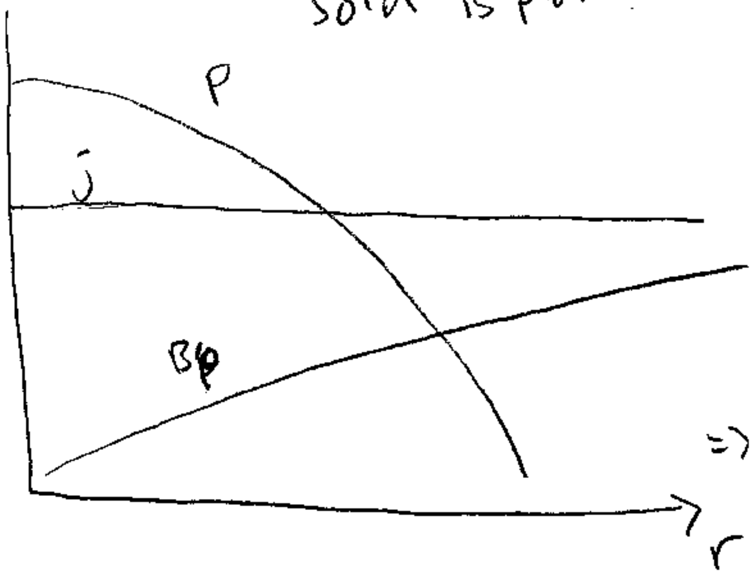
plugging into (302) $\Rightarrow$

$$\frac{d}{dr}\left(p + \frac{4\pi^{\cancel{2}}}{c^2}\frac{j^2 r^2}{\underset{2}{\cancel{8\pi}}}\right) + \frac{j^2 r \pi}{c^2} = 0$$

or $$\frac{d}{dr} p = -\frac{2\pi}{c^2} r j^2$$

$$\Rightarrow \quad \boxed{p = p_0 - \frac{\pi r^2}{c^2} j^2} \quad (\text{for constant } j) \qquad (305)$$

soln is plotted below: note that p drops as B_φ increases suggesting that pressure is concentrated by the "hoop pinch force" of the B_φ field.

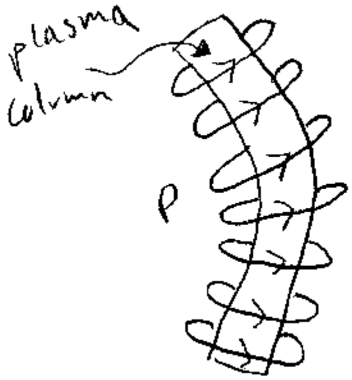


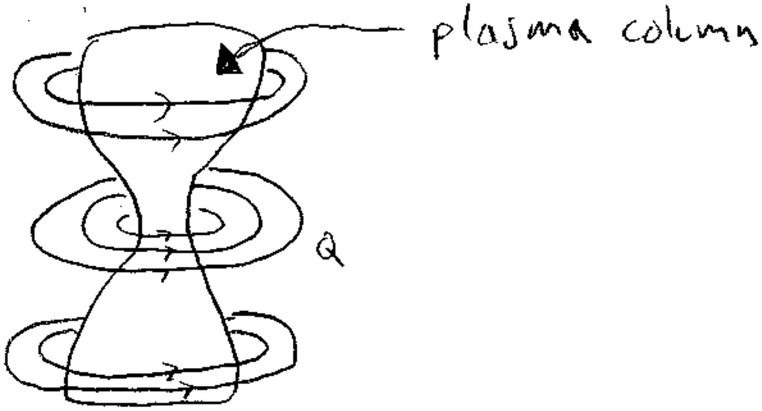
$\Rightarrow$ MAGNETIC COLLIMATION OF JETS IN ASTRO!

Apr 24
cont

Stability of plasma columns

physical considerations allow one to intuit the stability or instability of a plasma column. Detailed calcs required, but consider the perturbations below:





Crowding of B-field lines at point P enhances magnetic pressure there and pushes plasma column so as to enhance the kink → kink grows, system unstable. $\boxed{\text{KINK INSTABILITY}}$

In the second fig., B_θ at Q is larger than its value away from the perturbed pinch. The extra tension pinches further and system is unstable to $\boxed{\text{SAUSAGE INSTABILITY}}$

An Axial field can supress these instabilities: As the kink bends the field column, the axial field tension resists the bending. Similarly, for the sausage case, the magnetic pressure associated with the axial field resists the pinching. $B_{axial} \gtrsim B_\theta$ is required to stabilize the instabilitres.

Fusion & plasma confinement

Dominant fusion reaction desired is
2 deuterium atoms → tritium or Helium + energy

Coulomb forces repulse the deuterium atoms so the must have high enough relative velocity to penetrate coulomb barrier to fuse. This requires hot plasma

→ But high temp deuterium ($> 10^7$ K) canot be easily confined. It would burn container walls if too dense. And, if too diffuse, it would quickly lose heat content with the wall. ⇒ hope is to confine plasma with magnetic fields. Push for fusion devices was initiated by US, UK, USSR after WW II. ⟶

Expectation was that commercial production possible in a few years but its 60 years later and still a long way off. Confining was more difficult than expected and a lot of energy is always lost in heating & setting up the configurations.

Inorder to get sufficient energy out of fusion, plasma must be confined for a time such that the product of the number density and confinement time τ, satisfies

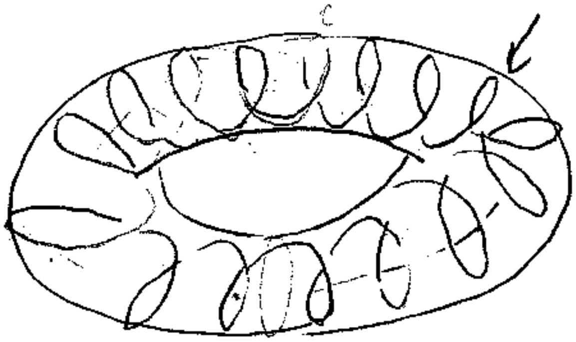
$n\tau > 10^{16}$ sec cm^{-3}. (called Lawson criterion).

Typical magnetic devices have $n\tau \approx \left(\frac{10^{15}}{cm^3}\ 0.1\,sec\right) \approx \frac{10^{14}}{cm^3}$ sec too small by about 2-orders of magnitude.

In the laser lab, the idea is to make n very large even though τ is short. $\tau \sim 10^{-10}$ sec so n has to be 10^{26} cm^{-3}, but such high densities are not yet reached. The system falls short, in part due to the Rayleigh-Taylor Instability.

Magnetic confinement still seems like best hope. Devices are typically toroidal plasma columns, which avoid edges by allowing plasma to close on itself.

Key reason for the difficulty in confinement is plasma instabilities, which induce plasma in the core region to diffuse prematurely to the walls.



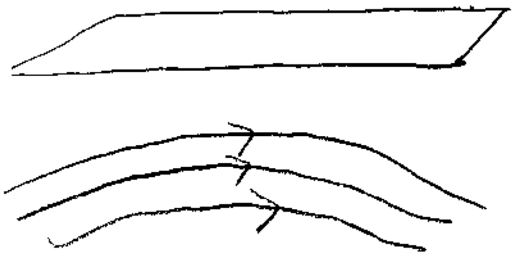
torus confinement
e.g. TOKOMAK

<u>core</u> is region inside the pinch where plasma is confined between the inner and outer walls.

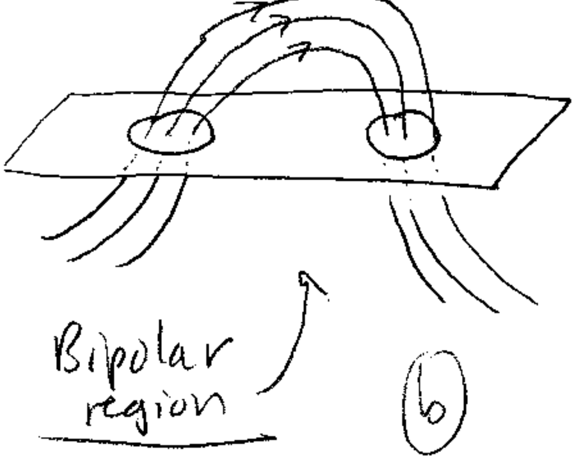
---

polar magnetic regions and buoyany

Hale (1908) realized sunspots were associated with magnetic fields, and in 1919 noticed that often two large sunspots appear side by side with opposite polarities. The obvious explanation is that the dual spots represent places where magnetic field penetrates the solar surface:

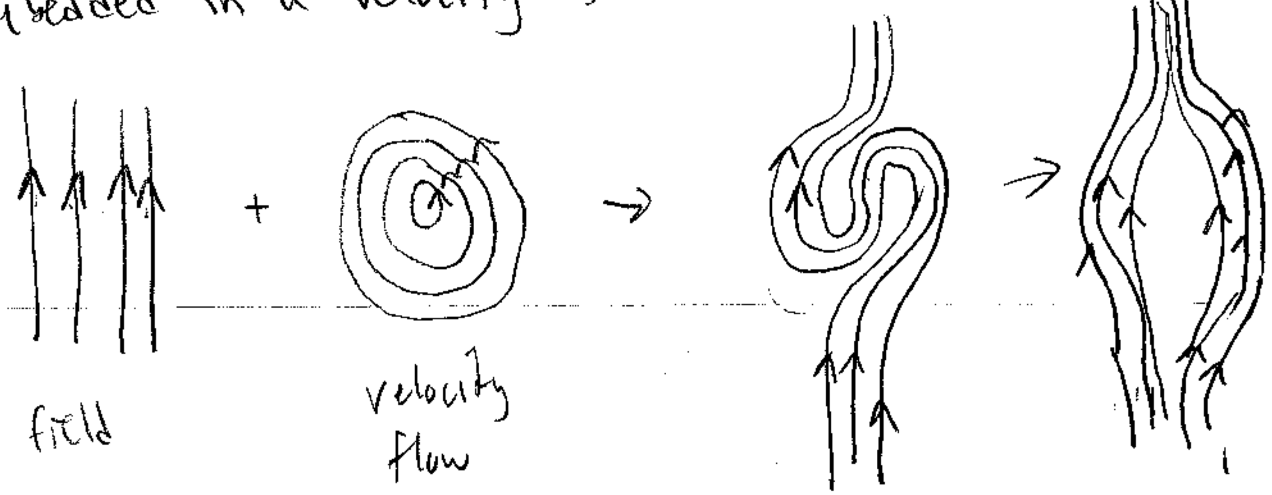


(a)

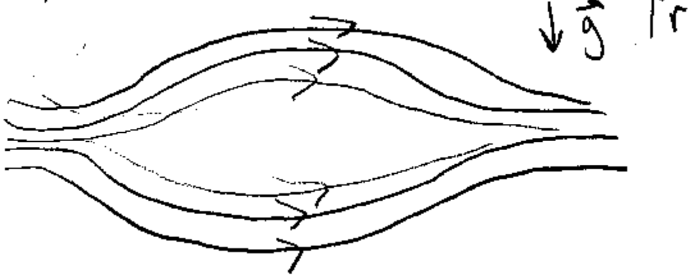


(b)

Inside convection zone, field is pushed toward boundaries of convection cells for example, consider a region of field embedded in a velocity flow:



tilt the last figure, and imagine it is imbedded in sun:



we can see that we have segregated flux tubes.

the top part can represent fig ⓐ on the previous page. Now, why should such a structure become buoyant and rise through solar surface to corona?

Consider a horizontal flux tube with gas pressure p_i inside the tube and let p_e be the external pressure →

equation of motion without velocity field but with gravity, pressure, and $\vec{B}$ is

$$\frac{1}{4\pi}(\vec{B}\cdot\vec{\nabla})\vec{B} = \nabla\left(\frac{B^2}{8\pi}+P\right) - \rho\vec{g}$$

consider flux tube of strength $\vec{B} = B_0\hat{x}$ in vertically stratified atmosphere. Left side vanishes.

In this situation. If tube is in pressure balance with surroundings then

$$P_e = P_i + \frac{B^2}{8\pi} \qquad (306)$$

where P_e is external pressure and P_i is internal gas pressure. Then $P_i < P_e$. If tube is in thermal equilib with surroundings then $\rho_i < \rho_e$ or

$$n_i kT = P_i = P_e - \frac{B^2}{8\pi} = n_e kT - \frac{B^2}{8\pi}$$

$$\Rightarrow n_i = n_e - \frac{B^2}{8\pi kT}, \qquad (307)$$

thus $F_{buoy} = (n_e - n_i)m_H \overset{\text{grav. force}}{g} V = \frac{B_0^2 m_H g \overset{\text{tube volume}}{V}}{8\pi kT}$ (308)

is the upward force. Now $\frac{kT}{mg}$ is scale height so

$$F_{buoy} = \frac{B^2 V}{8\pi H} \qquad (309)$$

After rising distance H, tube gets kinetic energy

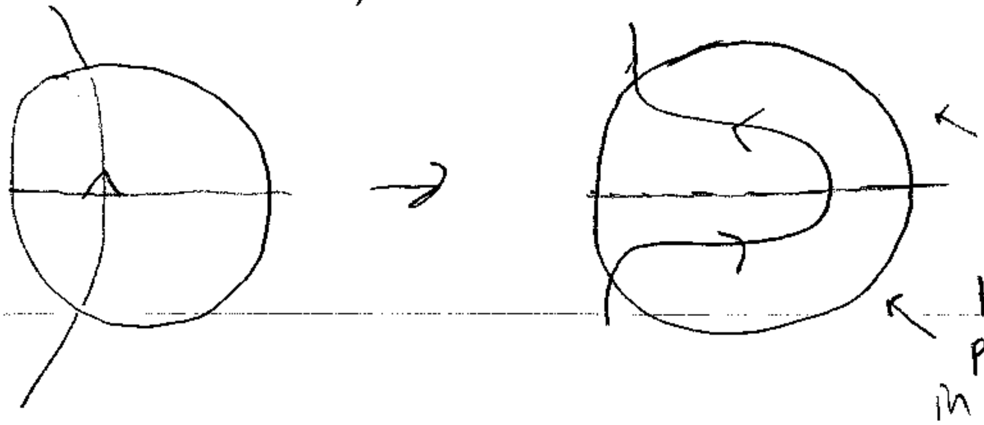
$F_{buoy}\cdot H = \frac{B^2}{8\pi}V = \frac{1}{2}\rho_i V u^2$, so u = velocity of tube is $\simeq u = \left(\frac{B^2}{4\pi\rho_i}\right)^{1/2} = V_{A,tube}$ = Alfvén speed of tube (310) when the temperatures inside and outside e tube are not equal, the entire tube may not rise up, since then ρ_i at that location may not necessarily be $< \rho_e$, since $P_i < P_e$ can be satisfied by $\rho_e < \rho_i$ if $T_i < T_e$. (buoyancy requires $\rho_i < \rho_e$).

On the sun, most bipolar regions are roughly aligned parallel to the solar equator In northern hemisphere, when + polarities are to ight of negative polarity, in the south, − polarities re to the right of + polarities. Thus each of the northern & southern hemispheres typically show an pposite sign of leading & trailing polarity system, and the pattern reverses every 11 years:

e.g.

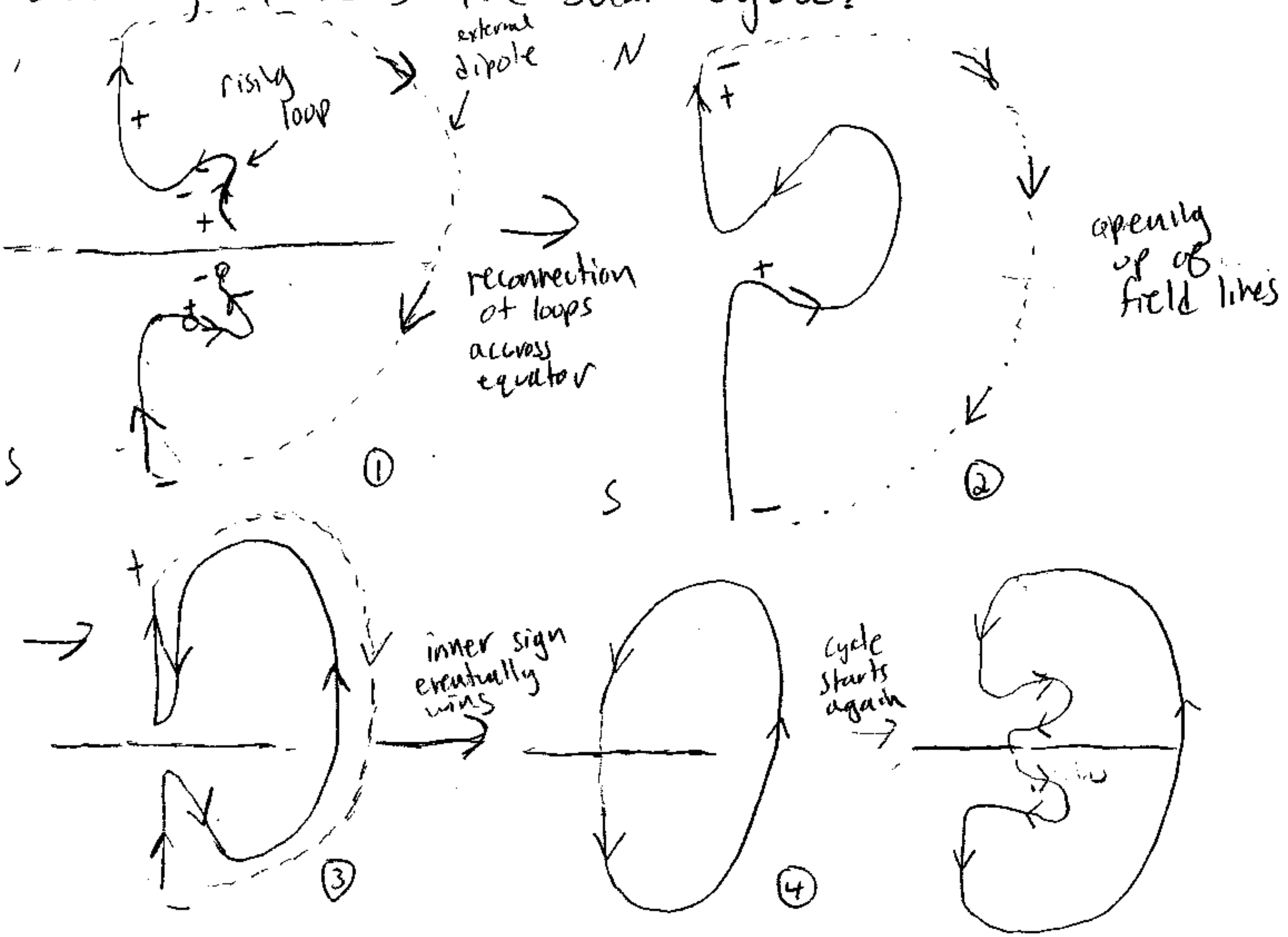
How can this situation arise? :

First, the sun is not rotating uniformly, but differentially, faster at equator, so that



thus buoyancy would naturally produce opposite trailing and leading polarity patterns in each hemisphere as on previous page.

Second, there is the solar cycle:



Angular momentum transport & magnetic fields:

Magnetic fields can help transport angular momentum. To see this, first prove a theorem:

Ferraro's law of isorotation: consider rotating object symmetric around rotation axis. Using cylindrical coords, this implies

$$V = r\Omega(r,z)\,\hat{e}_\theta \qquad (311)$$

independent of θ. Suppose object has axisymmetric poloidal field, frozen into plasma. Steady state is possible only if Ω is constant along field lines:

proof: A poloidal (r,z) field independent of θ can be written as curl of vector potential A_θ and in the form: (cylindrical coords)

$$\vec{B} = \left(\nabla \times \left(\frac{1}{r}\psi(r,z)\right)\hat{e}_\theta\right) \qquad (312)$$

then $B_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}$, $\quad B_z = \frac{1}{r}\frac{\partial\psi}{\partial r} \qquad (313)$

ψ is $\perp$ to field lines. Now let dr, dz represent displacements along streamlines of $\vec{B}$ (i.e. curves which have tangents $\parallel$ $\vec{B}$)

$\lrcorner$

then $$\frac{dr}{B_r} = \frac{dz}{B_z} \quad . \qquad (314)$$

From above we then have

$\frac{\partial \psi}{dr} dr + \frac{\partial \psi}{\partial z} dz = 0$ so that ψ is constant along streamlines of $\vec{B}$.

now use induction equation in steady state with no diffusivity:

$$\nabla \times (\vec{v} \times \vec{B}) = 0 \qquad (315)$$

for (312) & (311)

$$\Rightarrow \quad \nabla \times \left(r\Omega \hat{e}_\theta \times \left(\nabla \times \frac{1}{r} \psi \hat{e}_\theta \right) \right)$$

$$- \nabla \times \left(r\Omega \hat{e}_\theta \times \frac{1}{r}\frac{\partial \psi}{\partial z} \hat{e}_r \right) + \nabla \times \left(r\Omega \hat{e}_\theta \times \frac{1}{r}\frac{\partial \psi}{\partial r} \hat{e}_z \right)$$

$$+ \nabla \times \left(\frac{\cancel{r}\Omega}{\cancel{r}} \frac{\partial \psi}{\partial z} \hat{e}_z \right) + \nabla \times \left(\frac{\cancel{r}\Omega}{\cancel{r}} \frac{\partial \psi}{\partial r} \hat{e}_r \right)$$

$$\left[\frac{\partial}{\partial z}\left(\Omega \frac{\partial \psi}{\partial r} \right) - \frac{\partial}{\partial r}\left(\Omega \frac{\partial \psi}{\partial z} \right) \right] \hat{\theta} = 0$$

$$= \frac{\partial \Omega}{\partial z}\frac{\partial \psi}{\partial r} - \frac{\partial \Omega}{\partial r}\frac{\partial \psi}{\partial z} = 0 \quad \Rightarrow \Omega = f(\psi)$$

thus, $\Omega = f(\psi)$

and this means that the angular velocity is constantant along field lines, since ψ is a constant along field lines.

If Ω were to vary along field lines then poloidal lines would be continuously stretched to produce toroidal lines, and steady state is not possible without dissipation. When field lines are stretched, work is done on them. If field is strong, then field resists deformation and tries to impose rigid rotation.

Now this helps to explain why B-fields can transport ℓ momentum. We will consider examples of magnetic breaking and jets

→