Spherically Symmetric Accretion

... object of mass M accreting spherically from large gas cloud

- goal is to predict $\dot{M}$ for a given $\rho(r_0 \gg r_*)$, $T(r_0 \gg r_*)$ and surface boundary conditions on r_*

- take spherical coords, and ignore ang. momentum, so assume only radial dependence of variables

- steady flow

$\Rightarrow$ mass conservation

$$\nabla \cdot (\rho v) = 0$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \rho v_r\right) = 0 \quad \rightarrow \quad \text{integrate}$$

(123)

$$\Rightarrow \quad r^2 \rho v = \text{constant}$$

$\rightarrow$

Note: $4\pi r^2 \rho V_r = \dot{M}$

$V_r < 0 \Rightarrow$ accretion $\dot{M}$ = accretion rate

$V_r > 0 \Rightarrow$ ejection, wind, $\dot{M}$ = ejection rate

→ Next consider momentum equation (Euler equation). Take $\vec{F}$ = gravitation force.

$\vec{F} = -\dfrac{GM\rho}{r^2}$, so momentum

equation becomes:

$$V_r \frac{dV_r}{dr} + \frac{1}{\rho}\frac{dP}{dr} + \frac{GM}{r^2} = 0 \qquad (124)$$

Consider now the polytropic relation

$P = K\rho^\gamma$ $\left(1 < \gamma < 5/3 \right)$

1 ↑ isothermal

$5/3$ ↑ adiabatic for monatomic gas

$P = K\rho^\gamma$ ↑ (this replaces the energy equation)

for a real accretor, neither $\gamma = 1$ nor $\gamma = 5/3$ (121)
are exactly satified. The value $\gamma \to 5/3$ is approached if the heating or cooling times are much larger than the infall time at that radius.

If we find $P(r)$, $\rho(r)$ then we get $v(r)$ and we can use ideal gas relation $T(r) = \frac{m_H \mu P}{\rho k_b}$, (125)
+ get temperature.

$$\left(\mu \equiv \frac{\frac{1}{2} n m_p + \frac{1}{2} n m_e}{n m_p} = \frac{1}{2}\right)$$

really need to integrate (124) but we can proceed quite far without integrating:
First write

$$\frac{dP}{dr} = \frac{dP}{d\rho}\frac{d\rho}{dr} = c_s^2 \frac{d\rho}{dr} \qquad (126)$$

thus (124) becomes $\longrightarrow$

$$V_r \frac{dV_r}{dr} + \frac{C_s^2}{\rho}\frac{d\rho}{dr} + \frac{GM}{r^2} = 0 \qquad (126a)$$

but (123) $\Rightarrow$

$$\frac{d \ln(r^2 \rho v)}{dr} = 0$$

$$\Rightarrow \frac{1}{\rho}\frac{d\rho}{dr} = -\frac{1}{r^2 V_r}\frac{d}{dr}(r^2 V_r)$$

using this in (126a) $\Rightarrow$

$$V_r \frac{dV_r}{dr} - \frac{C_s^2}{r^2 V_r}\frac{d(r^2 V_r)}{dr} + \frac{GM}{r^2} = 0$$

$$\downarrow$$

$$\frac{1}{2}\frac{dV_r^2}{dr} - \frac{C_s^2}{V_r}\frac{dV_r}{dr} - \frac{2C_s^2}{r} + \frac{GM}{r} = 0$$

$$\boxed{\frac{1}{2}\left(1 - \frac{C_s^2}{V_r^2}\right)\frac{dV_r^2}{dr} = -\frac{GM}{r^2}\left[1 - \frac{2C_s^2 r}{GM}\right]} \qquad (127)$$

useful equation

at large r, right side of (60) is positive because $c_s \to c_s(\infty)$ there.

On left side of (127),

$$\frac{dv_r^2}{dr} < 0 \quad \text{for } r \to \infty$$

because gas is at rest there. ($v_r \to 0$)

$\Rightarrow$ $c_s^2 / v_r^2 > 1$ for large r,

that is, flow is subsonic at $r \to \infty$ for desired accretion solution.

As gas moves to smaller r, the factor $\left(1 - \frac{2c_s^2 r}{GM}\right)$ increases, provided that c_s^2 increases more slowly than $\frac{1}{r}$ as r is decreased. Note that this factor vanishes at $r = \frac{GM}{2c_s^2(r_s)} \equiv r_s$, the <u>sonic radius</u>

$\rightarrow$

Thus provided c_s^2 increases slowly with r, the desired accretion soln will be supersonic $v_r^2 > c_s^2$ at small r (from left side of (127)).

Note that approximating $c_s^2(r_s) \simeq c_s^2(r \gg r_s)$ we can use typical values for galaxy: $c_s^2(r_\infty) \simeq \frac{kT_{ism}}{m_p} \simeq 10^{12}\left(\frac{T}{10^4\,K}\right)\frac{cm^2}{s^2}$

then

$$\Rightarrow \quad r_s = \frac{GM}{2c_s^2} \simeq \frac{(10^{-7})(6\times10^{39})}{2\;10^{12}} \simeq 3\times10^{20}\,cm$$
$$\simeq \underline{100\,pc}$$

$$100\,pc \gg r_g \simeq \frac{GM}{c^2} = \left(\frac{10^{-7}}{10^{21}}\,6\times10^{39}\right) \simeq \underline{6\times10^{11}\,cm}$$

thus the sonic radius $\gg$ gravitational radius for B.H. in Galaxy

which helps justify assumption

$c_s(r_s) \simeq c_s(r_\infty)$ since we can measure the temperatures at $r = 100\,pc$ and at $r \simeq$ few kpc and they don't differ hugely. →

To be more rigorous, we really need to impose conditions at central object (star or B-H) to fully specify soln.

Note again (12.7). at $r = r_s$

$V_r^2 = C_s^2$ or $\dfrac{dV_r^2}{dr} = 0$. Thus there are different possible solutions, depending bdry conditions. There are 6 classes of solutions. The two most interesting are:

1) $V_r^2(r_s) = C_s^2(r_s)$ $\quad V_r^2 \to 0$ as $r \to \infty$

(a) $V_r^2 < C_s^2$, $r > r_s$

(b) $V_r^2 > C_s^2$, $r < r_s$

} accretion soln

2) $V_r^2(r_s) = C_s^2(r_s)$ $\quad V_r^2 \to 0$ as $r \to 0$

(a) $V_r^2 > C_s^2$ $\quad r > r_s$

(b) $V_r^2 < C_s^2$, $r < r_s$

} ejection solution

$\longrightarrow$

Type (1) solution is called <u>Bondi Accretion</u>: flow goes from subsonic to supersonic as material falls inward.

Now let us go back and integrate (124)

$$\frac{V_r^2}{2} + \int \frac{dP}{\rho} - \frac{GM}{r} = \text{constant} \qquad (128)$$

$$dP = K\gamma\rho^{\gamma-1} d\rho \Rightarrow \int \frac{dP}{\rho} = \int K\gamma\rho^{\gamma-2} d\rho$$

$\Rightarrow (128) \Rightarrow$

$$\frac{V_r^2}{2} + \frac{K\gamma}{\gamma-1}\rho^{\gamma-1} - \frac{GM}{r} = \text{constant} \qquad (129)$$

but $K\gamma\rho^{\gamma-1} = \frac{\gamma P}{\rho} = C_s^2$ (adiabatic sound speed)

$\Rightarrow (129) \Rightarrow$

$$\boxed{\frac{V_r^2}{2} + \frac{C_s^2}{\gamma-1} - \frac{GM}{r} = \text{const}} \qquad (130)$$

(62) is <u>Bernouilli constant</u>

Since <u>type ① soln</u> $V_r \to 0$ as $r \to \infty$ and $\frac{GM}{r} \to 0$ at $r \to \infty$ (and $V_r = C_s$ at $r = r_s$)

we have from (130) that the constant

is $\frac{C_s^2(\infty)}{\gamma - 1}$, so at the sonic radius

$$C_s^2(r_s)\left[\frac{1}{2} + \frac{1}{\gamma-1} - 2\right] = \frac{C_s^2(\infty)}{\gamma - 1}$$

$$\boxed{C_s(r_s) = C_s(\infty)\left(\frac{2}{5-3\gamma}\right)^{\frac{1}{2}}} \qquad (131)$$

(for $\gamma \sim 1.4$ or so, this confirms $C_s(r_s) \simeq C_s(\infty)$)

Now mass conservation gives

$\rho V_r r^2 = \text{constant} \Rightarrow$

$$\dot{M} = 4\pi r^2 \rho(-V_r) = 4\pi r_s^2 \rho(r_s) C_s(r_s) \qquad (132)$$

and since

$$C_s \propto \left(\frac{P}{\rho}\right)^{1/2} \propto \rho^{\frac{\gamma-1}{2}} \Rightarrow \frac{\rho(r_s)}{\rho(\infty)} \simeq \left[\frac{C_s(r_s)}{C_s(\infty)}\right]^{\frac{2}{\gamma-1}} \qquad (133)$$

using (131) and (133) in (132)

$$\dot{M} = 4\pi r_s^2 \rho(\infty) \left[\frac{2}{5-3\gamma}\right]^{\frac{1}{\gamma-1}} c_s(\infty) \left(\frac{2}{5-3\gamma}\right)^{1/2}$$

$$= 4\pi r_s^2 \rho(\infty) c_s(\infty) \left[\frac{2}{5-3\gamma}\right]^{\frac{1+\gamma}{2(\gamma-1)}} \qquad (134)$$

but $r_s^2 \equiv \frac{G^2M^2}{4c_s^4(r_s)} \Rightarrow \frac{G^2M^2}{4c_s^4(\infty)}\left(\frac{5-3\gamma}{2}\right)^2$

( ) $\Rightarrow$

$$\dot{M} = \frac{\pi G^2 M^2 \rho(\infty)}{c_s^3(\infty)} \left(\frac{2}{5-3\gamma}\right)^{\left(\frac{1+\gamma}{2(\gamma-1)} - \frac{4(\gamma-1)}{}\right)}$$

$$\boxed{\dot{M} = \frac{\pi G^2 M^2 \rho(\infty)}{c_s^3(\infty)} \left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{2(\gamma-1)}}} \qquad (135)$$

next compute

$v(r) \rightarrow$

Using $\dot{M} = 4\pi r^2 \rho V_r$

we can find $V_r(r)$ from $c_s^2 \propto \frac{P}{\rho} \propto \rho^{\gamma-1}$: (previously derived)

$$V_r(r) = \frac{\dot{M}}{4\pi r^2 \rho(r)} = \frac{\dot{M}}{4\pi r^2 \rho(\infty)} \left[\frac{c_s(\infty)}{c_s(r)}\right]^{2/\gamma-1} \qquad (136)$$

then substituting into Bernouilli integral

$$\frac{V^2(r)}{2} + \frac{c_s^2(r)}{\gamma-1} - \frac{GM}{r} = \frac{c_s^2(\infty)}{\gamma-1} \quad \text{(Bernouilli Integral)} \qquad (137)$$

gives $c_s(r)$ and thus $v(r)$.

The fractional exponents require numerical solution, but can infer behavior analytically.

→ At large r, quantities have asymptotic values $V_r \to 0$

→ At small r, $v(r)$ increases and surpasses c_s at r_s, and then is balanced by $\frac{GM}{r}$

From (137), $\frac{v^2(r)}{2} \simeq \frac{c_s^2(r)}{\gamma-1}$ when approximately

$$r \simeq \frac{GM}{c_s^2(\infty)} \qquad (138)$$

→

Lets analyze this a bit further:

If V_r approaches free-fall near the object then how does $\rho(r)$ depend?

$$4\pi r^2 \rho \underbrace{V_r}_{\frac{1}{r^{1/2}} \text{ if freefall}} = \underbrace{\dot{M}}_{\text{constant}}$$

$$\Rightarrow \underline{\rho \propto r^{-3/2} \qquad r < r_s}$$

(for $\gamma \simeq 1.4$)

$$\text{en} \quad T \propto C_s^2 \propto \frac{P}{\rho} \propto \rho^{\gamma-1} \Rightarrow r^{-\frac{3}{2}(\gamma-1)} \sim r^{-0.6}$$

(Note when radiation losses are considered at smaller r, γ is smaller and T may change more slowly, but lets stick with $\gamma \simeq 1.4$ here)

Since $C_s^2 \propto r^{-0.6}$, when we plug into left side of Bernoulli integral, the C_s^2 term increases less rapidly than the $\frac{GM}{r}$ term as we move to smaller r.

This justifies ignoring the C_s^2 term at small r and considering only balance between V^2 and $\frac{GM}{r}$.

thus our conclusion that

$\frac{v_r^2}{2} \sim \frac{GM}{r}$ at small r is

justified and this balance begins

at $r \leq r_{acc} \simeq \frac{GM}{c_s^2(\infty)}$ (equation 138)

accretion radius.

The radius at which

thermal energy = grav. binding energy

For $r \gg r_{acc}$: gravity has little effect

For $r \ll r_{acc}$: gravity <u>dominates</u>