Classical Mechanics - Module 10

1. Consider two masses (each of mass M) connected by a spring to each other and by springs to fixed positions. Motion is only allowed along one dimension. (This is exactly the same system that is discussed in chapter 12.2 of the lecture notes on coupled oscillations.) Let each of the two oscillator springs have a force constant κ and let the force constant of the coupling spring be κ_{12}. Let x_1 and x_2 be the coordinates as described in the textbook.

   (a) Draw a picture of the two masses displaced by a small amount. Using the picture, try to make sense of the equations of motion as given in the text:

   $$M\ddot{x}_1 + (\kappa + \kappa')x_1 - \kappa' x_2 = 0, :::: M\ddot{x}_2 + (\kappa + \kappa')x_2 - \kappa' x_1 = 0$$

   (b) Each of the trial solutions is written in the form $Be^{i\omega t}$. Why are the trial solutions written this way? Are there any other ways to write the trial solution?

   (c) For a nontrivial solution to exist for the pair of simultaneous equations resulting from the substitution of the trial solution, the determinant of the coefficients of B_1 and B_2 must vanish. Why must this be the case? Is a similar statement true when considering three masses? What about n masses?

   (d) Suppose you had the actual two-mass system sitting in front of you. How could you create antisymmetric motion? How could you create symmetric motion? Can you describe each of these motions using a set of suitable initial conditions?

2. Two particles, each with mass m, move in one dimension in a region near a local minimum of the potential energy where the potential energy is approximately given by

   $$U = \frac{1}{2}k(7x_1^2 + 4x_2^2 + 4x_1x_2)$$

   where k is a constant.

   (a) Determine the frequencies of oscillation.

   (b) Determine the normal coordinates.

3. What is degeneracy? When does it arise?

4. The Lagrangian of three coupled oscillators is given by:

   $$\sum_{n=1}^{3}\left[\frac{m\dot{x}_n^2}{2} - \frac{kx_n^2}{2}\right] + k'(x_1x_2 + x_2x_3).$$

   Find $x_2(t)$ for the following initial conditions (at $t = 0$):

   $$(x_1, x_2, x_3) = (x_0, 0, 0), :::::: (\dot{x}_1, \dot{x}_2, \dot{x}_3) = (0, 0, v_0).$$

5. A mechanical analog of the benzene molecule comprises a discrete lattice chain of 6 point masses M connected in a plane hexagonal ring by 6 identical springs each with spring constant κ and length d.

   a) List the wave numbers of the allowed undamped longitudinal standing waves.

   b) Calculate the phase velocity and group velocity for longitudinal travelling waves on the ring.

   c) Determine the time dependence of a longitudinal standing wave for a angular frequency $\omega = 2\omega_{cutoff}$, that is, twice the cut-off frequency.