Classical Mechanics - Module 3

1. Given below are a list of statements followed by a list of reasons related to harmonic motion. For each of the statements, determine the reason(s) that make that statement true. You may do this in small groups or as one large group–the TA will decide what works best for your workshop.
   Statements:

   - We can neglect the higher order terms in the Taylor expansion of $F(x)$.
   - The restoring force is a linear force.
   - F_0 must vanish.
   - $(dF/dx)_0$ is negative and k is positive.
   - We can write $F(x)$ as a Taylor series expansion.

   Reasons:

   - $F(x)$ depends only on x.
   - A position of stable equilibrium exists and we call this point the origin of our coordinate system.
   - $F(x)$ has continuous derivatives of all orders.
   - The restoring force is directed toward the equilibrium position.
   - We consider only small displacements.

2. Second-order ordinary differential equations are an important part of the physics of the harmonic oscillator.

   (a) What do each of the following terms mean with respect to differential equations?

      i. Ordinary
      ii. Second-order
      iii. Homogeneous
      iv. Linear

   (b) Give a mini-lesson on how to solve second-order differential equations by working through the following examples. Don't just provide a solution; explain the steps leading up to the solution.

      i. $y'' + 5y' + 6y = 0$
      ii. $y'' + y' + y = 0$
      iii. $y'' + 4y' + 4y = 0$
      iv. $y'' - 3y'^{2x}$

v. $y'' - 3y' - 4y = 2\sin x$

3. Harmonic oscillations occur for many different types of systems and it is important to recognize when the equations for harmonic motion apply. Three different systems are described below. Each system can be approximately described using the equations for harmonic motion. Break up into three groups–one group per system. For your group's system, answer the following questions:

   (a) What approximations are necessary for this system to exhibit harmonic oscillations?

   (b) What is the differential equation that governs the motion of this system? Use Newton's second law to arrive at this equation.

   (c) What is the solution to the differential equation that you found in part (b)?

   (d) What is the natural frequency of oscillations?

   Here are the three systems:

   - A mass m is tied to a massless spring having a spring constant k. The system oscillates in one dimension along a horizontal fricitonless surface.
   - A particle of mass m is attached to a weightless, extensionless rod to form a pendulum. The length of the rod is L and the system oscillates in a single plane.
   - A tube is bent into the shape of a U and is partially filled with a liquid of density ρ. The cross-sectional area of the tube is A and the length of the tube filled with liquid is L. The liquid is initially displaced so that it is higher on one side of the tube than the other.

   Once each group has answered all of the questions, share the results with the entire class.

4. Consider a mass m attached to a spring of spring constant k. The spring is mounted horizontally so that the mass oscillates horizontally on a frictionless surface. The spring is attached to the wall on the right and the mass is initially moved to the right of its equilibrium position (compressing the spring) by a distance s and released. Working individually, determine how (if at all) the period of the motion would be affected by each of the changes below. Once you have answered each part on your own, compare your answers with a classmate.

   (a) The spring is replaced with a stiffer spring.

   (b) The mass is initially displaced a distance s to the left and released.

   (c) The mass is replaced with a heavier mass.

(d) The mass is initially displaced a distance r ($r < s$) to the right and released.

5. When you were first introduced to simple harmonic motion, you used the formula $m\ddot{x} = -kx$ to find the position of the oscillating mass as a function of time. This assumes that the origin is defined to be the equilibrium point. What happens if this is not the case? What would the equation of motion look like? How would the position of the oscillating mass as a function of time change?

6. For each of the situations described below, give a rough sketch of the state space diagram ($\dot{x}$ versus x) that represents the motion of each object. All of the motion takes place along the x-axis.

   (a) An eggplant is at rest at a point on the $+x$ axis.

   (b) A monkey on a skateboard skates with constant speed in the negative x direction.

   (c) A racecar moving in the $+x$ direction undergoes constant acceleration until it abruptly stops.

   (d) A cantaloupe undergoes simple harmonic motion. The initial location of the cantaloupe is at a point on the $+x$ axis.

7. Consider a simple harmonic oscillator consisting of a mass m attached to a spring of spring constant k. For this oscillator $x(t) = A\sin(\omega_0 t - \delta)$.

   (a) Find an expression for $\dot{x}(t)$.

   (b) Eliminate t between $x(t)$ and $\dot{x}(t)$ to arrive at one equation similar to that for an ellipse.

   (c) Rewrite the equation in part (b) in terms of x, $\dot{x}$, k, m, and the total energy E.

   (d) Give a rough sketch of the phase space diagram ($\dot{x}$ versus x) for this oscillator. Also, on the same set of axes, sketch the phase space diagram for a similar oscillator with a total energy that is larger than the first oscillator.

   (e) What direction are the paths that you have sketched? Explain your answer.

   (f) Would different trajectories for the same oscillator ever cross paths? Why or why not?

8. Consider a damped, driven oscillator consisting of a mass m attached to a spring of spring constant k.

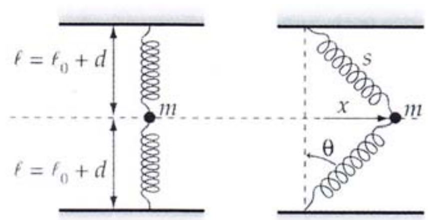
   (a) What is the equation of motion for this system?

(b) Solve the equation in part (a). The solution consists of two parts, the complementary solution and the particular solution. When might it be possible to safely neglect one part of the solution?

(c) What is the difference between amplitude resonance and kinetic energy resonance?

(d) How might phase space diagrams look for this type of oscillator? What variables would affect the diagram?

9. Consider the system of a mass suspended between two identical springs as shown.



If each spring is stretched a distance d to attach the mass at the equilibrium position the mass is subject to two equal and oppositely directed forces of magnitude κd. Ignore gravity. Show that the potential in which the mass moves is approximately

$$U(x) = \left\{\frac{\kappa d}{l}\right\} x^2 + \left\{\frac{\kappa(l-d)}{4l^3}\right\} x^4$$

Construct a state-space diagram for this potential.

10. Find the extremal of the functional

$$J(x) = \int_1^2 \frac{\dot{x}^2}{t^3} dt$$

that satisfies $x(1) = 3$ and $x(2) = 18$. Show that this extremal provides the global minimum of J.