Classical Mechanics - Module 7

1. Listed below are several statements concerning central force motion. For each statement, give the reason for why the statement is true. If a statement is only true in certain situations, then explain when it holds and when it doesn't. The system referred to below consists of mass m_1 located at r_1 and mass m_2 located at r_2.

   - The potential energy of the system depends only on the difference $r_1 - r_2$, not on r_1 and r_2 separately.
   - The potential energy of the system depends only on the magnitude of $r_1 - r_2$, not the direction.
   - It is possible to choose an inertial reference frame in which the center of mass of the system is at rest.
   - The total energy of the system is conserved.
   - The total angular momentum of the system is conserved.

2. A particle of mass m moves in a potential $U(r) = -U_0 e^{-\lambda^2 r^2}$.

   (a) Given the constant l, find an implicit equation for the radius of the circular orbit. A circular orbit at $r = \rho$ is possible if
   $$\left(\frac{\partial V}{\partial r}\right)\bigg|_{r=\rho} = 0$$
   where V is the effective potential.

   (b) What is the largest value of l for which a circular orbit exists? What is the value of the effective potential at this critical orbit?

3. A particle of mass m is observed to move in a spiral orbit given by the equation $r = k\theta$, where k is a constant. Is it possible to have such an orbit in a central force field? If so, determine the form of the force function.

4. The interaction energy between two atoms of mass m is given by the Lennard-Jones potential, $U(r) = \epsilon\left[(r_0/r)^{12} - 2(r_0/r)^6\right]$

   (a) Determine the Lagrangian of the system where r_1 and r_2 are the positions of the first and second mass, respectively.

   (b) Rewrite the Lagrangian as a one-body problem in which the center-of-mass is stationary.

   (c) Determine the equilibrium point and show that it is stable.

   (d) Determine the frequency of small oscillations about the stable point.

5. Consider two bodies of mass m in circular orbit of radius $r_0/2$, attracted to each other by a force $F(r)$, where r is the distance between the masses.

   (a) Determine the Lagrangian of the system in the center-of-mass frame (Hint: a one-body problem subject to a central force).

   (b) Determine the angular momentum. Is it conserved?

   (c) Determine the equation of motion in r in terms of the angular momentum and $|\mathbf{F}(r)|$.

   (d) Expand your result in (c) about an equilibrium radius r_0 and show that the condition for stability is, $\frac{F'(r_0)}{F(r_0)} + \frac{3}{r_0} > 0$

6. Consider two charges of equal magnitude q connected by a spring of spring constant k' in circular orbit. Can the charges oscillate about some equilibrium? If so, what condition must be satisfied?

7. Consider a mass m in orbit around a mass M, which is subject to a force $F = -\frac{k}{r^2}\hat{r}$, where r is the distance between the masses. Show that the Runge-Lenz vector $A = p \times L - \mu k \hat{r}$ is conserved.