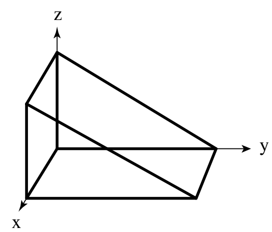
Classical Mechanics - Module 9

1. Three objects are described below. Break up into three groups, one group per object, and determine the inertia tensor.

   (a) A very thin sheet with a mass density $\sigma = Cxy$ where C is a positive constant. The sheet lies in the xy plane and its sides are both of length a.

   (b) An inclined-plane shaped block of mass M is oriented with one corner at the origin as shown.

   (c) An equilateral triangle made up of three thin rods of length l and uniform mass density ρ.



2. Consider the objects described in problem 1.

   (a) For the first object (the thin sheet), determine the principal moments of inertia.

   (b) For the second object (the inclined plane), determine the principal axes.

   (c) For the third object (the equilateral triangle), determine the products of inertia.

3. Consider the inertia tensor.

   (a) What are the advantages of diagonalizing the inertia tensor?

   (b) How can the inertia tensor be diagonalized?

   (c) What can you say about a tensor that is real and symmetric?

4. A hollow spherical shell has a mass m and radius R.

   (a) Calculate the inertia tensor for a set of coordinates whose origin is at the center of mass of the shell.

   (b) Now suppose that the shell is rolling without slipping toward a step of height h, where $h < R$. The shell has a linear velocity v. What is the angular momentum of the shell relative to the tip of the step?

   (c) The shell now strikes the tip of the step inelastically (so that the point of contact sticks to the step, but the shell can still rotate about the tip of the step). What is the angular momentum of the shell immediately after contact?

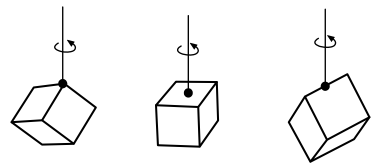
   (d) Finally, find the minimum velocity which enables the shell to surmount the step. Express your result in terms of m, g, R, and h.

5. The vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ constitute a set of orthogonal right-handed axes. The vectors $\hat{x} + \hat{y} - 2\hat{z}$, $-\hat{x} + \hat{y}$, and $\hat{x} + \hat{y} + \hat{z}$ are also perpendicular to one another.

   (a) Write out the set of direction cosines relating the new axes to the old.

   (b) How are the Eulerian angles defined? Describe this transformation by a set of Eulerian angles.

6. A torsional pendulum consists of a vertical wire attached to a mass which can rotate about the vertical axis. Consider three torsional pendula which consist of identical wires from which identical homogeneous solid cubes are hung. One cube is hung from a corner, one from midway along an edge, and one from the middle of a face as shown. What are the ratios of the periods of the three pendula?



7. A dumbbell comprises two equal point masses M connected by a massless rigid rod of length $2A$ which is constrained to rotate about an axle fixed to the center of the rod at an angle θ as shown in the figure. The center of the rod is at the origin of the coordinates, the axle along the z-axis, and the dumbbell lies in the $x-y$ plane at $t=0$. The angular velocity ω is a constant in time and is directed along the z axis.

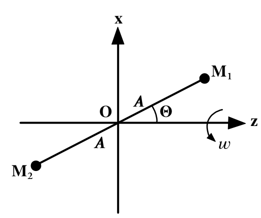
a) Calculate all elements of the inertia tensor. Be sure to specify the coordinate system used.

b) Using the calculated inertia tensor find the angular momentum of the dumbbell in the laboratory frame as a function of time.

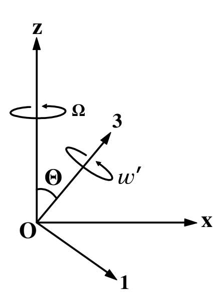
c) Using the equation $L = r \times p$, calculate the angular momentum and show that it it is equal to the answer of part (b).

d) Calculate the torque on the axle as a function of time.

e) Calculate the kinetic energy of the dumbbell.



8. A heavy symmetric top has a mass m with the center of mass a distance h from the fixed point about which it spins and $I_1 = I_2 \neq I_3$. The top is precessing at a steady angular velocity Ω about the vertical space-fixed z axis. What is the minimum spin ω' about the body-fixed symmetry axis, that is, the 3 axis assuming that the 3 axis is inclined at an angle $\theta = \theta$ with respect to the vertical z axis. Solve the problem at the instant when the $z, x, 3, 1$ axes all are in the same plane as shown in the figure.

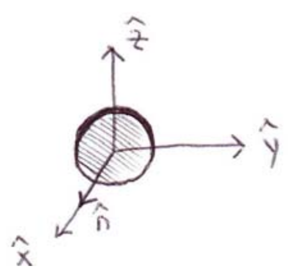


9. Consider an object with the center of mass is at the origin and inertia tensor,

$$I = I \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   (a) Determine the principal moments of inertia and the principal axes. Guess the object.
   (b) Determine the rotation matrix R and compute $R^\dagger I R$. Do the diagonal elements match with your results from (a)? Note: columns of R are eigenvectors of I.
   (c) Assume $\omega = \frac{\omega}{\sqrt{2}}(\hat{x} + \hat{z})$. Determine L in the rotating coordinate system. Are L and ω in the same direction? What does this mean?
   (d) Repeat (c) for $\omega = \frac{\omega}{\sqrt{2}}(\hat{x} - \hat{y})$. What is different and why?
   (e) For which case will there be a non-zero torque required?
   (f) Determine the rotational kinetic energy for the case $\omega = \frac{\omega}{\sqrt{2}}(\hat{x} - \hat{y})$?

10. Consider a wheel (solid disk) of mass m and radius r. The wheel is subject to angular velocities $\omega_A = \omega_A\,\hat{n}$ where $\hat{n}$ is normal to the surface and $\omega_B = \omega_B\,\hat{z}$.



   (a) Choose a set of principal axes by observation.
   (b) Determine the angular velocities and angular momentum along the principal axes. Note: $I_1 = \frac{1}{2}mr^2$ and $I_2 = I_3 = \frac{1}{4}mr^2$.
   (c) Determine the torque.
   (d) Determine the rotation matrix that rotates the fixed coordinate system to the body coordinate system.

11. Determine the principal moments of inertia of an ellipsoid given by the equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

12. Determine the principal moments of inertia of a sphere of radius R with a cavity of radius r located ϵ from the center of the sphere.

13. Three equal masses m form the vertices of an equilateral triangle of side length L. The masses are located at $\left(0, 0, \frac{L}{\sqrt{3}}\right)$, $\left(0, \frac{L}{2}, -\frac{L}{2\sqrt{3}}\right)$, and $\left(0, -\frac{L}{2}, -\frac{L}{2\sqrt{3}}\right)$, such that the center-of-mass is located at the origin.

   (a) Determine the principal moments of inertia and principal axes.
   Now consider the same system rotated 45° about the $\hat{z}$-axis. The masses are located at $\left(0, 0, \frac{L}{\sqrt{3}}\right)$, $\left(-\frac{L}{2\sqrt{2}}, \frac{L}{2\sqrt{2}}, -\frac{L}{2\sqrt{3}}\right)$, and $\left(\frac{L}{2\sqrt{2}}, -\frac{L}{2\sqrt{2}}, -\frac{L}{2\sqrt{3}}\right)$, respectively.
   (b) Determine the principal moments of inertia and principal axes.
   (c) Could you have answered (b) without explicitly determining the inertia tensor? How?