# P235 - PROBLEM SET 3

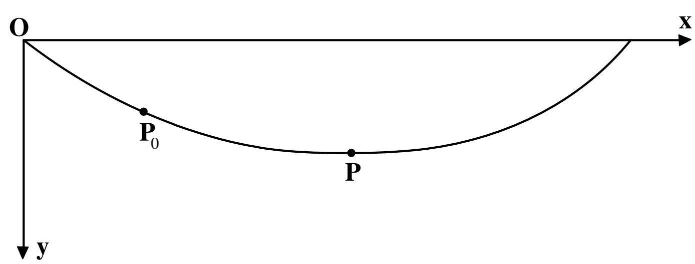
To be handed in by **1800 hr on Friday, 25 September 2015**.

1. Find the extremal of the functional
$$J(x) = \int_0^\pi (2x \sin t - \dot{x}^2) dt$$
that satisfies $x(o) = x(\pi) = 0$. Show that this extremal provides the global maximum of $J$.

2. Find and describe the path $y = y(x)$ for which the the integral $\int\limits_{x_1}^{x_2} \sqrt{x}\sqrt{1+(y')^2}dx$ is stationary.

3. Find the dimensions of the parallelepiped of maximum volume circumscribed by a sphere of radius $R$.

4. Consider a single loop of the cycloid having a fixed value of $a$ as shown in the figure. A car released from rest at any point $P_0$ anywhere on the track between $O$ and the lowest point $P$ , that is, $P_0$ has a parameter $0 < \theta_0 < \pi$.




   (a) Show that the time $T$ for the cart to slide from $P_0$ to $P$ is given by the integral
$$T(P_0 \to P) = \sqrt{\frac{a}{g}} \int\limits_{\theta_0}^{\pi} \sqrt{\frac{1-\cos\theta}{\cos\theta_0 - \cos\theta}} d\theta$$
   (b) Prove that this time $T$ is equal to $\pi\sqrt{a/g}$ which is independent of the position $P_0$.
   (c) Explain qualitatively how this surprising result can possibly be true.

5. Consider a medium for which the refractive index $n = \frac{a}{r^2}$ where $a$ is a constant and $r$ is the distance from the origin. Use Fermat's Principle to find the path of a ray of light travelling in a plane containing the origin. Hint, use two-dimensional polar coordinates with $\phi = \phi(r)$. Show that the resulting path is a circle through the origin.

6. Find the shortest path between the $(x, y, z)$ points $(0, -1, 0)$ and $(0, 1, 0)$ on the conical surface
$$z = 1 - \sqrt{x^2 + y^2}$$
What is the length of this path? Note that this is the shortest mountain path around a volcano.

7. Show that the geodesic on the surface of a right circular cylinder is a segment of a helix.