PHY 407
Homework 11

1. a) Prove the Thomas-Reiche-Kuhn sum rule

$$\sum_{n'} (E_{n'} - E_n) \, |\langle n'|X|n\rangle|^2 = \frac{\hbar^2}{2m}, \tag{1}$$

where X is the coordinate operator and $|n\rangle$ represent the eigenstates of the Hamiltonian

$$H = \frac{P^2}{2m} + V(X). \tag{2}$$

b) Test the sum rule on the n-th state of the harmonic oscillator.

2. Consider the Runge-Lenz operator defined as

$$M_i = \frac{1}{2\mu}\epsilon_{ijk} \left(p_j L_k - L_j p_k\right) - \frac{e^2}{r} r_i, \tag{3}$$

where L_i denotes the three orbital angular momentum operators. Given that the Hamiltonian for the Hydrogen atom has the form

$$H = \frac{\mathbf{p}^2}{2\mu} - \frac{e^2}{r}, \tag{4}$$

show that

a) $[M_i, H] = 0$.

b) $[M_i, L_j] = i\hbar\epsilon_{ijk} M_k$.

c) $[M_i, M_j] = -\frac{2i\hbar}{\mu}\epsilon_{ijk} H L_k$.

3. Find the lowest energy eigenstate (ground state) of the hydrogen atom in the coordinate representation starting from the operator formalism of $O(4)$ symmetry discussed in class.