## Astronomy 203 Problem Set #4: Solutions

## 19 October 1999

1. a. Show that the plate scale at the detector in Figure 1 is  $f_1/f_2$  times the plate scale at the telescope focus.

According to the thin lens equation,

$$\frac{1}{o_1} + \frac{1}{i_1} = \frac{1}{f_1} + \frac{1}{i_1} = \frac{1}{f_1} \quad , \tag{1}$$

the distance  $i_1$  to the image formed by the first lens is formally infinite, and the corresponding magnification  $m_1 = -i_1/f_1$  is zero. But wait: let's just let the first object distance *approach* the focal length  $f_1$  to keep things finite; then since the object distance for the next lens is  $o_2 = d - i_1 \cong -i_1$ , we get for the second image position

$$\frac{1}{o_2} + \frac{1}{i_2} = -\frac{1}{i_1} + \frac{1}{i_2} = \frac{1}{f_2} \implies i_2 = \frac{-i_1 f_2}{-i_1 - f_2} \cong f_2 \quad , \tag{2}$$

so the overall magnification is

$$m = m_1 m_2 = \frac{i_1}{f_1 - i_1} \frac{f_2}{f_1} = -\frac{f_2}{f_1} \quad . \tag{3}$$

Now let the angle between the axis and a point nearby on the sky be called  $\theta$ , and the distances to that off-axis spot in the focal plane be called  $y_1$  for the telescope focal plane and  $y_2$  for the final ("detector") focal plane. Then

$$PS_2 = \frac{d\theta}{dy_2} = \frac{d\theta}{dy_1} \frac{dy_1}{dy_2} = PS_1 \frac{1}{m} = -PS_1 \frac{f_1}{f_2} \quad . \tag{4}$$

(Q.E.D.)



Figure 1: collimator and camera lenses for reimaging; on-axis (solid lines) and off-axis (dashed lines) shown.

b. Suppose the two ray bundles in Figure 1 represent light from two point objects separated by a small angle  $\theta$  in the sky, and the effective focal length of the telescope is f. Show that the (small) angle between the two bundles of rays in the collimated portion of the beam is  $\theta' = (f/f_1)\theta$ .

In the terms used in part a, the telescope plate scale is  $PS_1 = d\theta/dy_1 = 1/f$ . The plate scale of the first lens is  $PS'_1 = d\theta'/dy_1 = 1/f_1$ . Thus if the ray incident on the telescope at angle  $\theta$  lands a distance  $y_1$  from the center of the focal plane, and comes out on the other side of the first lens at angle  $\theta'$ , then

$$\theta' = PS'_1 y_1 = PS'_1 \left(\frac{\theta}{PS_1}\right) = \frac{f}{f_1} \theta \quad .$$
(5)

2. The arrangement of the lenses in Figure 1, planoconvex with the curved surfaces facing the collimated light, was chosen on purpose, to minimize the SA of these lenses. Use RayTrace to demonstrate that this is true. Take the lens focal lengths to be 20 cm and 30 cm and to be separated by 50 cm. Plot spot diagrams at the detector focus, and measure the RMS spot size at the paraxial focus, for the lens shapes shown in the figure, for the lenses reversed (flat sides facing the collimated beams) and for the lenses replaced with equiconvex lenses of the same focal lengths.

You had to choose your own appropriate combination of index, curvature, and trace parameters here. I used distances in millimeters, took AP = 10 for all optical elements, n = 1.5 for the glass in the lenses, and 1000 rays per trace. RayTrace doesn't mind using zero thickness for the lenses, so that's what I told it to do; then the following thicknesses for the back surfaces of the lenses were 500 mm and 300 mm, and the object distance, set on the trace menu for Near Field, to be DI = 200. According to the thin lens equation and RayTrace's (different) sign convention, then, the curvature radii of the lenses need to be as follows:

	First lens front	First lens back	Second lens front	Second lens back
Original (as in Figure 1)	0	-100 mm	+150 mm	0
Lenses reversed	+100 mm	0	0	-150 mm
Equiconvex lenses	+200 mm	-200 mm	+300 mm	-300 mm

The results are shown in Figure 2. It turns out that the lens configuration of Figure 1 is better by a factor of about 4 than the lenses-reversed configuration, and by a factor of 1.4 than the equiconvex-lens arrangement, judging by the RMS (or even Sm\_RMS) spot sizes. Thus the lenses in collimator-camera pairs should be planoconvex, and have their curved sides facing collimated light.

- 3. The numerical inputs for this problem were taken from the example Pan-Cinor treated in Kingslake's Lens design fundamentals, pages 63-66.
  - a. The focal lengths of the lenses in Figure 3 are  $f_a$ ,  $f_b$ , and  $f_c$ , from left to right. Show that the back focal distance  $i_c$  is given for  $\Delta = 0$  by

$$i_{\rm c} = \frac{\left[dD - f_{\rm b}(D - d)\right]f_{\rm c}}{dD - f_{\rm b}(D - d) - f_{\rm c}D - f_{\rm b}f_{\rm c}}$$



Figure 2: results of raytraces for problem 2. All of the spot diagrams are plotted on the same scale, 0.004 mm per tic.

The object lies at  $o_a \rightarrow \infty$ , so  $i_a = f_a$ . This makes the next object virtual, as drawn. The object distance is  $o_b = -D$ , and the image distance and magnification are

$$i_{b} = \frac{o_{b}f_{b}}{o_{b} - f_{b}} = \frac{Df_{b}}{D - f_{b}} ,$$

$$m_{b} = -\frac{i_{b}}{o_{b}} = -\frac{f_{b}}{D - f_{b}} .$$
(6)

The third object distance is  $o_c = d - i_b$ , so the final image distance and magnification are

$$i_{c} = \frac{o_{c}f_{c}}{o_{c}-f_{c}} = \frac{(d-i_{b})f_{c}}{d-i_{b}-f_{c}} = \frac{\left(d - \frac{Df_{b}}{D-f_{b}}\right)f_{c}}{d - \frac{Df_{b}}{D-f_{b}} - f_{c}} = \frac{(dD - df_{b} - Df_{b})f_{c}}{dD - df_{b} - Df_{b} - Df_{c} + f_{b}f_{c}}$$

$$= \frac{\left[dD - f_{b}(D-d)\right]f_{c}}{dD - f_{b}(D-d) - f_{c}D - f_{b}f_{c}} , \qquad (7)$$

$$m_{c} = -\frac{i_{c}}{o_{c}} = -\frac{f_{c}}{d - i_{b} - f_{c}} = -\frac{f_{c}}{d - \frac{Df_{b}}{D-f_{b}} - f_{c}} = -\frac{(D - f_{b})f_{c}}{dD - f_{b}(D-d) - f_{c}D - f_{b}f_{c}} .$$

The first of these results is what we were asked to show. The second result, and the magnification by the second lens in Equation (6), will be useful in the next part of the problem.

b. Show similarly that the plate scale is



Figure 3: the Pan-Cinor zoom.

$$PS = -\frac{dD - f_{b}(D - d) - f_{c}D - f_{b}f_{c}}{f_{a}f_{b}f_{c}}$$

The plate scale of the first lens is  $PS_a = 1/f_a$ . The next two stages each change the plate scale by a factor equal to the reciprocal of their magnification (see Equation (4)), so we use the results from part a to obtain

$$PS = PS_a \frac{1}{m_b m_c} = \frac{1}{f_a} \frac{D - f_b}{f_b} \frac{dD - f_b (D - d) - f_c D - f_b f_c}{(D - f_b) f_c} = \frac{dD - f_b (D - d) - f_c D - f_b f_c}{f_a f_b f_c} \quad . \tag{8}$$

(Q.E.D.)

c. A certain zoom has  $f_a = 7.15959$  cm,  $f_b = -1.95959$  cm,  $f_c = 3.35410$  cm, D = 4.15959 cm and d = 1.69451 cm. Replace d and D by  $d + \Delta$  and  $D + \Delta$  in the equations above, and plot the image displacement,  $\delta(\Delta) = i_c(\Delta) + d + \Delta - [i_c(0) + d]$ , and the plate scale as a function of  $\Delta$  from -0.5 to 2.5 cm. Show thereby that the image is displaced by at most 0.068 cm, and that the plate scale increases by a factor of 3.0, as  $\Delta$  runs from zero to 2 cm.

The result is shown in Figure 4. At  $\Delta = 0$ , 1 and 2 cm the image displacement is  $\delta = 0$  — that is, it is in perfect focus — and in between it reaches extremes of 0.068 cm (at  $\Delta = 0.4$  cm) and -0.036 cm (at  $\Delta = 1.5$  cm). That is to say, the image doesn't move very much, and stays in pretty good focus as the lens is zoomed. Meanwhile the plate scale changes from 0.096 rad cm<sup>-1</sup> to 0.287 rad cm<sup>-1</sup>, so the image gets larger by a factor of 3.0.

Now you know a little about how a zoom lens works. To see how one goes about *designing* one, look at the very nice discussion of the thin-lens layout of optically-compensated zooms in Kingslake's book.

4. UR infrared astronomers frequently use their newest infrared camera, built by Profs. Bill Forrest and Judy Pipher, at the Wyoming Infrared Observatory. WIRO has a classical Cassegrain optimized for infrared performance. The primary mirror is a 2340 mm diameter paraboloid with focal length 4800.1 mm. The secondary has diameter 202.8 mm and focal lengths 431.7 mm and 5380.0 mm. Its edge comprises the aperture



Figure 4: image displacement  $\delta(\Delta)$  and plate scale of the Pan-Cinor, as functions of the displacement  $\Delta$  of the first and third lenses.

stop. 269.5 mm past the Cassegrain focus there is an achromatic doublet lens with focal length 76.37 mm. The detector array sits at the final focus.

a. *Calculate the outer diameter and position of the entrance pupil.* 

The entrance pupil is the image of the secondary seen through the primary:

$$i = \frac{of}{o - f} = \frac{(4800.1 - 431.7)4800.1}{4800.1 - 431.7 - 4800.1} \text{ mm} = -48572.5 \text{ mm} ,$$

$$m = -\frac{i}{o} = -\frac{f}{o - f} = -\frac{4800.1}{-431.7} = 11.1 .$$
(9)

Thus we have a *virtual* entrance pupil, with diameter 11.1 times that of the secondary (2255 mm), lying 48573 mm behind the primary.

b. Calculate the outer diameter and position of the exit pupil. Is it real or virtual? Would it make a good Lyot stop?

The exit pupil is the image of the secondary seen through the camera's lens:

$$i = \frac{of}{o-f} = \frac{(5380 + 269.5)76.37}{5380 + 269.5 - 76.37} \text{ mm} = 77.42 \text{ mm} ,$$

$$m = -\frac{i}{o} = -\frac{f}{o-f} = -\frac{76.37}{5380 + 269.5 - 76.37} = -0.0137 .$$
(10)

This is a real image; its diameter is 0.0137 that of the secondary, or 2.8 mm. It would make a good Lyot stop, and you will indeed find one in its position in the camera.