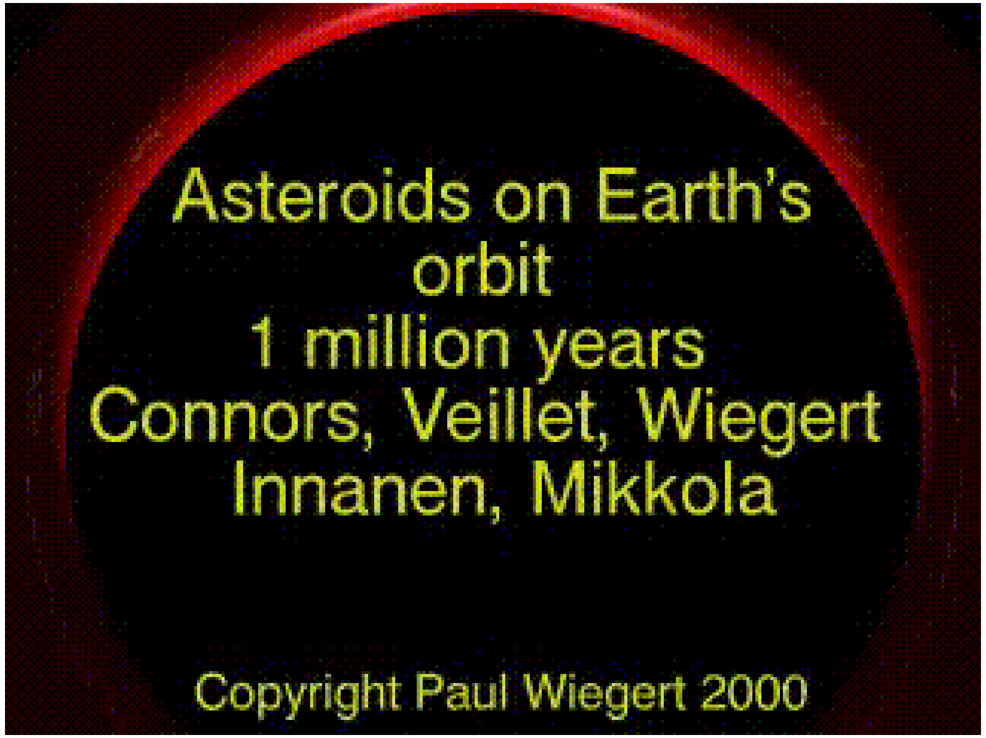
Today in Astronomy 111: Kepler's Laws, perturbations and orbital resonances

- Kepler's Laws
- More than two gravitating bodies: perturbations by planets on each other's orbits
- Mean motion resonances
- Asteroids and their orbits
- Asteroids as a chief example of the effect of perturbations on orbital stability and instability



Simulation of perturbation by the inner planets of the orbits of asteroids initially spread evenly along the Earth's orbit, by Paul Wiegert (U. W. Ontario) and colleagues.

Kepler's Laws

Before Newton discovered the laws of motion and gravity, Johannes Kepler noticed the following facts about planetary orbits, which had been measured with great accuracy by his mentor Tycho Brahe:

1. Each planet follows an elliptical orbit, with the Sun at one focus.

2. The line between the Sun and each planet sweeps out equal areas in equal times, no matter where the planet is, along its orbit.

3. The square of a planet's orbital period is proportional to the cube of its orbital semimajor axis.

All these empirical facts can be derived from Newton's laws.

- That is, Newton's theories are validated by Tycho's observations, as Newton himself noted.

  To wit:

Kepler's Laws (continued)

- Kepler's first law: we showed, last class, that elliptical orbits are consistent with gravity and the laws of motion, and the relationship between the parameters of the ellipse and the conserved mechanical quantities.

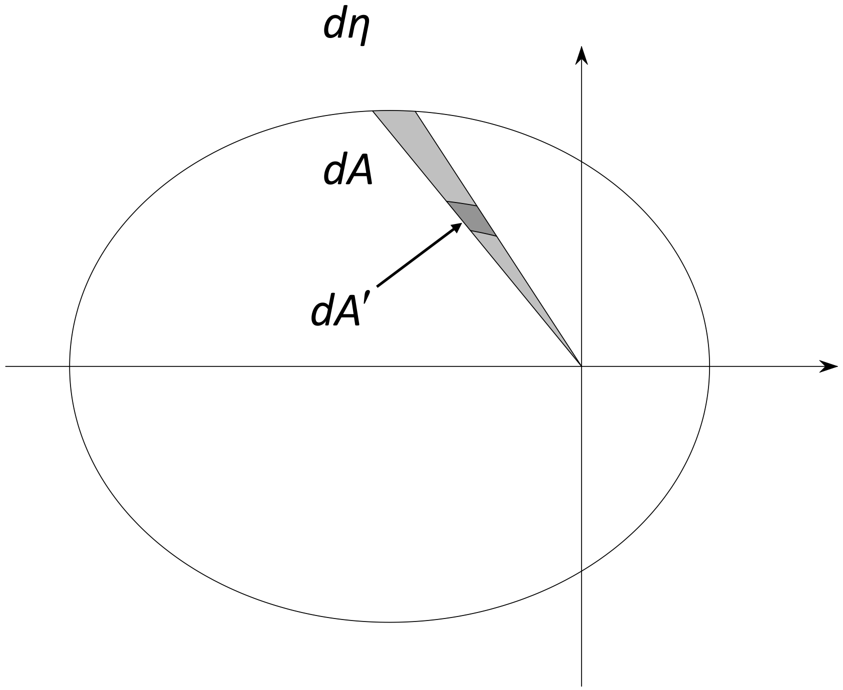
- Kepler's second law (equal areas):

$$dA' = dr'\left(r'd\eta\right)$$

$$dA = d\eta\int_0^r r'dr' = \frac{r^2}{2}d\eta$$

$$\frac{dA}{dt} = \frac{r^2}{2}\frac{d\eta}{dt} = \frac{1}{2}rv_\eta$$

$$= \frac{h}{2} = \boxed{\frac{1}{2}\sqrt{GMa\left(1-\varepsilon^2\right)} = \text{constant},}$$ since h is constant.



Kepler’s Laws (continued)

- Kepler’s third law (period and semimajor axis):

$$\frac{dA}{dt}=\frac{h}{2}$$

Integrate over one orbital period:

$$\int_0^A dA' = \frac{h}{2}\int_0^P dt$$

$$A=\frac{h}{2}P=\pi ab$$

$$P^2=\frac{4\pi^2a^2b^2}{h^2}=\frac{4\pi^2a^4\left(1-\varepsilon^2\right)}{GMa\left(1-\varepsilon^2\right)}=\boxed{\frac{4\pi^2}{GM}a^3}\ .$$

Tycho and Kepler, Newton and Copernicus

- Thus the Kepler/Tycho results are consistent with the predictions of Newtonian dynamics, and can be regarded as the first experimental (pre-)validation of Newton's theories.

- They also supply a **crucial missing piece** of the Copernican model of the solar system:

  - Copernicus hypothesized circular orbits for the planets. Measurements like Tycho's were accurate enough to rule out circular heliocentric orbits.

  - And the Ptolemaic geocentric theory, with ~15 epicycles per planet, was thus in much better agreement with the observations.

    - Copernicus had only one free parameter per planet: the orbital radius.

    - A theory with more free parameters can always fit experimental results better than one with fewer free parameters.

  - So, as neither theory had an explanation for why anything was orbiting anything else, a contemporary of Copernicus's could **maintain** – not just **assert** – that the Ptolemaic theory was better, as it gave better agreement with observations.

Elliptical orbits and center of mass

As in the case of circular orbits, allowing the two masses to be finite in ratio changes the equations for elliptical orbits very little, if one just introduces the center of mass and reduced mass.

- The results for positions, velocities and conserved quantities:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad , \quad \boldsymbol{r}_1 = -\frac{\mu}{m_1}\boldsymbol{r} \quad , \quad \boldsymbol{r}_2 = \frac{\mu}{m_2}\boldsymbol{r} \quad ,$$

$$\boldsymbol{v}_1 = -\frac{\mu}{m_1}\boldsymbol{v} \quad , \quad \boldsymbol{v}_2 = \frac{\mu}{m_2}\boldsymbol{v} \quad ,$$

$$E = \frac{1}{2}\mu v^2 - \frac{G\mu\left(m_1 + m_2\right)}{r} = -\frac{G\mu\left(m_1 + m_2\right)}{2a} \quad ,$$

$$L = \mu\sqrt{G\left(m_1 + m_2\right)a\left(1 - \varepsilon^2\right)} \quad .$$

Elliptical orbits and center of mass (continued)

- And for a few other useful relations:

$$\frac{dA}{dt} = \frac{h}{2} = \frac{1}{2}\sqrt{G\left(m_1 + m_2\right)a\left(1 - \varepsilon^2\right)}$$

$$P^2 = \frac{4\pi^2}{G\left(m_1 + m_2\right)}a^3$$

$$v^2 = G\left(m_2 + m_1\right)\left(\frac{2}{r} - \frac{1}{a}\right)$$

Recall: $P = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$ for circular orbits.

Perturbation of orbits by other planets

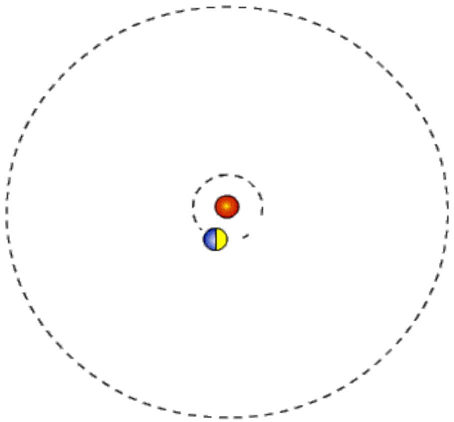
In the **two-body** problem considered hitherto, each body is subjected to a purely $1/r^2$ force centered at the other body.

- When there is any deviation from the perfect inverse square law, orbits are no longer perfect ellipses.
- Such departures would be produced, for instance, by additional bodies in the system, or by nonspherical symmetry of the mass density in one or other of the bodies.
- In our solar system these effects are small enough that orbits are usually very close to elliptical. We call the small additional effects **perturbations.**
- The main effect of perturbations is that the orbits don't exactly close: each successive orbit is very slightly different from the last.

Perturbation of orbits by other planets (cont'd)

- Because of perturbation and slight departure from closed elliptical orbits, the **orientations** of planetary orbits slowly change. We call this effect **perihelion precession**.
  - Nice pedagogical treatment accessible to ≥ juniors: [Berche & Medina 2024].
- The best example is Mercury, with a perihelion precession of 5600 arcsec/century, first explained up to the very last step by [Le Verrier (1845)].

Perihelion precession



Perihelion advance, arcsec century^{-1}	**Origin**
5025.65 ± 0.50	Precession of the equinoctes
277.86 ± 0.68	Perturbation by Venus
153.58 ± 0.00	Perturbation by Jupiter
90.04 ± 0.08	Perturbation by Earth
7.30 ± 0.01	Perturbation by Saturn
2.73 ± 0.02	Perturbation by the other planets and solar asphericity
5557.18 ± 0.85	Total, perturbations and equinox precession
5599.74 ± 0.41	Observations
42.56 ± 0.94	Difference
42.98 ± 0.03	GR spacetime warping (e.g. [Einstein 1915])

The many-body problem

The two-body problem is of a class mathematical physicists call **completely integrable**. That means that there is **one conserved quantity per degree of freedom**.

- For point masses: $t \leftrightarrow E$

  $x, y, z \leftrightarrow p_x = mv_x, p_y, p_z.$

- Unique solutions can be obtained for completely integrable systems, so orbits can be predicted exactly for the two-body system.

- For more than two bodies, even 3, **there is no general analytical solution**.

- Even the **restricted three-body problem** (two massive, one massless particle) is complicated, as we shall see.

- Among the unpredictable or hard-to-predict consequences:

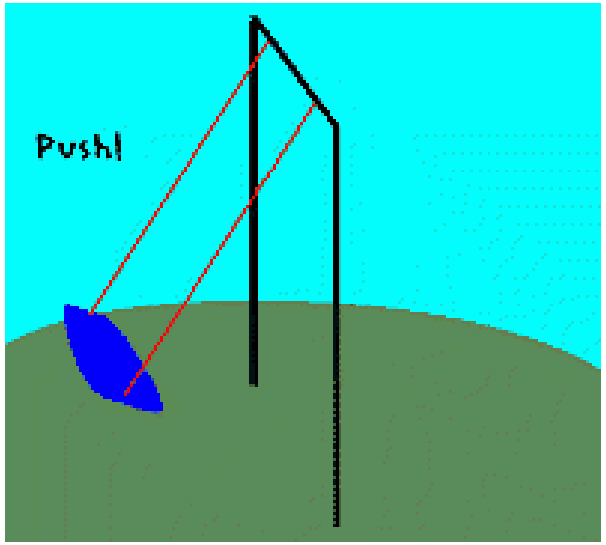
  - Cumulative effects of perturbations (**resonances**)

  - Non-deterministic solutions ("chaos")

Orbital resonances

- Periodic perturbations on an object, such as the nearby passage of a planet, will add up if they are **in phase** with the orbital motion of the object.

- If there are out of phase, then the perturbations will cancel on average and there will be no net change in the orbit of the object.



Orbital resonances (continued)

- Even small perturbations can be important:
  - The solar system is about 4.6 billion years old.
  - This means that everything in the inner Solar system has had more than a billion orbits.
  - A billion small perturbations, if they add **constructively** (i.e. mostly in phase), can have large effects on the orbit of a small planet or asteroid.
- The leading cause in our solar system of small perturbations in phase with orbital motion are the **mean-motion resonances** of three-body systems.

Mean-motion resonances

Consider the three-body system consisting of the Sun, a particle (large or small), and a perturbing planet.

- When do perturbations from the planet on the particle add constructively? That is, how could they be in phase with the orbital motion?
  - One way is for an integer times the orbital period of the planet is equal to an integer times the orbital period of the particle.
  - Define the **mean motion** to be the average value of orbital angular velocity ω during a period:

$$\bar{\omega} \equiv \frac{2\pi}{P} = \sqrt{\frac{GM}{a^3}}$$

Here, $M = M_\odot$, since it's our Solar system we're discussing.

Mean-motion resonances (continued)

- If the mean motion of the particle, ϖ_p, and that of the perturbing planet, ϖ_q, are related by

$$i\varpi_p \cong j\varpi_q \quad ,$$

or equivalently $$jP_p \cong iP_q \quad ,$$

Note that we don't need *exact* equality for there to be a big effect over many orbits.

  - where *i* and *j* are integers,
- then any possible orientation of the three bodies will recur
  - every *j* orbits of the particle
  - and every *i* orbits of the perturber.
- If any of those orientations comprise a perturbation, then it is manifestly in phase with the orbit. We would call this one the ***"i:j* resonance."**

Mean-motion resonances (continued)

- Recall that

$$P = \frac{2\pi a^{3/2}}{\sqrt{GM}}$$

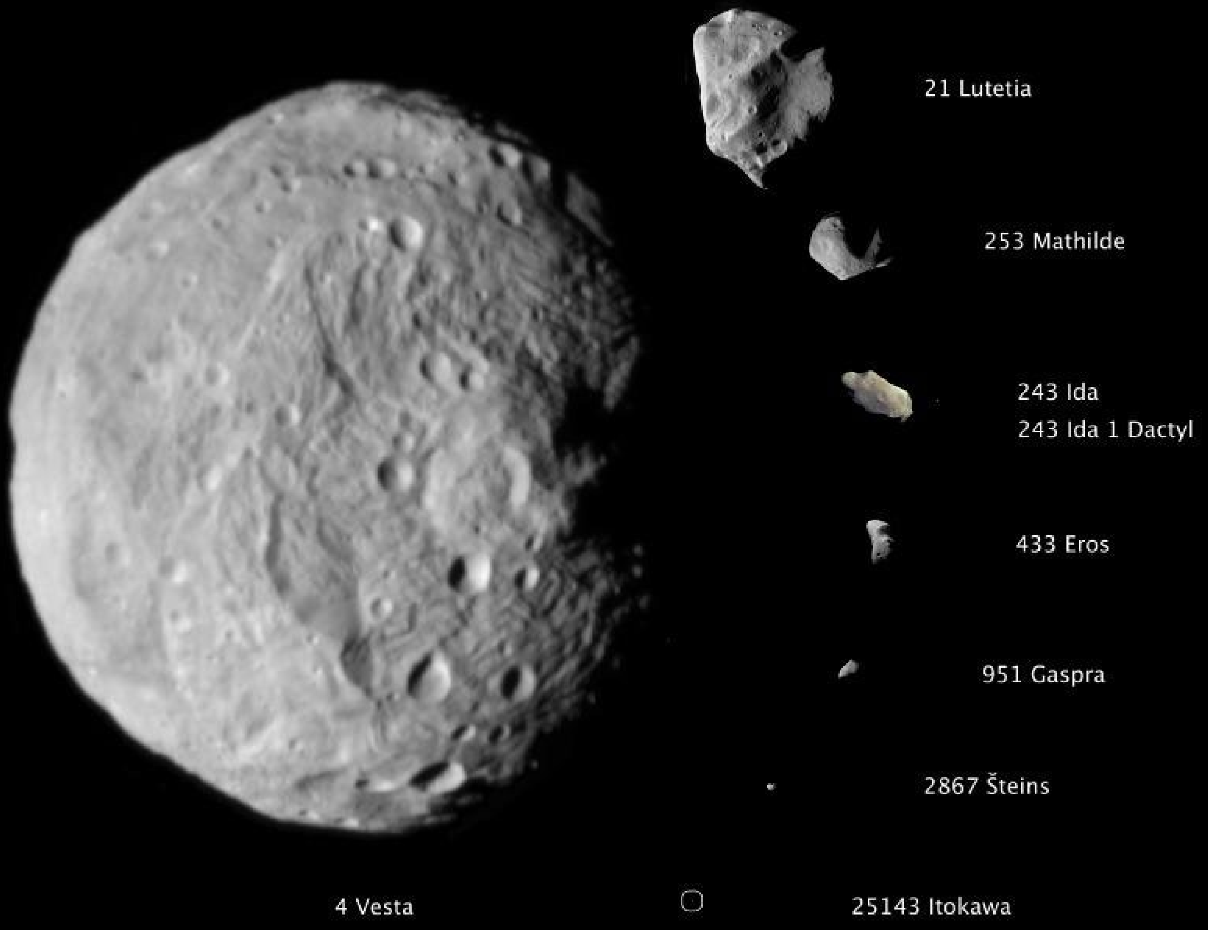
(Kepler's third law), so the semimajor axis ratio for a resonance is

$$\boxed{\left(\frac{a_p}{a_q}\right) = \left(\frac{P_p}{P_q}\right)^{2/3} \approx \left(\frac{i}{j}\right)^{2/3} .}$$

- The largest convenient collection of small particles in orbit around the Sun, on which to see the effect of planetary perturbations, is the **asteroids,** a collection of small rocky bodies lying mostly between the orbits of Mars and Jupiter.

Vital statistics of main-belt asteroids

Total mass	~2.3×10^{24} gm $\left(3.9\times10^{-4}M_\oplus\right)$
Number	~200,000 with diameter $\geq$ 1 km
Mass range	$10^{14}-10^{24}$ gm
Diameter range	1-1000 km
Density range	$0.5-7$ gm cm^{-3}
Orbits:	
Major axis range	$1.7-3.7$ AU
Eccentricity range	$0.05-0.9$
Inclination range	$0-30°$
Typical separation	10^7 km (0.07 AU)



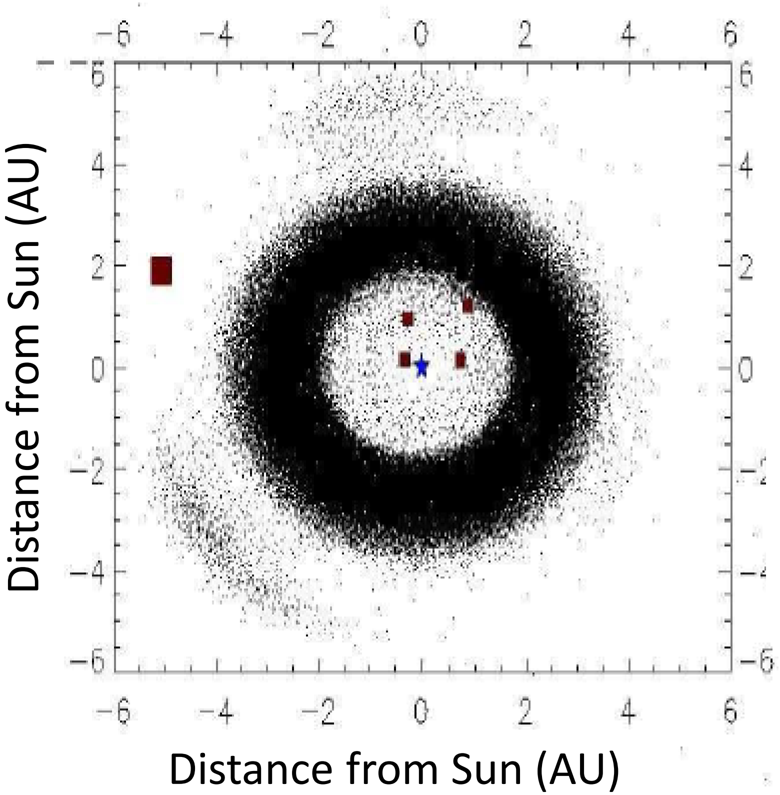
Eight asteroids seen at close range, shown on a common scale (*Dawn*/JPL/NASA)

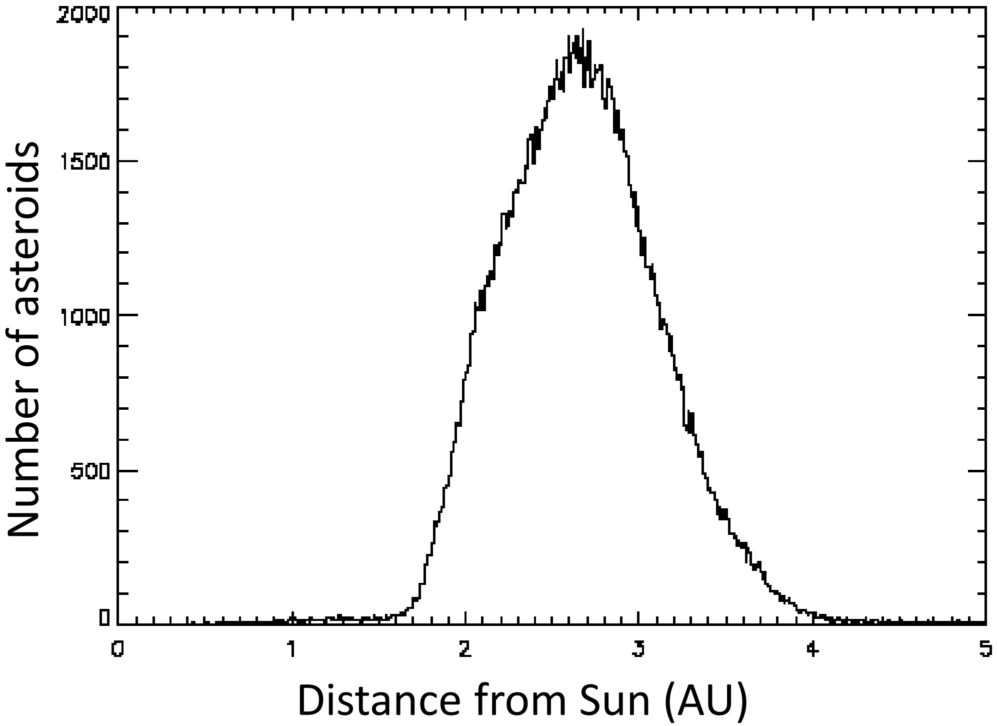
Visits to the asteroids

- 16 asteroids so far, in 13 missions. (67P/C-G is a comet.)
- One successful sample return; another on its way home.
- Five more missions on the way: 3 NASA, one JAXA, one CNSA, one ESA.
- Next is NASA's Lucy: launched in 2021, visiting six Trojan asteroids 2025-2033.
- Then NASA's Psyche, to visit that asteroid in October 2029.

Mission	Launch	Visited (flyby unless otherwise indicated)
Galileo (NASA)	1989	951 Gaspra, 243 Ida (+ Dactyl, the first asteroidal moon)
NEAR (NASA)	1996	253 Mathilde (flyby 1997), 433 Eros (orbit 2000). Laid to rest on Eros at end of mission but survived impact and continued to communicate.
Cassini (NASA)	1997	2685 Masursky
Deep Space 1 (NASA)	1998	9969 Braille
Stardust (NASA)	1999	5535 Annefrank
Hayabusa (JAXA)	2003	25143 Itokawa – orbiter, lander, sample return but the sampler didn't work
Rosetta (ESA)	2004	2867 Steins (2008), 21 Lutetia (2010), 67P/Churyumov-Gerasimenko (2014-2016); laid to rest on 67P/C-G.
Philae (ESA)	2004	Rosetta's lander, arrived 2014 on 67P/C-G.
Dawn (NASA)	2007	4 Vesta (2011-2012), 1 Ceres (2015) – orbited both, still in orbit about Ceres.
Chang'e (CNSA)	2010	4179 Toutatis
Hayabusa 2 (JAXA)	2014	Orbiter/lander landed twice on 162173 Ryugu (2018,2019) and brought samples back to Earth (2020). Deployed three additional landers, one impactor, and another orbiter to observe the impact (2019).
OSIRIS-REx (NASA)	2016	101855 Bennu, orbit 2018-, sample return to Earth Sunday, 24 September 2023. Then on to 99942 Apophis (2029).
DART (NASA)/LICIACube (ASI)	2021	65803 Didymos, or more particularly its moon Dimorphos. Impact at Δv = 6.6 km/sec on 26 September 2022.

The main asteroid belt

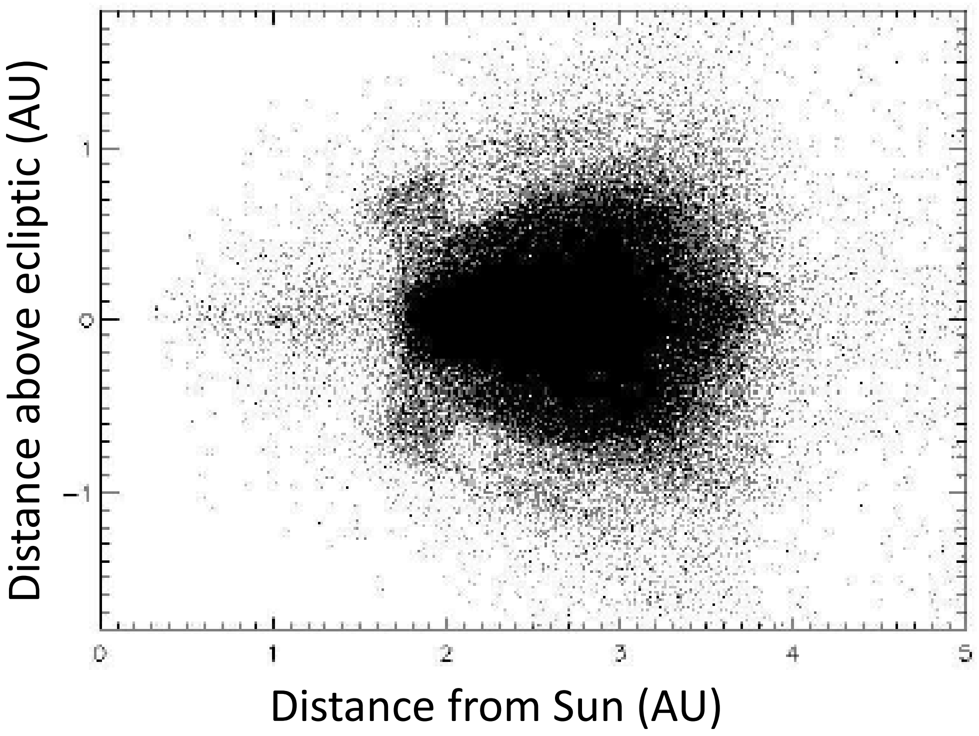


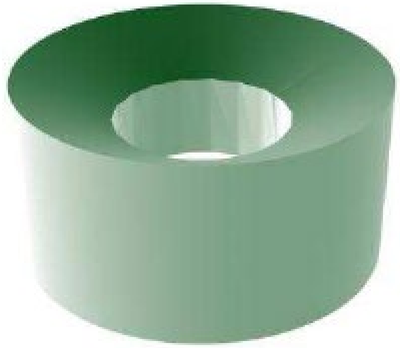


Snapshot of distribution of main-belt asteroids, as would be viewed from the ecliptic pole (Bidstrup et al. 2005).

- The Sun’s position is labelled with a blue star, and the five inner planets by red blocks.

The main asteroid belt (continued)





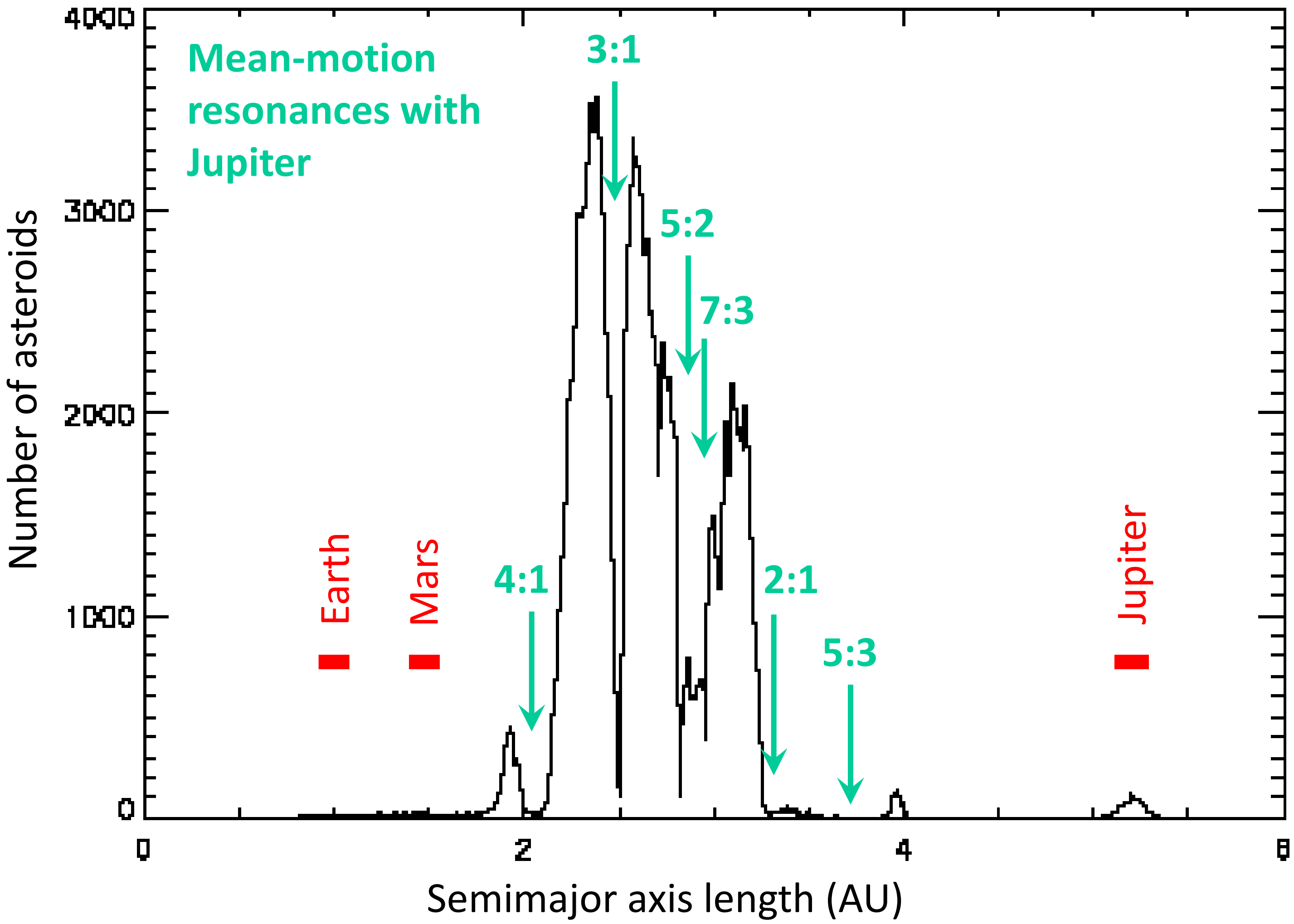
Snapshot of distribution of main-belt asteroids above and below the ecliptic (left), and perspective sketch of volume occupied by the main belt (right) (Bidstrup et al. 2005).

Resonances resulting in instability

Within the asteroid belt, particles often change orbits due to gravitation of other particles they encounter.

- Most particles that are knocked thereby into **strong mean motion resonances with Jupiter** have their orbits slowly increase in eccentricity until they cross the orbit of a planet.
  - We'll discuss next class why these encounters increase an asteroid's orbital eccentricity, under the rubric of Hohmann transfer orbits.
- An encounter with that planet will eventually eject the particle from the inner Solar system.
- This process is evident in the distribution of semimajor axes among the asteroids: the number of particles in the Jovian mean-motion resonances is very small.
  - These sharp dips in the distribution, discovered in 1866, are called the **Kirkwood gaps** (see Kirkwood 1887).

Resonances resulting in instability (continued)



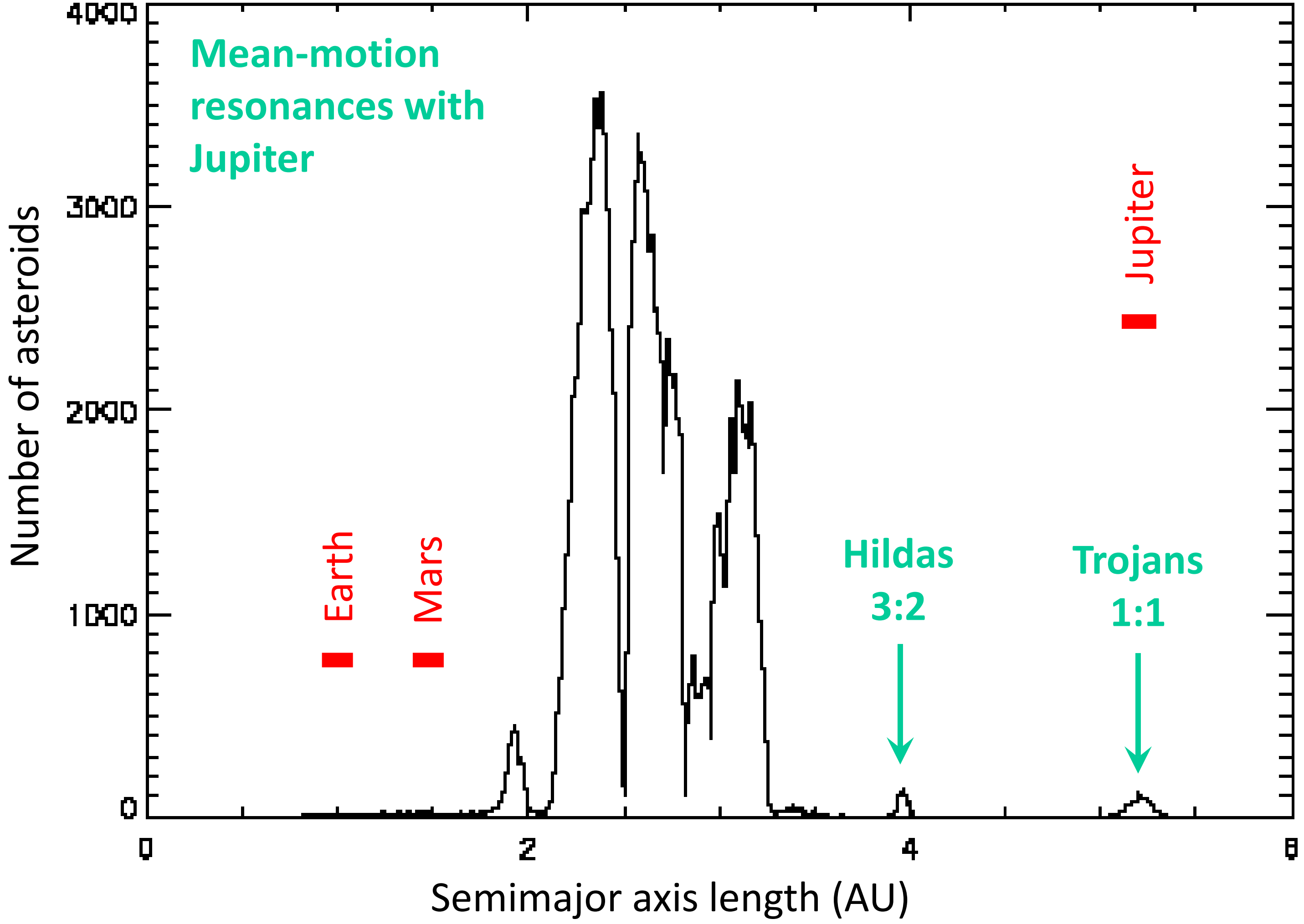
The Kirkwood gaps
([Bidstrup *et al.* 2005](#))

Resonances resulting in stability

But *some* locations in some of the mean-motion resonances offer protection against being scattered away by a planet.

- Suppose, for instance, that an encounter resulted in an asteroid in an orbit just like Jupiter's, but placed opposite the Sun from Jupiter.
    - Then it stays away from Jupiter, and out of the greatest scattering danger.
- The Hildas, Thule, and Trojan groups are examples of this effect:
    - they lie in Jovian orbital resonances, but keep well away from Jupiter all the time.
- We will study this in more detail on Thursday, under the rubric of the restricted three-body problem.

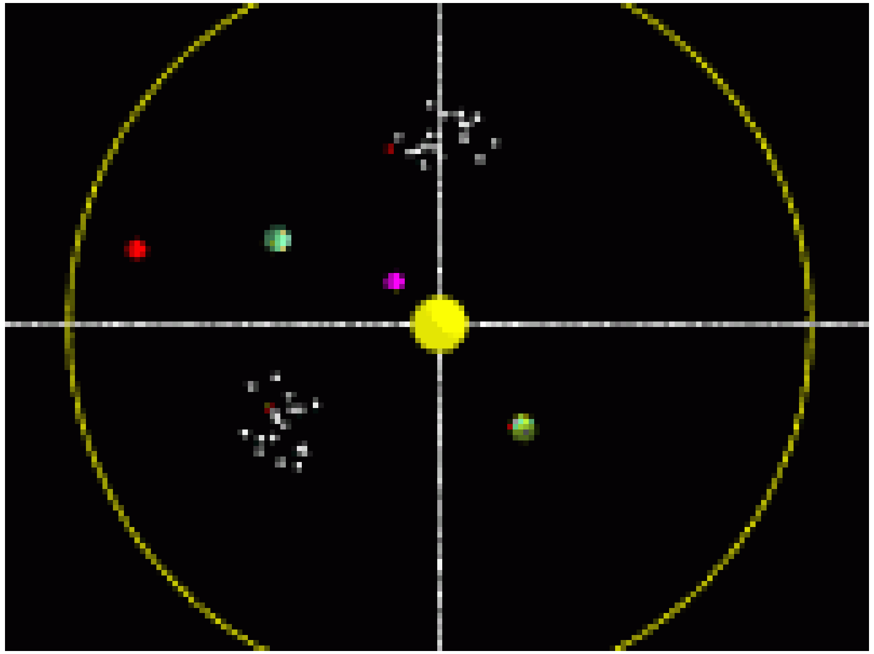
Resonances resulting in stability (continued)



[Bidstrup *et al.* 2005]

Resonances resulting in stability (continued)

- Probably the best known of these stable resonances is the set inhabited by Jupiter's Trojan asteroids, which will be visited by the NASA Lucy mission.
- Trojans are in a pair of 1:1 resonances that we will learn to call the fourth and fifth Lagrange points, L4 and L5.
- All the Solar system's planets have such resonances. Most potentially can trap asteroids in them.
  - Two Earth Trojans have been found so far, 2010 TK_7 and 2020 XL_5.
  - Both at L4, ahead of Earth in orbit.



Simulation of hypothetical Earth Trojan asteroids, by Paul Wiegert (U. W. Ontario) and colleagues.