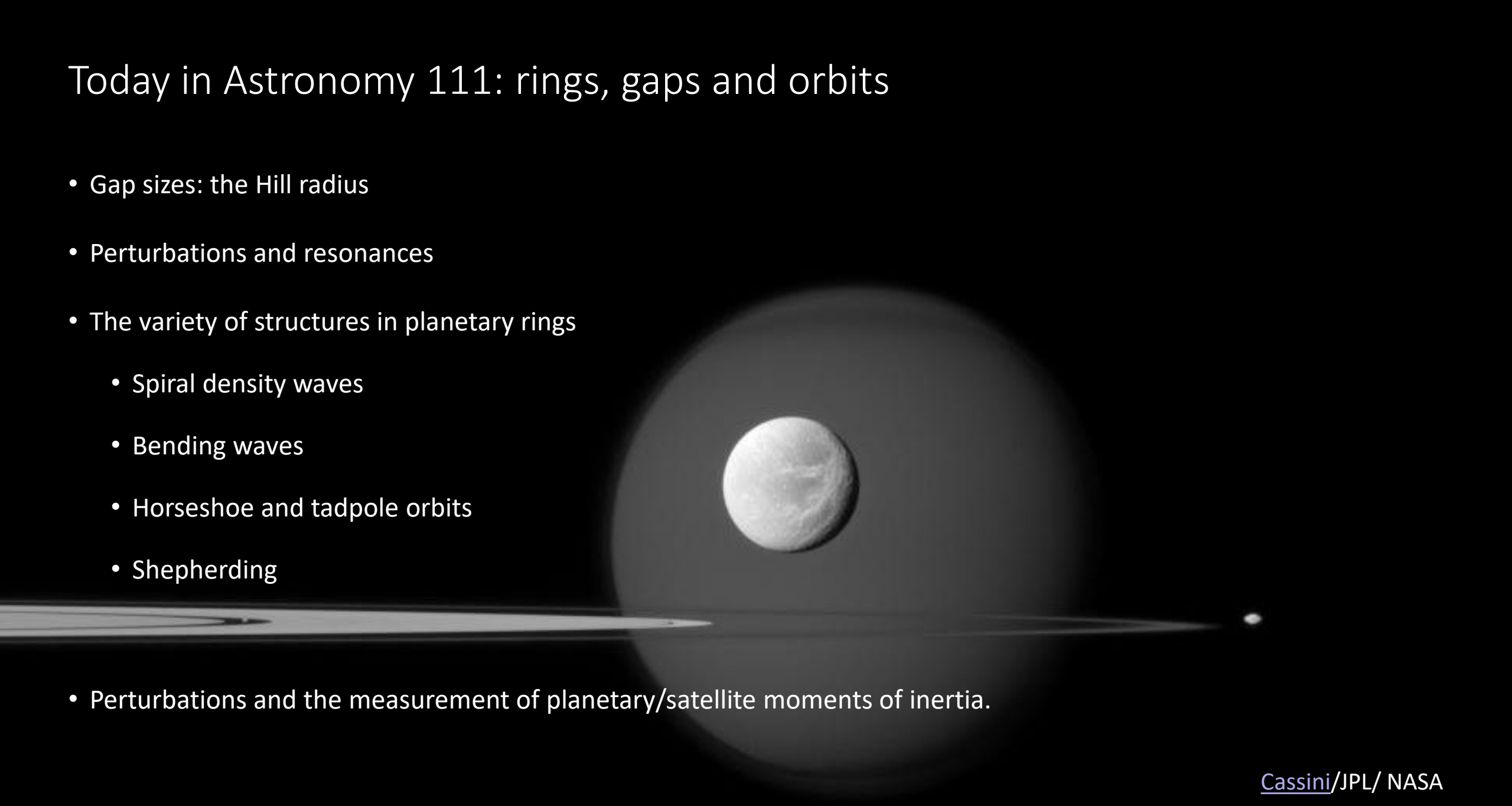
Today in Astronomy 111: rings, gaps and orbits

- Gap sizes: the Hill radius
- Perturbations and resonances
- The variety of structures in planetary rings
  - Spiral density waves
  - Bending waves
  - Horseshoe and tadpole orbits
  - Shepherding
- Perturbations and the measurement of planetary/satellite moments of inertia.



Cassini/JPL/ NASA

The Hill radius

Next, to explain with tides, are the gaps in rings. For what distance around a moon's orbit does its perturbations clear out a gap?

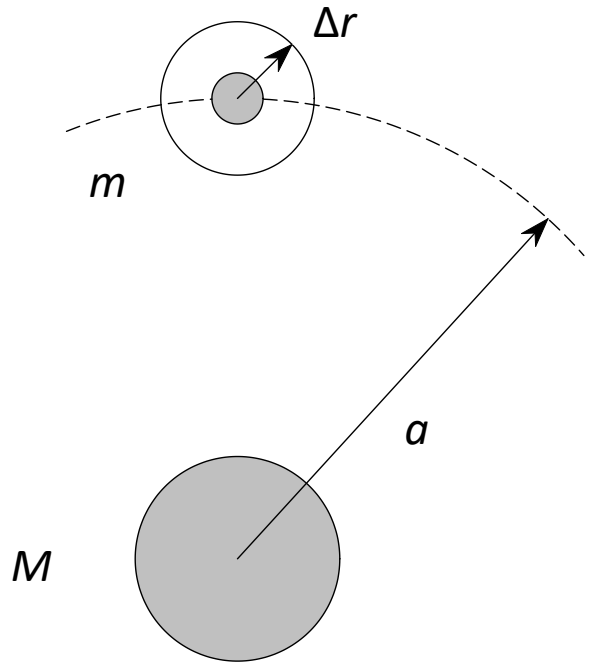
- We can judge this by comparison of the planetary tides in the moon's vicinity with the moon's gravitational acceleration.

- As before with the Roche limit, acceleration due to the planet is

$$g_{eff} = -\frac{GM}{r^2} + \frac{GM}{a^3} r$$

- At the location of the satellite, the tidal acceleration in the radial direction, at distance Δr from the moon, is approximately

$$g_p \cong \Delta r \frac{dg_{eff}}{dr}(a) = \left(\frac{2GM}{a^3} + \frac{GM}{a^3} \right) \Delta r = \frac{3GM}{a^3} \Delta r$$



The Hill radius (continued)

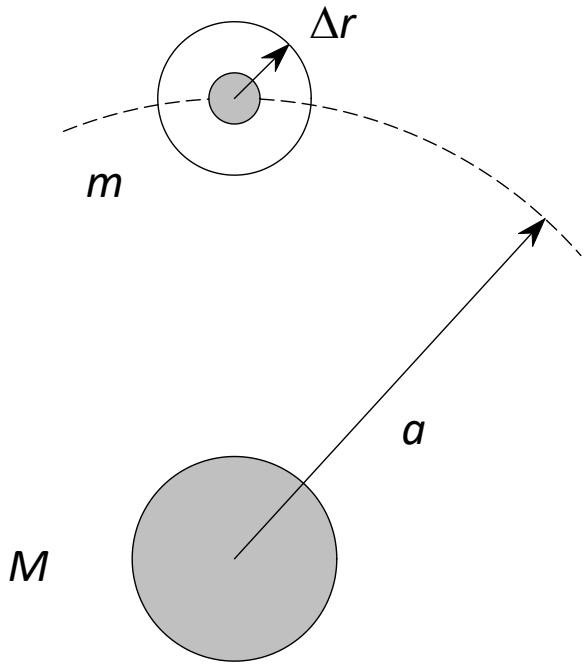
- Along the radial direction from the parent body through the moon's center, the magnitude of gravitational acceleration due to the moon is

$$g_m = Gm/\Delta r^2$$

- Set $g_m = g_p$, and a special radius $\Delta r = r_H$ is obtained:

$$\frac{3GM}{a^3}\Delta r = \frac{Gm}{\Delta r^2} \Rightarrow \boxed{\Delta r = a\left(\frac{m}{3M}\right)^{1/3} \equiv r_H.}$$

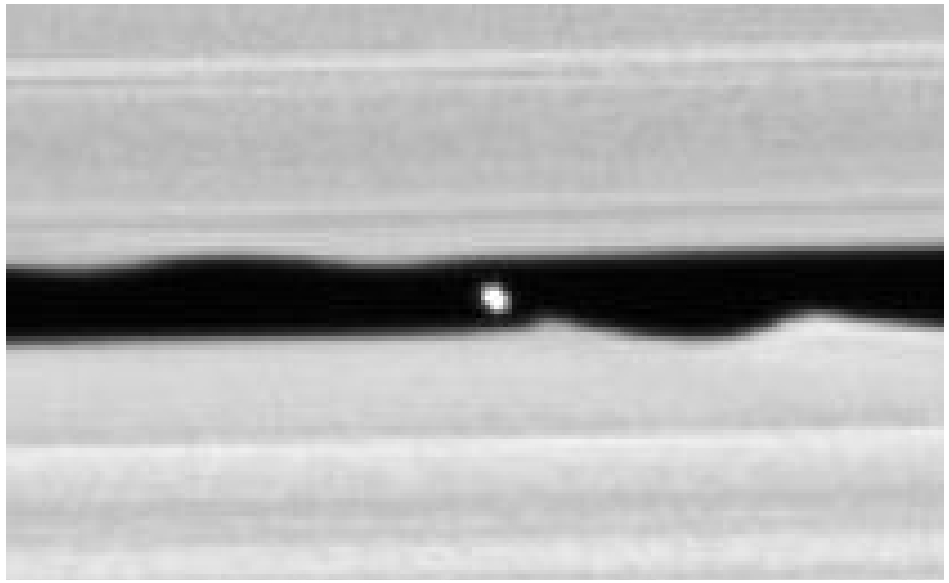
Hill radius



- As the moon orbits the planet, its Hill radius traces out a torus within which ring particles will be kicked out by the moon's perturbations: a gap surrounds the moon's orbit.

The Hill radius (continued)

- Example: Take a spherical body with diameter 7 km and density 1 gm cm^{-3}, …
  - Made of ice, like Saturn's ring particles; same size as Daphnis.
- … r = 136530 km away from Saturn.
  - Radius of the Keeler gap.
- This gives $\boldsymbol{r_H}$ **= 6.5 km**, so this moon should perturb the orbits of smaller ones very strongly within this distance of its orbit, and clear a gap about 13 km wide, centered on the moon's orbit.
- Which is just about right.



Cassini image of the Keeler gap and Daphnis; JPL/NASA.

Orbital perturbations

And now, to generate the complex but regular structure in rings, we list the different kinds of orbital perturbations.

- As we have learned, in the simple case of transfer orbits, exerting forces or impulses (momentum changes) on satellites changes their orbits in straightforwardly-predictable ways.

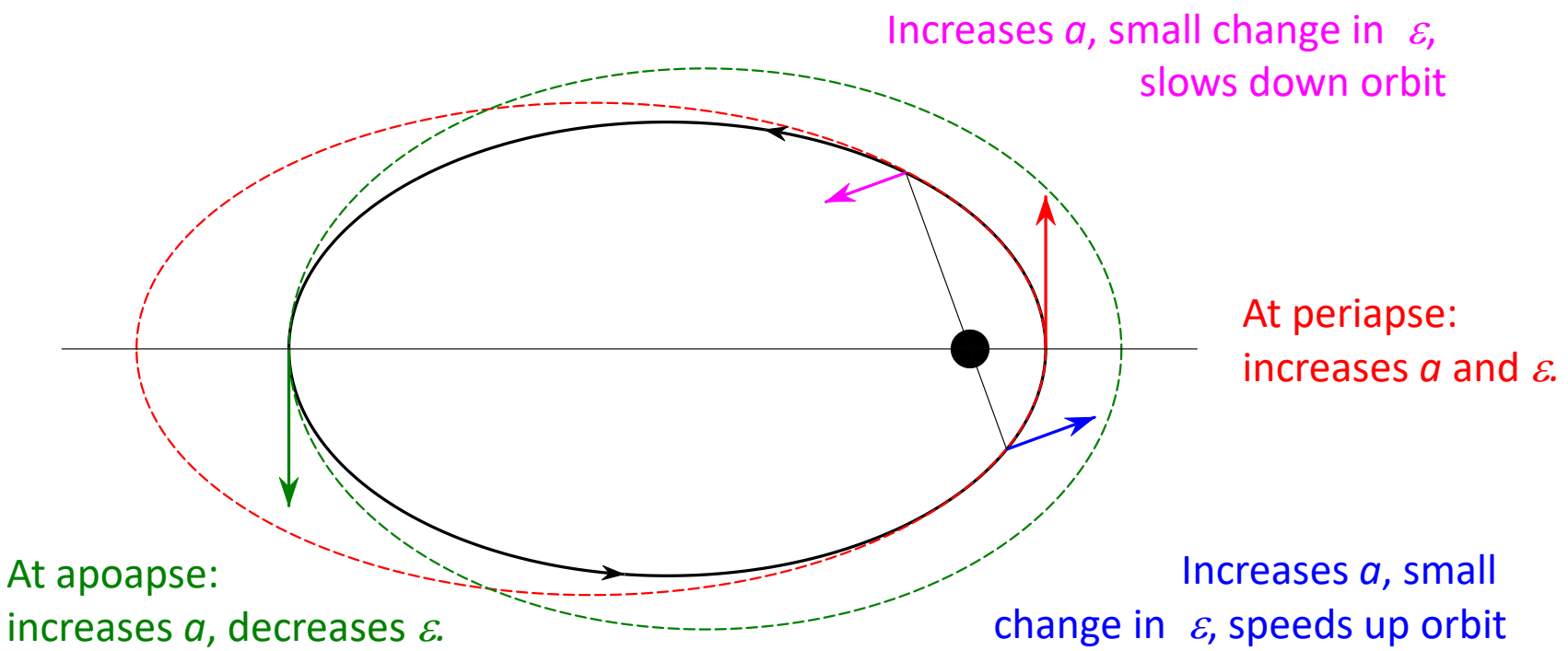
- Of course, straightforward here does not mean simple. Depending upon where in the orbit an impulse is applied, and in what direction, various of the orbital parameters can be changed, and in general all change.

- The form of the equations of motion, though, can be used to identify points at which the impulse affects just one or two of the orbital parameters.

- See table 11.3 (page 225) in the supplementary textbook by Consolmagno & Schaefer (CS) for the formulas in full regalia, but we won't be using the formulas.

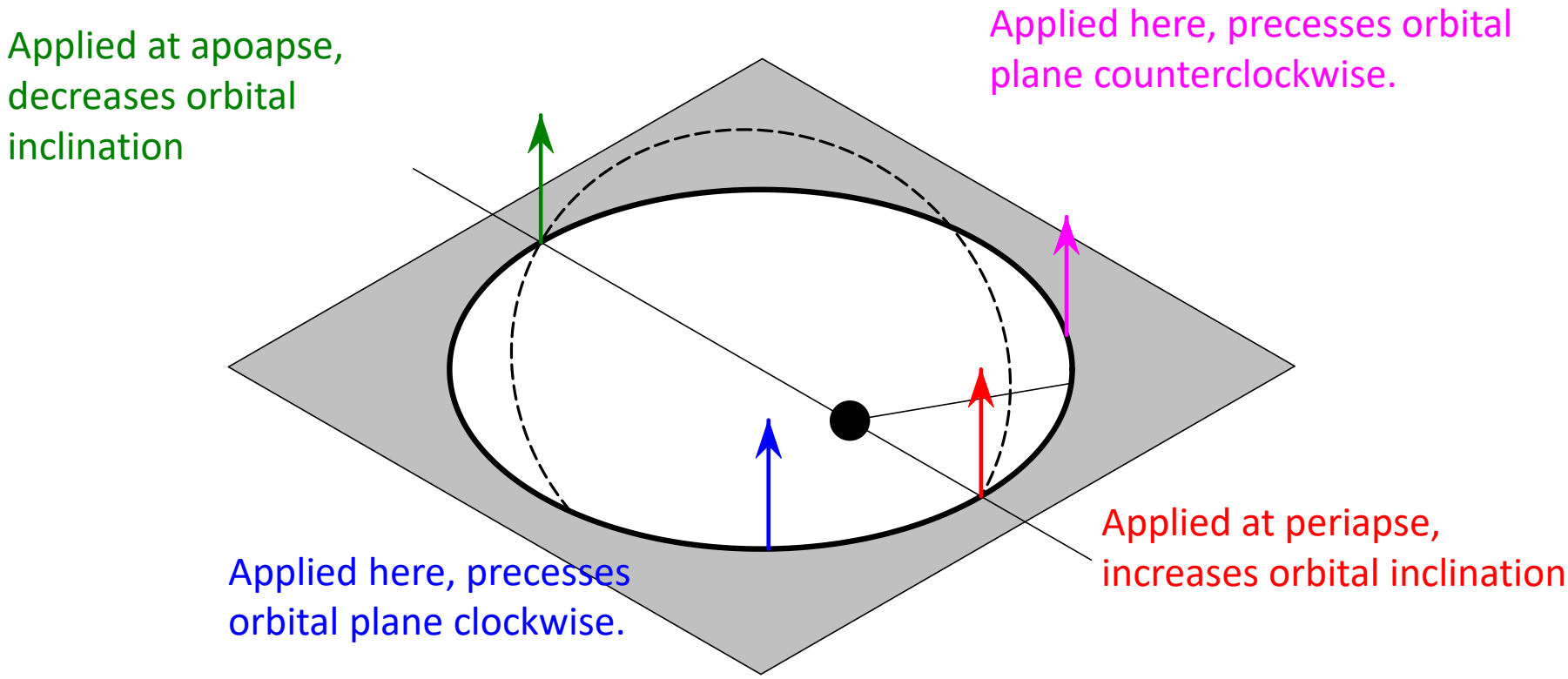
A guide to perturbations

- The only perturbative impulses we have considered hitherto have been **transverse** accelerations: those in the plane of the orbit, perpendicular to the radius *r*.



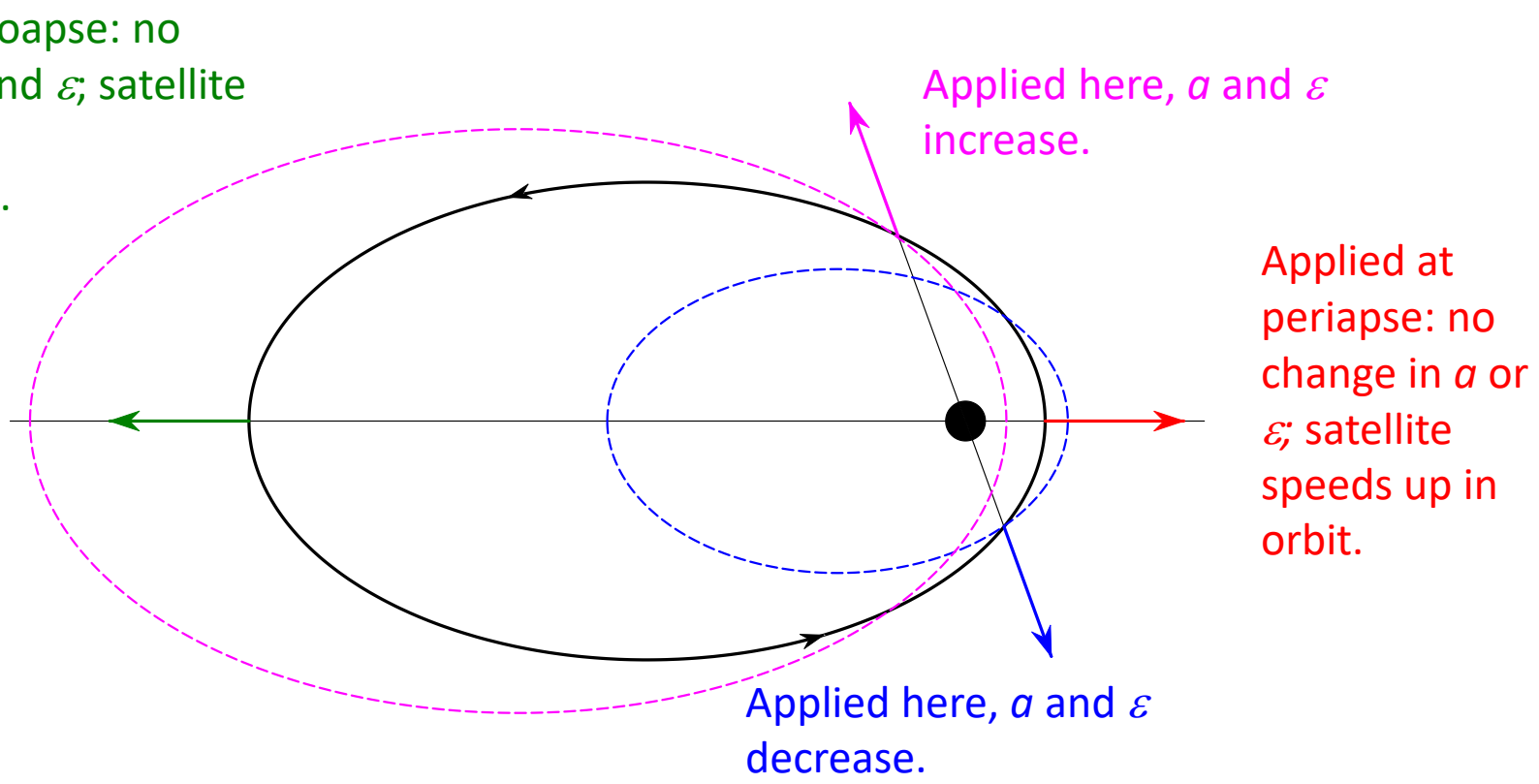
A guide to perturbations (continued)

- **Normal** accelerations (i.e. perpendicular to the initial orbital plane) change the inclination and precession rate.



A guide to perturbations (continued)

- **Radial accelerations** are rather counterintuitive around periapse and apoapse.



Orbital resonances and waves

If perturbations take place periodically – i.e. at regular intervals – then their effects can build over time as a series of them can add constructively.

- Periodic normal perturbations can drive vertical oscillations in the orbits of small bodies: **simple harmonic motion**, with the gravity of the disk of ring particles, and the vertical component of the planet's gravity, acting as the restoring force.

- Radial perturbations can drive radial oscillations, also a simple harmonic motion.

    - Combined with revolution, radial oscillations cause bodies to appear to travel in circles with their centers in uniform Keplerian motion, like **epicycles** in the Ptolemaic solar system.

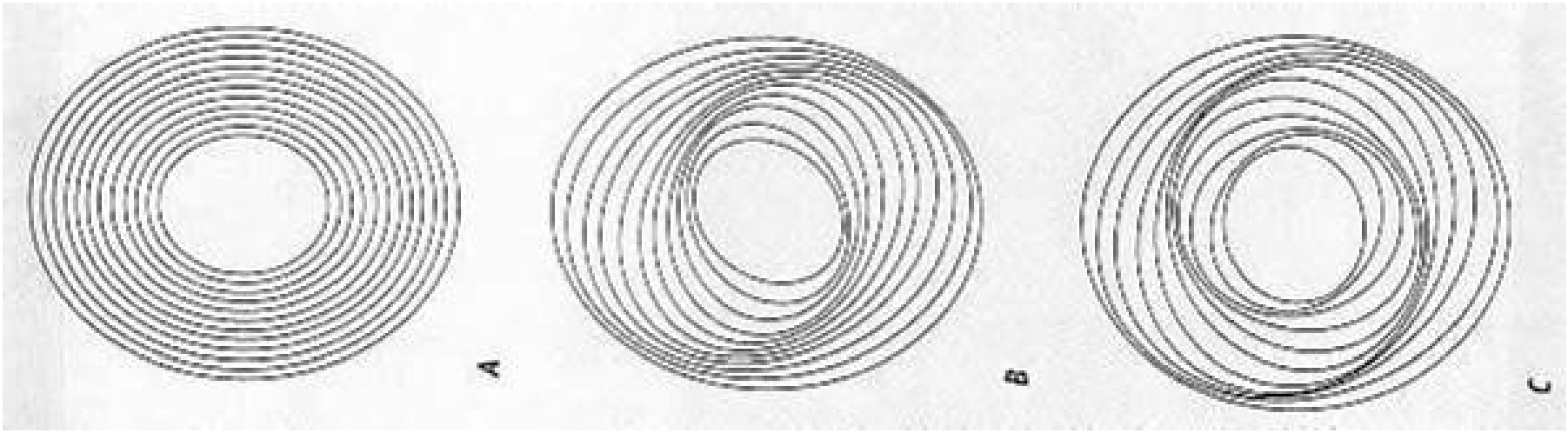
- If at a certain location for which the orbital angular frequency is Ω, the natural angular frequency of vertical oscillations is ν and that of radial oscillations is κ, and the angular frequency at which the perturbation occurs is Ω_p, then the perturbation is **resonant** if $h\nu + i\kappa + j\Omega + k\Omega_p$ is near zero, with h, i, j, k all integers.

- Waves are patterns in the structure in density or position created by these orbital resonances.

Example: spiral density waves

- Radial oscillations among ring particles, driven by orbital resonance with a given moon, are damped more (by collisions), the further away from a moon one looks.

- Damping thus spreads some of the moon's angular momentum into the ring. This moves material away from the moon, where it can pile up with material that was already there: a wave in the **density** of the ring particles.

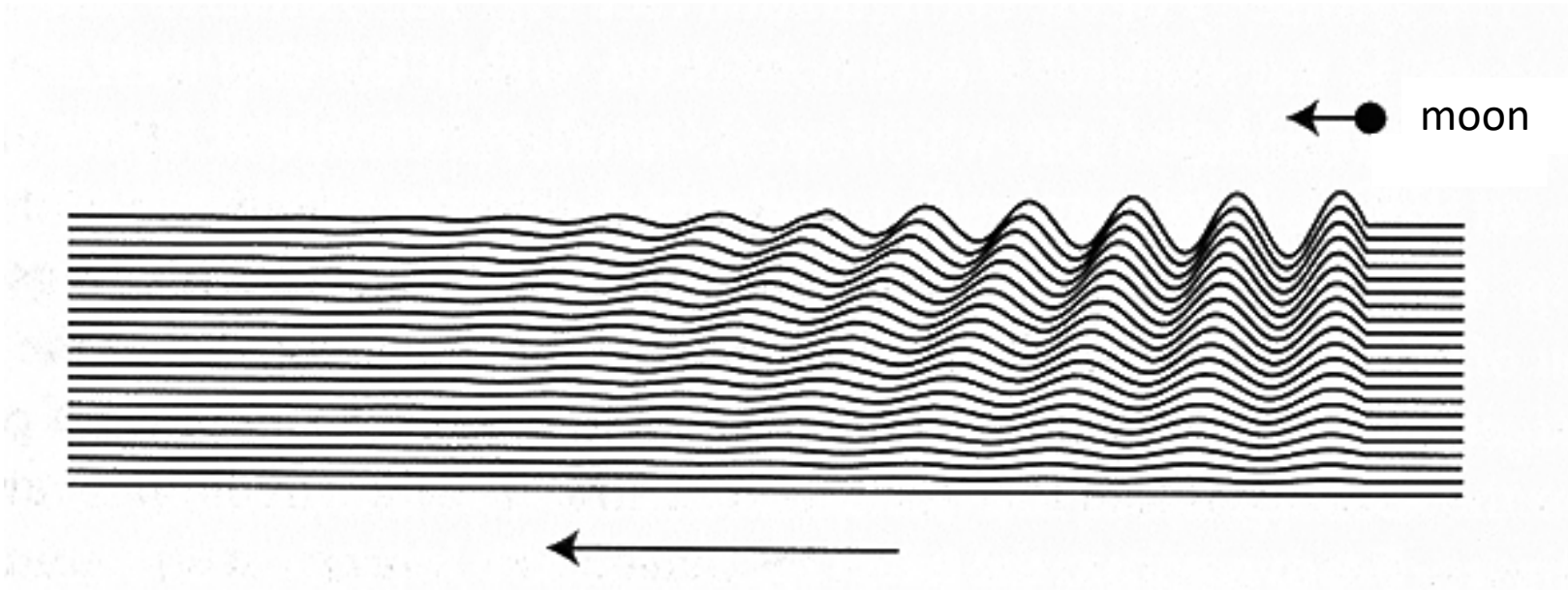


Example: bending waves

The same reasoning applies to vertical oscillations; the result is a ripple-shaped **bending** of the ring-disk.

- Saturn's rings are even thinner than they look. They appear to have thickness greater than the size of ring particles only because they carry bending waves.

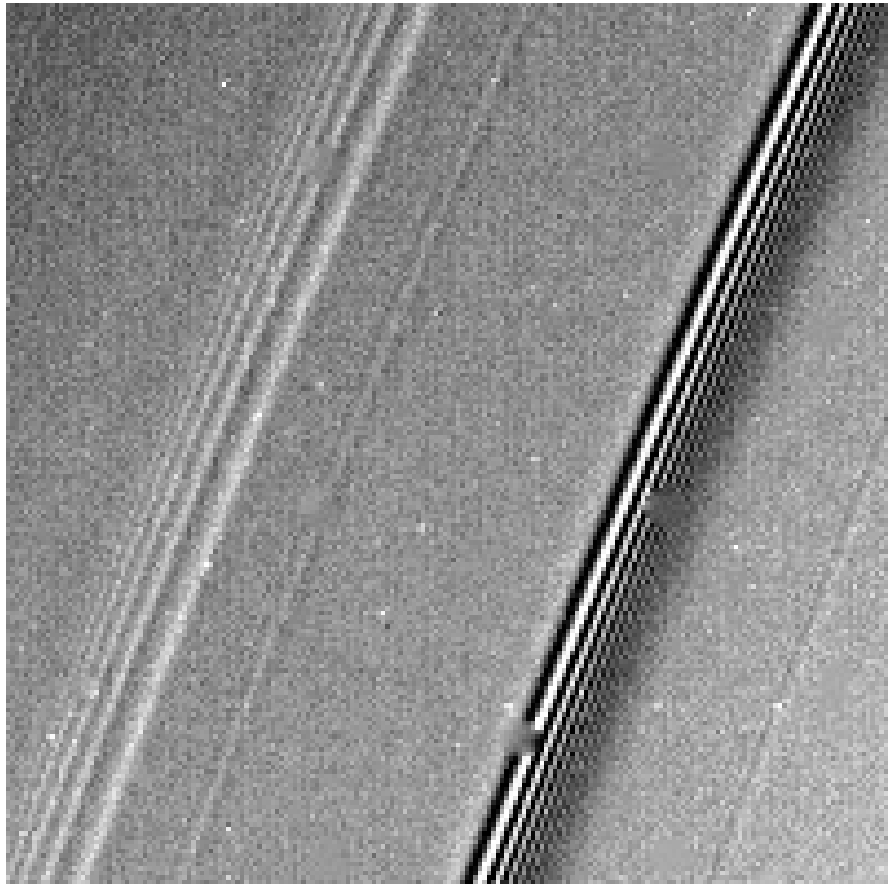
- Amplitudes of the bending waves can be thousands of times larger than the ring thickness.



Density and bending waves in Saturn's rings

This Voyager image captured a closeup of a pair of waves at the 5:3 orbital resonance with the moon Mimas.

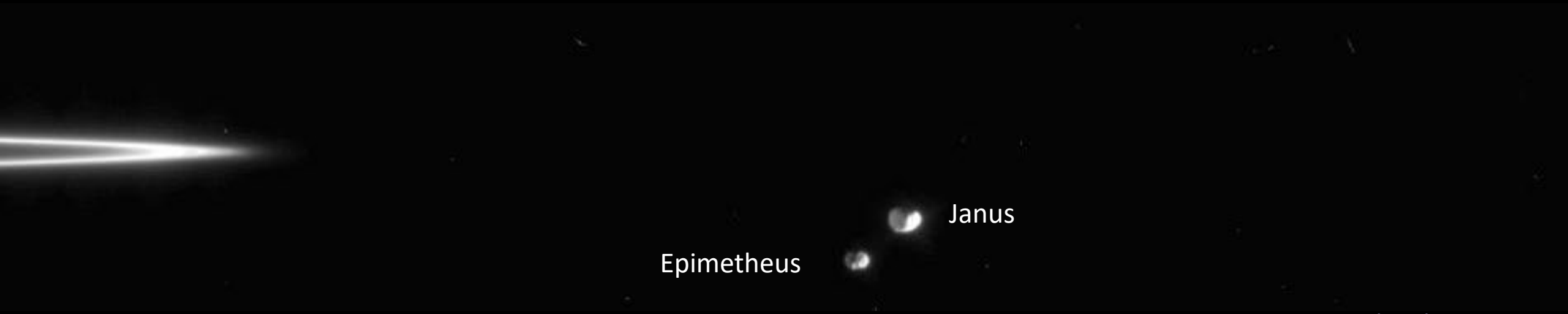
- 5:3 = 5 ring rotations, 3 Mimas revolutions.
- The inner wave (right) is a **spiral bending wave**, made of vertical positional oscillations and azimuthal density oscillations in the rings.
    - It is visible because the peaks cast shadows over the troughs.
- The outer wave (left) is a **spiral density wave**, in which particles are bunched together in a spiral pattern due to azimuthal density oscillations.



Horseshoe and tadpole orbits

Particles perturbed by a satellite, or even a pair perturbed by each other, can find themselves in orbits closely associated with the two-body potential and the five Lagrange points.

- Orbits around L4 and L5, if large enough, elongate parallel to the potential contours; these are called **tadpole orbits**.
- Orbits can enclose L3, L4, and L5, in which case they are called **horseshoe orbits**, famously exemplified by Janus and Epimetheus.

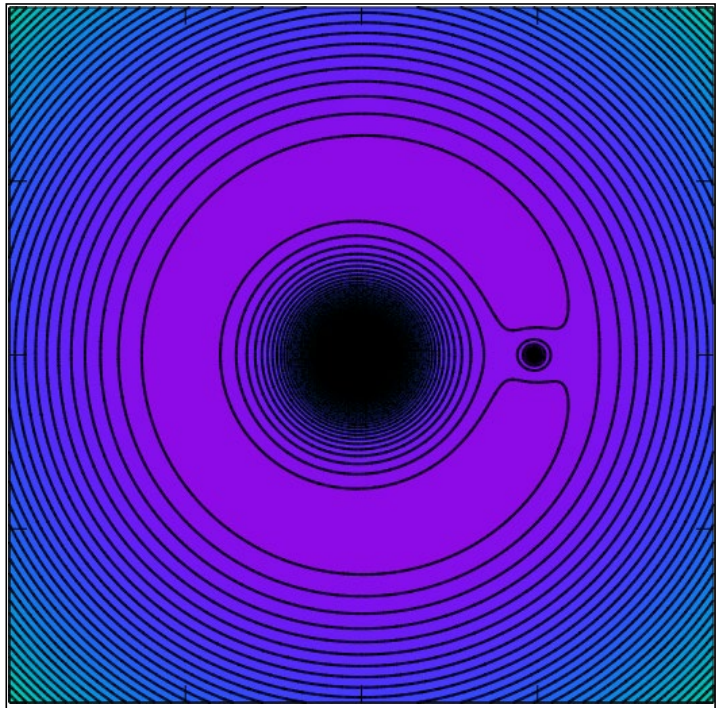


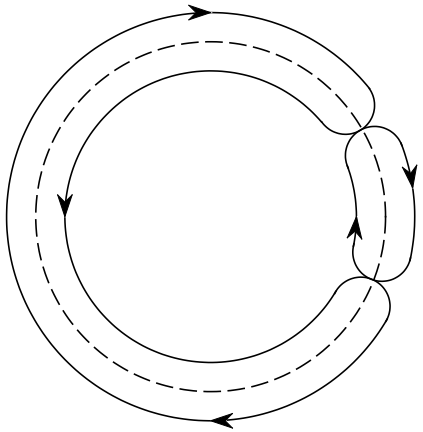
Cassini/JPL/NASA

Example horseshoe orbits

Schematic diagram of the Epimetheus-Janus-Saturn system: gravitational potential (left), orbits (right) in a frame revolving with the average mean motion of one of the satellites.

- The satellites execute their horseshoe patterns with the same period and **swap orbits** every (synodic) period.
- Click here for a simulation video.



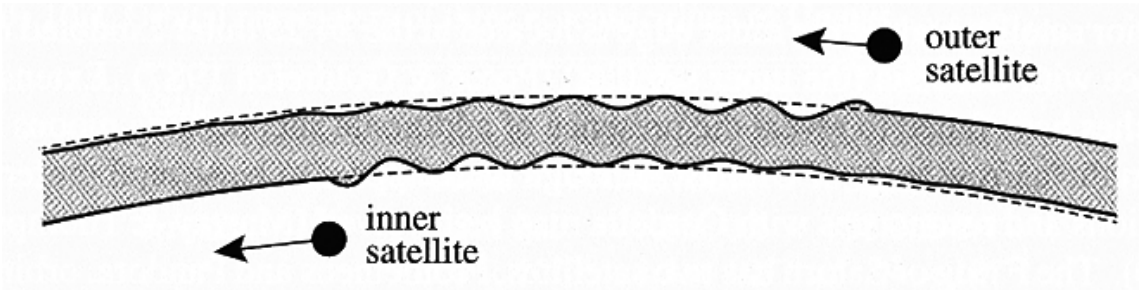


Shepherding and shepherd satellites

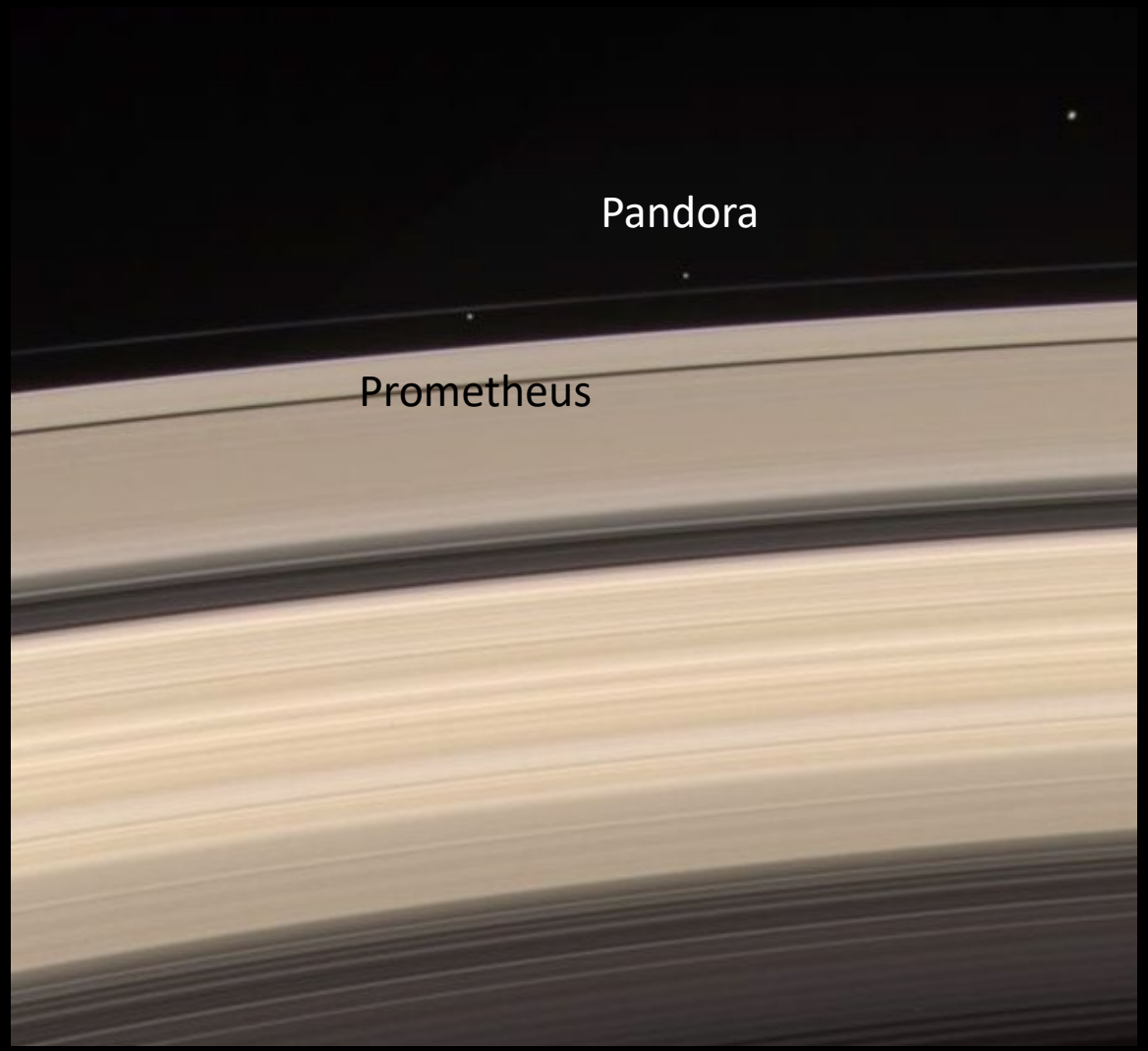
If two satellites orbit at similar radii, their orbital resonances can trap particles and drive density waves in the annulus between their orbits. This confinement is called **shepherding**.

- Lots of the fine-scale structure in the Saturn ring system is due to multiple small satellites with closely-spaced, nested orbits, shepherding the material in between.

- The rings so shepherded will show prominently the density-wave structure driven by their shepherds, resulting in “ripples” and “braids” such as those seen in Saturn’s F ring by the shepherd moons Pandora and Prometheus (next page).



Shepherd satellites (continued)



Cassini/JPL/ NASA

Perturbations and the measurement of planetary moments of inertia

Long ago, I promised a brief explanation of how we measure moments of inertia *I* of solar-system bodies. Now it's time. We can measure *I* because…

- …the physical principles governing for gravitational potential energy *U* and moment of inertia *I* are similar, and we can measure *U* directly from gravitational forces. For instance, for uniform density spheres,

$$U = -\frac{3}{5}\frac{GM^2}{R} \quad , \quad I = \frac{2}{5}MR^2 \quad .$$

See the following for *U*, and the lecture notes for 12 September 2024 for *I.*

- The similarity between gravitational potential energy and moment of inertia persists for non-spherical and non-uniform-density bodies.

- It provides us a means to **measure the moment of inertia *I* of a planet or moon** – uniform or differentiated – **by measurement of the details of the body's gravitational field** through energy or force.

Gravitational potential energy of a uniform-density sphere

Suppose a uniform-density planet were made by collecting a mass *M*, originally in the form of small particles lying at *r* = infinity.

- Consider the point at which a spherical mass *m* had been built up within a radius *r'*, and consider adding an infinitesimal increment *dm*:

$$dU = \boldsymbol{F} \cdot d\ell = \frac{Gmdm}{r^2} dr \quad \text{, where} \quad m = \rho \frac{4}{3} \pi r'^3 = M \frac{r'^3}{R^3} \quad \text{and}$$

$$dm = \rho r'^2 \sin\vartheta dr' d\vartheta d\varphi = \frac{3M}{4\pi R^3} r'^2 \sin\vartheta dr' d\vartheta d\varphi \quad \text{, so}$$

$$U = \int\int \frac{Gmdm}{r^2} dr = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \int_{\infty}^{r'} \frac{G}{r^2} \left(M \frac{r'^3}{R^3} \right) \left(\frac{3M}{4\pi R^3} r'^2 \sin\vartheta \right) dr dr' d\vartheta d\varphi$$

Not on homework or exams.

Gravitational potential energy of a uniform-density sphere (continued)

- Most of the terms can be brought outside the various integrals:

$$U = \frac{3GM^2}{4\pi R^6} \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin\vartheta \int_0^{R} dr' r'^5 \int_{\infty}^{r'} \frac{dr}{r^2}$$

$$\int_0^{2\pi} d\varphi = 2\pi \quad , \quad \int_0^{\pi} d\vartheta \sin\vartheta = -\cos\vartheta \Big|_0^{\pi} = -(-1-1) = 2$$

$$\int_{\infty}^{r'} \frac{dr}{r^2} = -\frac{1}{r}\Big|_{\infty}^{r'} = -\frac{1}{r'} \quad , \text{ so}$$

$$\int_0^{R} dr' r'^5 \int_{\infty}^{r'} \frac{dr}{r^2} = -\int_0^{R} dr' r'^4 = -\frac{R^5}{5}$$

A multivariable integral, and thus forbidden in ASTR 111 homework and exams. But it's a simple one.

Not on homework or exams.

Gravitational potential energy of a uniform-density sphere (continued)

- So

$$U = \frac{3GM^2}{4\pi R^6} \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin\vartheta \int_0^{R} dr' r'^5 \int_{\infty}^{r'} \frac{dr}{r^2}$$

$$= \frac{3GM^2}{4\pi R^6} \cdot 2\pi \cdot 2 \cdot \frac{R^5}{5}$$

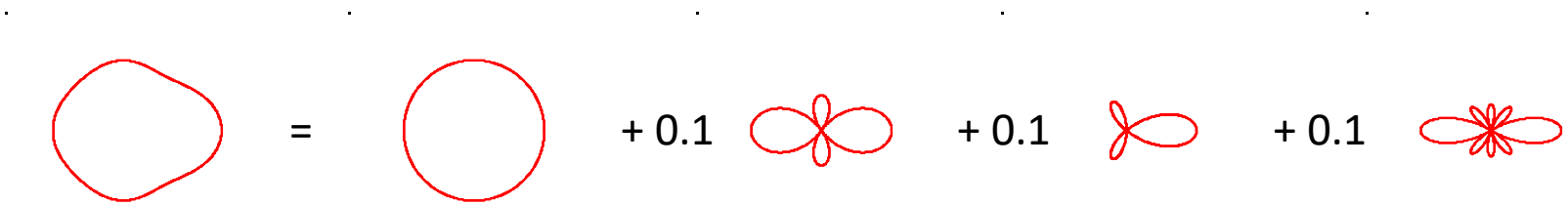
$$\boxed{= -\frac{3}{5}\frac{GM^2}{R}},$$

as advertised.

Not on homework or exams.

Describing gravity from a bumpy body, mathematically

- It is possible, and convenient, to describe the gravitational potential Φ – potential energy per unit orbiting mass, as on 1 October 2024, pp. 12 ff. – of a bumpy body as the sum of a series of regular, smooth functions. For instance:



- The conventional family of functions people use, depicted here, is the set of **Legendre polynomials**, $P_n(\cos\vartheta)$, of **order** n, expressed as functions of $\cos\vartheta$:

$$\Phi(r,\vartheta,\varphi) = -\frac{GM}{r}\left(1 + 0.1P_2(\cos\vartheta) + 0.1P_3(\cos\vartheta) + 0.1P_4(\cos\vartheta)\right)$$

You will use the Legendre polynomials extensively in PHYS 217, PHYS 218, and PHYS 237.

Describing gravity from a bumpy body, mathematically (continued)

- In the most general case, we would write an infinite sum of Legendre polynomials, each with their own coefficient J_n, expressing the magnitude of their contribution to the total:

$$\Phi(r,\vartheta,\varphi) = -\frac{GM}{r}\left[1 - \sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n J_n P_n(\cos\vartheta)\right] \quad ,$$

where R_e is the radius of the planet or moon at the equator.

  - **Don't worry about these hieroglyphics**; in ASTR 111 we're not going to be using Legendre polynomials or this formula except as a substitute for the "sum of pictures" on page 21.

  - Legendre polynomials are called "zonal harmonics" in some textbooks, such as CS.

Describing gravity from a bumpy body, mathematically (continued)

- If we put a satellite in orbit around this bumpy body, that orbit will not be perfectly elliptical or constant (i.e. closed).
- Instead, **its orbit is perturbed from stable elliptical shape by the collection of bumps**.
  - Usually the biggest bump is the first term in the infinite series, with J_2 and $P_2(\cos\vartheta)$.
  - In the decomposition shown on page 21, $J_2 = 0.1$.
- These perturbations may cause the satellite orbit to precess, or have its semimajor axis, eccentricity, or orientation change with time, according to pages 6-8 above.
- By watching the satellite in orbit we can measure the rates at which these orbital changes take place, and can thus determine where and how large the perturbations are.
- **This in turn can be used to work out what all the J_n are**.

Describing gravity from a bumpy body, mathematically (continued)

- It turns out that for bodies in hydrostatic equilibrium,

$$J_2 = \frac{I - I_\perp}{MR_e^2} = \frac{I - I_\perp}{I}\frac{I}{MR_e^2} = H\frac{I}{MR_e^2}$$

Moment of inertia about an axis perpendicular to its normal rotation axis.

where H (not to be confused with the isothermal scale height), a quantity called the **dynamic ellipticity,** can also be determined from the rate of precession of the satellite's orbit, as the force from a body with nonzero $I - I_\perp$ would count as a normal perturbation (page 7).

- Thus

$$\boxed{I = \frac{J_2}{H} MR_e^2}\ ,$$

where all of the quantities on the right-hand side can be determined from the details of the satellite's orbit: **the moment of inertia of the planet or moon is measured from the response of a satellite to its gravity**.

Describing gravity from a bumpy body, mathematically (continued)

- It is easiest to understand how this works for the case of a spacecraft-borne satellite making complete orbits around an actual planet, moon or asteroid, and indeed our best measurements of *I* come from orbiting deep-space probes.

- But it is not necessary for the satellite to be a spacecraft …

- nor is it necessary for the “orbit” to be complete or closed.

  - Many of the measured *I* for asteroids come from the details of very precise and accurate measurements of trajectory and velocity for the asteroid as it encounters a spacecraft that’s just passing by,

  - or even as it encounters another solar-system object, like Mars or its moons.