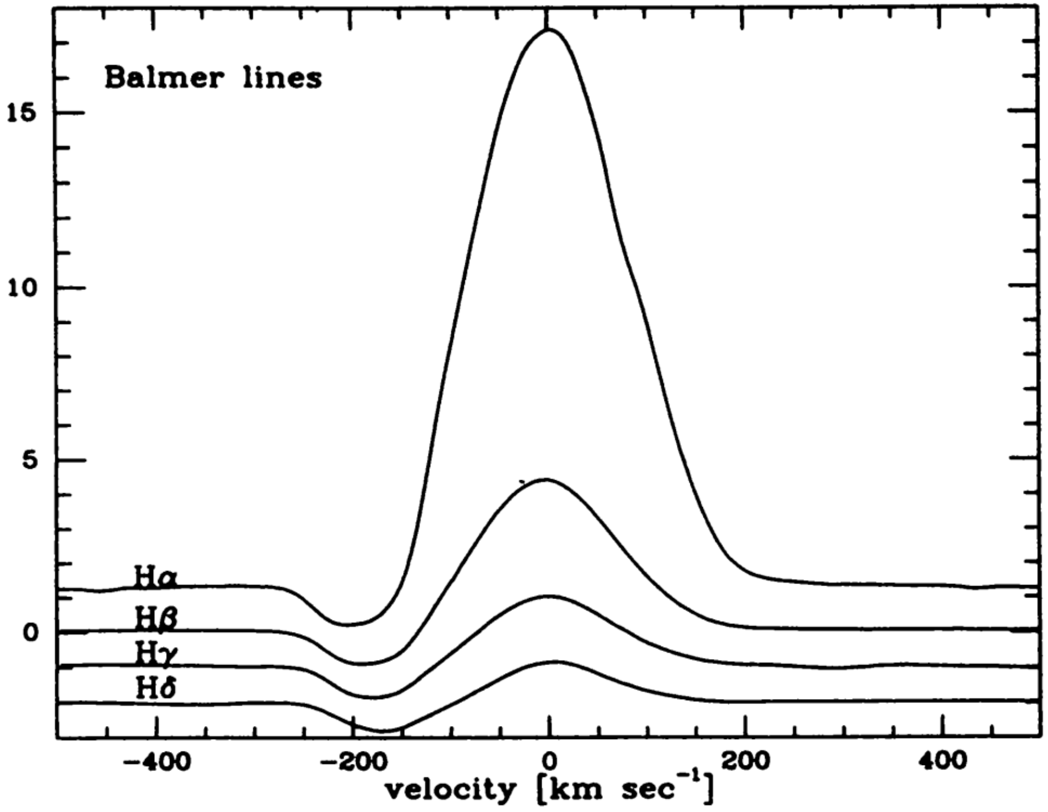
Today in Astronomy 241: radiation and opacity

Today's most important principles:

- The radiation field, its moments, and their relation to intensity, flux, and radiation pressure
- Opacity, absorption and optical depth
- **Reading**: C&O chapter 9, pp. 231-244



Visible-wavelength H I lines of the luminous blue variable star P Cygni (Stahl+1993), plotted as specific flux *vs.* radial velocity.

Note the blueshifted absorption and redshifted emission.

The radiation field

The radiation field of starlight is characterized by its **intensity**. Intensity *I* has dimensions of power per unit area, per unit solid angle.

- We are mostly concerned with the **specific** intensity, I_λ, of light incident on the unit area at angle ϑ. Specific intensity – connoted by the subscript λ – has dimensions of intensity per unit bandwidth (= wavelength interval):

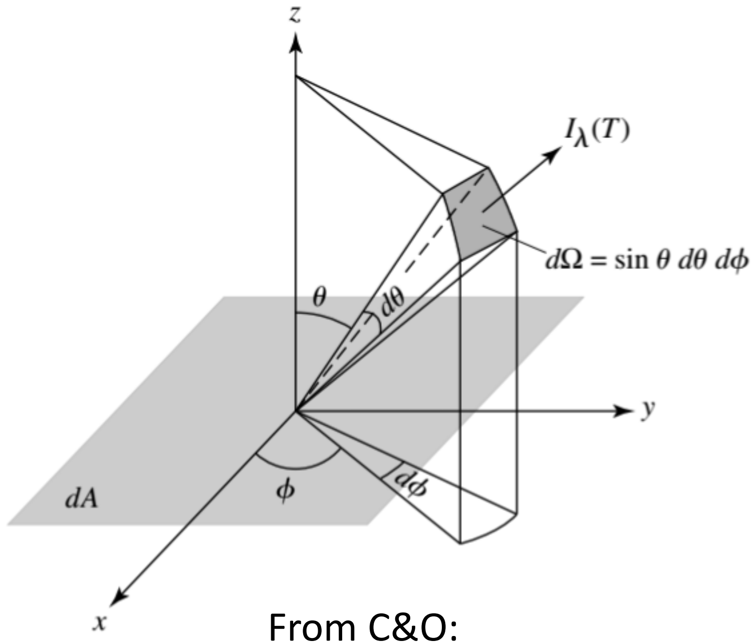
$$I_\lambda = I_\lambda(\lambda,\vartheta,\varphi) = \frac{d^3}{dtdAd\Omega}\frac{E_\lambda}{\cos\vartheta} \quad .$$

- And also with the **mean** intensity, which is the average of the specific intensity over solid angle:

$$\langle I_\lambda \rangle = \frac{1}{4\pi}\int I_\lambda d\Omega = \frac{1}{4\pi}\iint I_\lambda \sin\vartheta d\vartheta d\varphi \quad .$$

Recall:

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta d\vartheta = 4\pi$$



From C&O:

FIGURE 9.1 Intensity I_λ.

The radiation field (continued)

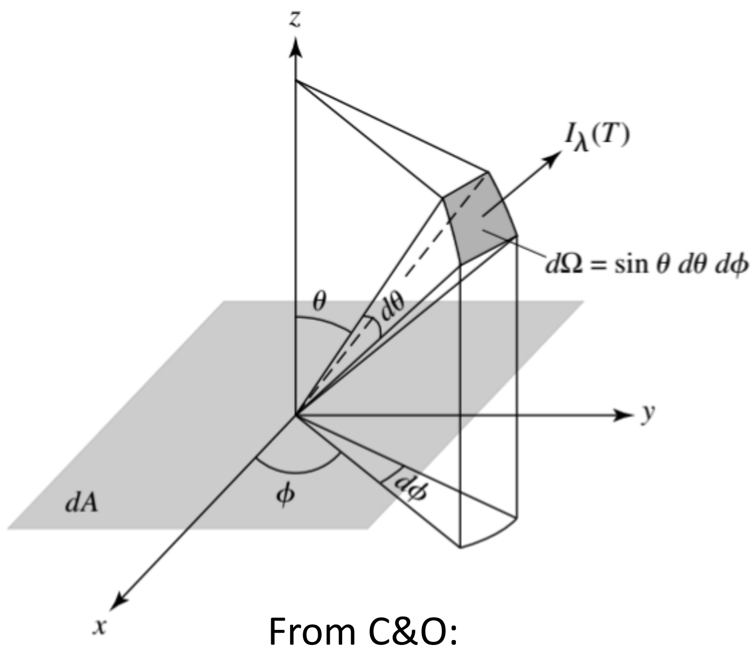
- Note that for **isotropic** radiation fields, $\langle I_\lambda \rangle = I_\lambda$.

- Good example of an isotropic radiation field: a **blackbody**, for which the specific intensity is the Planck function:

$$I_\lambda = B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda kT}\right) - 1}$$

- In spherical coordinates centered on the star, the nth **moment** of the radiation field is the integral of the specific intensity with $\cos^n \vartheta$:

$$M_\lambda^{(n)} = \frac{1}{4\pi} \int I_\lambda \cos^n \vartheta \, d\Omega = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} I_\lambda \cos^n \vartheta \sin\vartheta \, d\vartheta \, d\varphi$$



From C&O:

FIGURE 9.1 Intensity I_λ.

Moments of the radiation field

- In most radiative transfer books, the first three moments have standard symbols, *J*, *H*, and *K.* They are directly related to what we will call the mean intensity, specific flux, and specific radiation pressure, $\langle I_\lambda \rangle$, F_λ, and $P_{\mathrm{rad},\lambda}$.

$$J_\lambda = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} I_\lambda \sin\vartheta d\vartheta d\varphi$$

$$\langle I_\lambda \rangle = J_\lambda = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} I_\lambda \sin\vartheta d\vartheta d\varphi$$

$$H_\lambda = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} I_\lambda \cos\vartheta \sin\vartheta d\vartheta d\varphi$$

$$F_\lambda = 4\pi H_\lambda = \int_0^{2\pi}\int_0^{\pi} I_\lambda \cos\vartheta \sin\vartheta d\vartheta d\varphi$$

Called πF_λ in some books.

$$K_\lambda = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} I_\lambda \cos^2\vartheta \sin\vartheta d\vartheta d\varphi$$

$$P_{\mathrm{rad},\lambda} = \frac{4\pi}{c}K_\lambda = \frac{4\pi}{c}\int_0^{2\pi}\int_0^{\pi} I_\lambda \cos^2\vartheta \sin\vartheta d\vartheta d\varphi$$

Energy density and radiation pressure of the radiation field

- Photons in a radiation field with mean intensity $\langle I_\lambda \rangle$, regarded as a stream of photons each with energy hc/λ travelling at speed c, have **specific energy density** u_λ given as

$$u_\lambda = \frac{4\pi}{c}\langle I_\lambda \rangle = \frac{8\pi hc}{\lambda^5}\frac{1}{\exp\left(hc/\lambda kT\right)-1}$$

for isotropic blackbody radiation

- Photons carry momentum as well as energy (see e.g. ASTR 111, 16 November 2023), so the radiation field exerts pressure and force. The **specific** pressure is

$$P_{\mathrm{rad},\,\lambda} = \frac{1}{c}\int_0^{2\pi}\int_0^{\pi} I_\lambda \cos^2\vartheta \sin\vartheta d\vartheta d\varphi$$

pressure by photons, λ to λ+dλ

$$= \frac{4\pi}{3c} I_\lambda$$

isotropic

Radiation pressure and gas pressure

- (Total) radiation pressure is an integral of specific radiation pressure over all wavelengths.

$$P_{\mathrm{rad},\lambda} = \frac{4\pi}{3c} I_\lambda \qquad \text{if it's isotropic}$$

$$P_{\mathrm{rad}} = \frac{4\pi}{3c}\int_0^\infty I_\lambda d\lambda = \frac{4\pi}{3c}\int_0^\infty B_\lambda d\lambda = \frac{4\sigma}{3c}T^4 \qquad \text{if it's blackbody radiation}$$

- Total energy density in radiation:

$$u = \frac{4\pi}{c}\int_0^\infty \langle I_\lambda \rangle d\lambda = \frac{4\pi}{c}\int_0^\infty B_\lambda d\lambda = \frac{4\sigma}{c}T^4 = 3P_{\mathrm{rad}} \qquad \text{for isotropic blackbody radiation}$$

- Compare to an ideal gas, for which: $P = nkT \quad , \quad u = \frac{3}{2}nkT \quad , \quad u = \frac{3}{2}P \quad .$

Optical depth and absorption

- **Mean free path** of photons:

$$\ell = \frac{1}{n\sigma_\lambda} = \frac{1}{\rho\kappa_\lambda}$$

where n as usual is number density of absorbers – atoms or molecules – and σ is the **effective** cross-sectional area of the absorber: akin to the area of its shadow.

- Differential specific intensity absorbed:

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda ds$$

where ρ is mass density, κ_λ is called the absorption coefficient per unit mass or **opacity**, and s is along the direction photons travel.

- **Optical depth**: $d\tau_\lambda = -\kappa_\lambda \rho \, ds'$ $\quad \Delta\tau_\lambda = -\int_0^s \kappa_\lambda \rho \, ds'$

Today's in-class problems

First, do 9.2 (15 minutes).

Then, problem A (25 minutes).

A.i. Show that the differential elements of bandwidth for wavelength and frequency are related by $d\nu = -c\,d\lambda \big/ \lambda^2$.

A.ii Show that the Planck function expressed in frequency instead of wavelength is $B_\nu(\nu, T) = \dfrac{2h\nu^3}{c^2}\dfrac{1}{e^{h\nu/kT} - 1}$.

A.iii Demonstrate, however you like, that $\displaystyle\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \frac{\pi^4}{15}$.

A.iv Show that $\displaystyle\int_\infty^0 B_\lambda d\lambda = \int_0^\infty B_\nu d\nu = \frac{2\pi^4 k^4}{15h^3c^2}T^4 = \frac{\sigma}{\pi}T^4$.

Then, 9.4 (5 minutes).

Then, starting from the expression on page 7 and under the assumption that the absorbing medium is very cold, derive Equation 9.18 (15 minutes).

Hints for the last set of in-class problems

1. (8.4) Differentiate $n(v)$ with respect to v, set the result equal to zero, and solve for that special value of *v.*

2. (8.6) Only minor rearrangements required.

   a. Solve for T and plug in: T = 63800 K.

   b. Just evaluate Boltzmann factors, e.g. $N_3 = 1.74N$.

   c. As $T \to \infty$ the Boltzmann factor approaches 1, so the numbers in each state n are proportional to the statistical weights, $g_n = 2n^2 = 2, 8, 18, ...$ However, ionization will happen at some finite temperature, so only the lower states will actually reach this limit.

3. (8.9) Use Z_{I} = 2 and Z_{II} = 1, as in Example 8.1.4. Use this in the Saha equation to get

$$\left(\frac{N_{\mathrm{II}}}{N_{\mathrm{I}}}\right)^2 + \frac{m_p}{\rho}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_{\mathrm{I}}/kT}\left(\frac{N_{\mathrm{II}}}{N_{\mathrm{I}}}\right) - \frac{m_p}{\rho}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_{\mathrm{I}}/kT} = 0$$

   Then solve the quadratic and plug in. For T = 10000 K I got $N_{\mathrm{II}}/N_{\mathrm{total}} = \left(N_{\mathrm{II}}/N_{\mathrm{I}}\right) / \left[1 + \left(N_{\mathrm{II}}/N_{\mathrm{I}}\right)\right] = 0.70$.