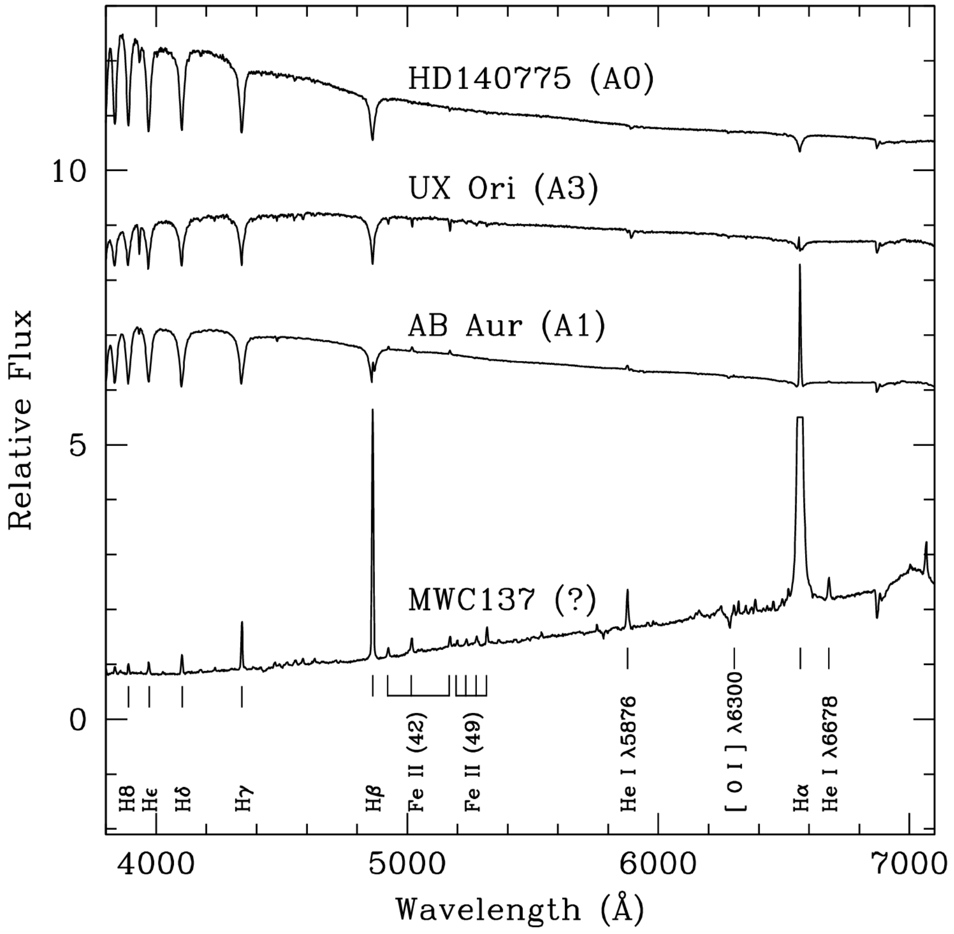
Today in Astronomy 241: radiative transfer

Today's most important principles

- Sources of opacity, and the Kramers constitutive relation for opacity
- The Rosseland mean opacity
- The radiative transfer equation in 1-D
- **Reading:** C&O chapter 9, pp. 244-258

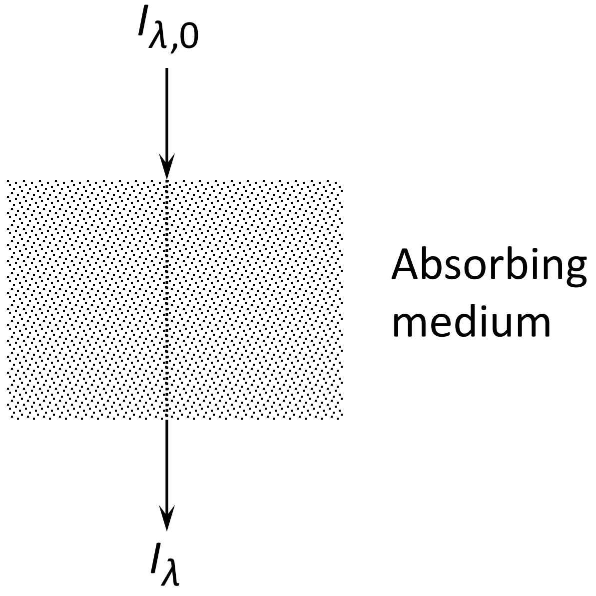


Emission-line stars (lower three spectra) compared to an ordinary A star (upper). From Hernandez+2004.

Pure absorption (*viz.* Equation 9.18)

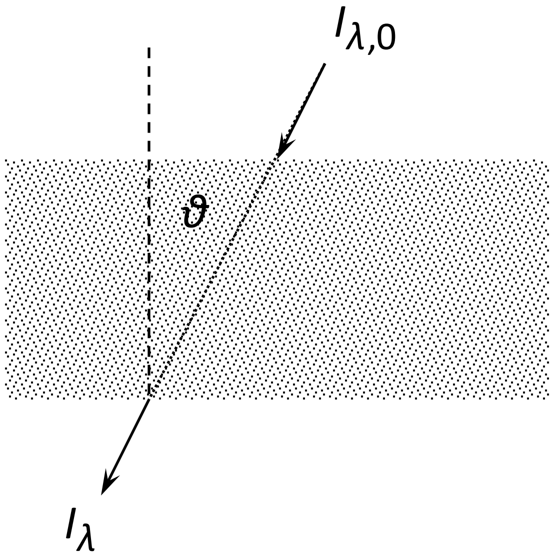
One dimension:

$$I_\lambda = I_{\lambda,0} e^{-\tau_\lambda}$$



Planar atmosphere:

$$I_\lambda = I_{\lambda,0} e^{-\tau_{\lambda,0} \sec\vartheta}$$



Sources of opacity

Opacity comes in four basic types.

1. Bound-bound: atomic and molecular spectral lines

- Can be measured in the lab, or calculated with quantum mechanics.
- Depends upon strengths of huge numbers of transitions among electronic states of atoms and molecules, and the value and variation of the relative abundances of those atoms and molecules.

Note: anything that **absorbs** can also **emit**, by the reverse process.

Sources of opacity (continued)

2. Bound-free: atomic photoionization, molecular photodissociation. Hydrogen with principal quantum number *n*, for example:

$$\sigma_{bf}(n) = \frac{1.31\times 10^{-15}\ \mathrm{cm}^2}{n^5}\left(\frac{\lambda}{500\,\mathrm{nm}}\right)^3$$

for wavelengths less than $-hc/E_n$.

- Also measurable or calculable with quantum mechanics.
- Also extremely complicated and abundance-dependent.
- For most stellar atmospheres, the photoionization of H^- is the dominant bound-free opacity source. (!)

Sources of opacity (continued)

3. Free-free: light absorbed by colliding ions (mostly electrons). In emission, this is known as thermal **bremsstrahlung**.

- Can be calculated quantum-mechanically (Bethe and Heitler, 1954) or classically (see, for instance, here).

- Complicated by multiplicity of positive ions usually present in stars.

Sources of opacity (continued)

4. Electron (Thomson) **scattering**:

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 6.65\times10^{-25}\ \mathrm{cm}^2$$

- Finally, a simple one.

This is all way too complicated for most uses, and sometimes unnecessarily so. Thus one often uses approximations for the averages of specific opacity sources, such as the

- **Kramers** opacities: an empirically-based set of approximations to opacity sources 1-4, and
- weighted mean total opacities, such as the **Rosseland mean opacity**.

Kramers-opacity constitutive relations

Bound-free: $$\overline{\kappa}_{bf} = \left(4.34\times10^{25}\ \frac{\text{cm}^4\ \text{K}^{3.5}}{\text{gm}}\right)\frac{g_{bf}}{t}Z(1+X)\frac{\rho}{T^{3.5}}$$

Free-free: $$\overline{\kappa}_{ff} = \left(3.68\times10^{22}\ \frac{\text{cm}^4\ \text{K}^{3.5}}{\text{gm}}\right)g_{ff}(1-Z)(1+X)\frac{\rho}{T^{3.5}}$$

Electron scattering: $$\overline{\kappa}_{es} = \left(0.2\frac{\text{cm}^2}{\text{gm}}\right)(1+X)$$

- Here X is the fraction of hydrogen by mass, Z the fraction of “metals” (elements heavier than helium) by mass, and t and the gs quantum-mechanical correction factors of order unity. There’s no simple form for bound-bound, though.

The Rosseland mean opacity, $\overline{\kappa}_R$

$\overline{\kappa}_R$ is a flux-weighted spectral-average opacity that emphasizes wavelengths where the absorption coefficient is small:

$$\frac{1}{\overline{\kappa}_R} = \int_0^\infty \frac{I_\lambda d\lambda}{\overline{\kappa}_\lambda} \Bigg/ \int_0^\infty I_\lambda d\lambda \quad ,$$

where $\overline{\kappa}_\lambda = \overline{\kappa}_{bb} + \overline{\kappa}_{bf} + \overline{\kappa}_{ff} + \overline{\kappa}_{es} \quad .$

- The calculations are complicated and lengthy, so Rosseland means have been computed comprehensively and accurately and provided as a central resource by several groups, notably the OPAL Project (Rogers & Iglesias 1992, 1997):

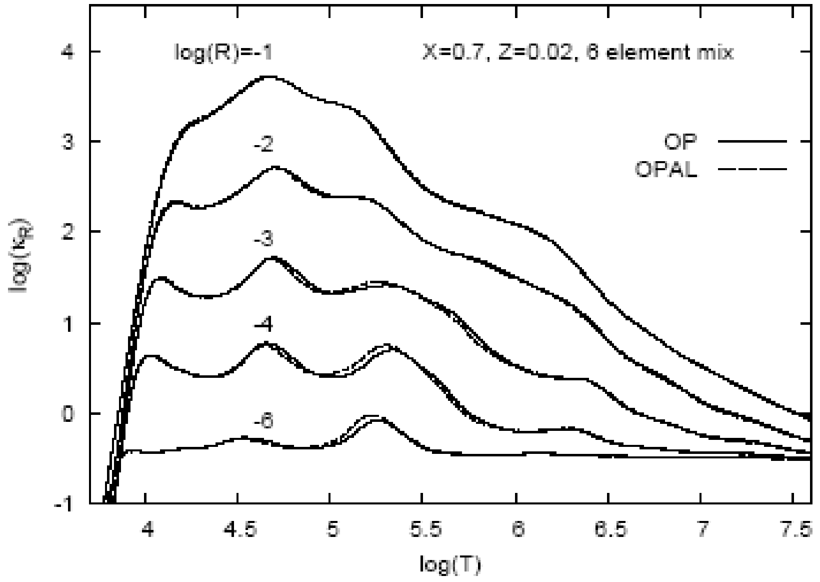


Figure 1. Comparisons of $\log(\kappa_R)$ from OP and OPAL for the 6-element mixture of Table 1. OP from [4], OPAL from [5]. Curves are labelled by values of $\log(R)$ where $R = \rho/T_6^3$, ρ is mass density in g cm^{-3} and T_6 is $10^6 \times T$ with T in K.

Seaton & Badnall 2004

Radiative transfer in 1-D

- Displacement after *N* steps of a random walk. See here, pp 23-26, for a familiar derivation:

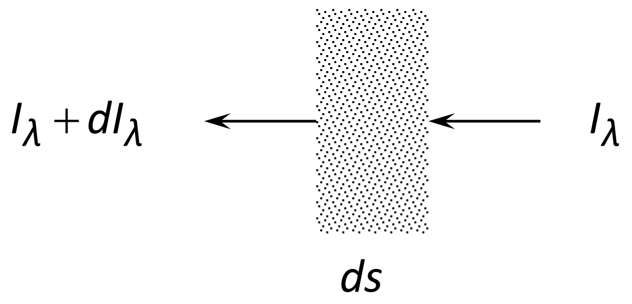
$$s = \ell\sqrt{N}$$

- Radiative transfer:

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda ds + j_\lambda \rho ds$$

$$-\frac{1}{\kappa_\lambda \rho}\frac{dI_\lambda}{ds} = I_\lambda - \frac{j_\lambda}{\kappa_\lambda}$$

$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda$$

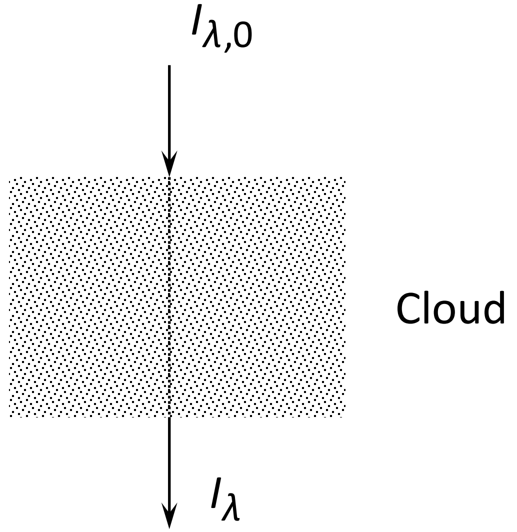


Today's in-class problems

B. **Part of limb darkening**. A small area *a* lies on a plane-parallel opaque surface, with a hemisphere of outer space above it. The *z* axis passes through its center.

i. Considering its projected area as seen from far away in direction ϑ, what is the effective solid angle into which it radiates? (Hint: it's not the hemisphere's 2π.)

ii. Revisit last classes' problem A, to obtain an expression for the total flux from a blackbody surface, comparing this to the expression used many times in earlier classes (e.g. ASTR 111).



C. Suppose that the source function – the ratio of the emission coefficient and the opacity – is equal to the Planck function, for a path through a constant-temperature cloud. Show that the solution of the radiative transfer equation for light passing through the cloud is

$$I_\lambda = I_{\lambda,0} e^{-\tau_{\lambda s}} + B_\lambda(\lambda, T)\left(1 - e^{-\tau_{\lambda s}}\right)$$

where $\tau_{\lambda s}$ is the total optical depth of the cloud at wavelength λ. How does this simplify in the limits of large or small optical depth?

Hints for the last set of in-class problems

1. (9.2) $n_\lambda = \frac{\lambda B_\lambda}{hc} = \frac{8\pi}{\lambda^4}\frac{1}{e^{hc/\lambda kT}-1}$ Substitute $x = hc/\lambda kT$, $dx = -d\lambda\, hc/\lambda^2 kT$:

$$n = \int_0^\infty n_\lambda d\lambda = -\frac{8\pi k^3T^3}{h^3c^3}\int_\infty^0 \frac{x^2dx}{e^x-1} = \frac{8\pi k^3T^3}{h^3c^3}\zeta(3) = 2.404\frac{8\pi k^3T^3}{h^3c^3}$$

$$nV = 2.404\frac{8\pi k^3T^3}{h^3c^3}\times 10^6 \text{ cm}^{-3} = 2.2\times 10^{15} \quad .$$

2. (A)

i. $d\nu = \left(\frac{d\nu}{d\lambda}\right)d\lambda = \frac{d}{d\lambda}\left(\frac{c}{\lambda}\right)d\lambda = -\frac{c}{\lambda^2}d\lambda.$ The minus sign reminds us that frequency and wavelength are inversely related.

ii. Energy is conserved, so $\int_0^\infty B_\nu d\nu = \int_\infty^0 B_\lambda d\lambda$:

$$\int_0^\infty B_\nu d\nu = \int_\infty^0 \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}d\lambda = -\int_\infty^0 \frac{2hc}{c^3}\frac{c^3}{\lambda^3}\frac{1}{e^{hc/\lambda kT}-1}\left(-\frac{cd\lambda}{\lambda^2}\right) = \int_0^\infty \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}d\nu \quad \text{, q.e.d.}$$

Hints for the last set of in-class problems (continued)

iii. If one integrates it numerically, one gets 6.49394, which to six figures also happens to be $\pi^4/15$. That will do as a demonstration, but will leave unsatisfied those who would like proof. In which case, look at the complete and rigorous derivation on Kevin Krisciunas's web site at Texas A&M.

iv. Just combine the results of iii and iv.

3. (9.4) Follows straightforwardly from 2.

4. $\frac{dI_\lambda}{ds} = -\kappa_\lambda \rho I_\lambda \quad \Rightarrow \quad \int \frac{dI_\lambda}{I_\lambda} = -\kappa_\lambda \rho \int ds \quad \Rightarrow \quad \ln I_\lambda = -\kappa_\lambda \rho s + C = -\tau_\lambda + C \quad \Rightarrow \quad I_\lambda(\tau_\lambda) = C' e^{-\tau_\lambda}$. Call $I_{\lambda,0} = I_\lambda(0)$, q.e.d.