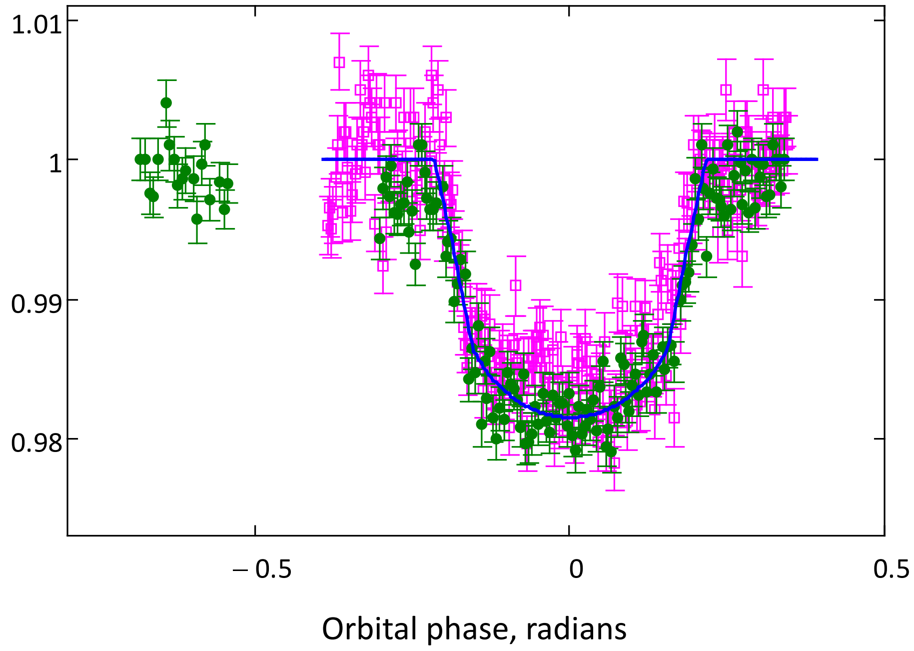
Today in Astronomy 241:

Two situations in which radiative transfer can be dealt with accurately but relatively simply:

- Expanding stellar atmospheres: the Sobolev approximation, escape-probability formalism, and large-velocity-gradient approximation
- Stellar limb darkening: power-series approximation to the source function
- **Reading**: C&O pp. 264-266; the supplement posted on our website; and Mandel & Agol 2002



Quadratic limb darkening (blue) applied to transit lightcurves of TOI-2046. Data (pink and green, for different observing nights) from last semester's ASTR 111 class.

Solutions to the radiative transfer equation

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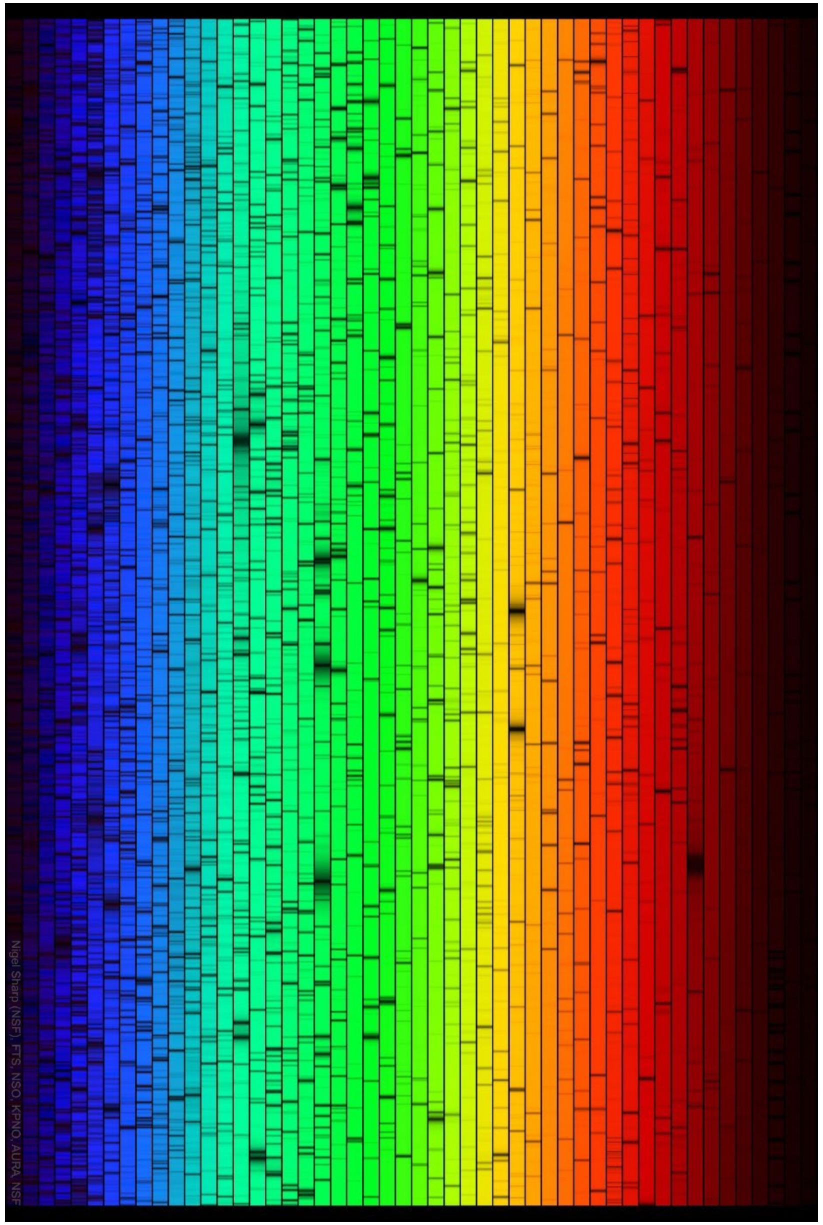


The goal of radiative transfer in stellar astrophysics is to link the observable properties of stars:

- Mass, luminosity, photospheric radius and effective temperature
- Absorption and emission spectrum
- Pulsation spectrum

to the underlying physical properties:

- Element abundances
- Physical structure of atmosphere and interior: density, temperature, ionization state as functions of radius
- Turbulent pressure, existence and extent of convection zones, magnetic fields and starspots.

Solutions to the radiative transfer equation (continued)

As we know,

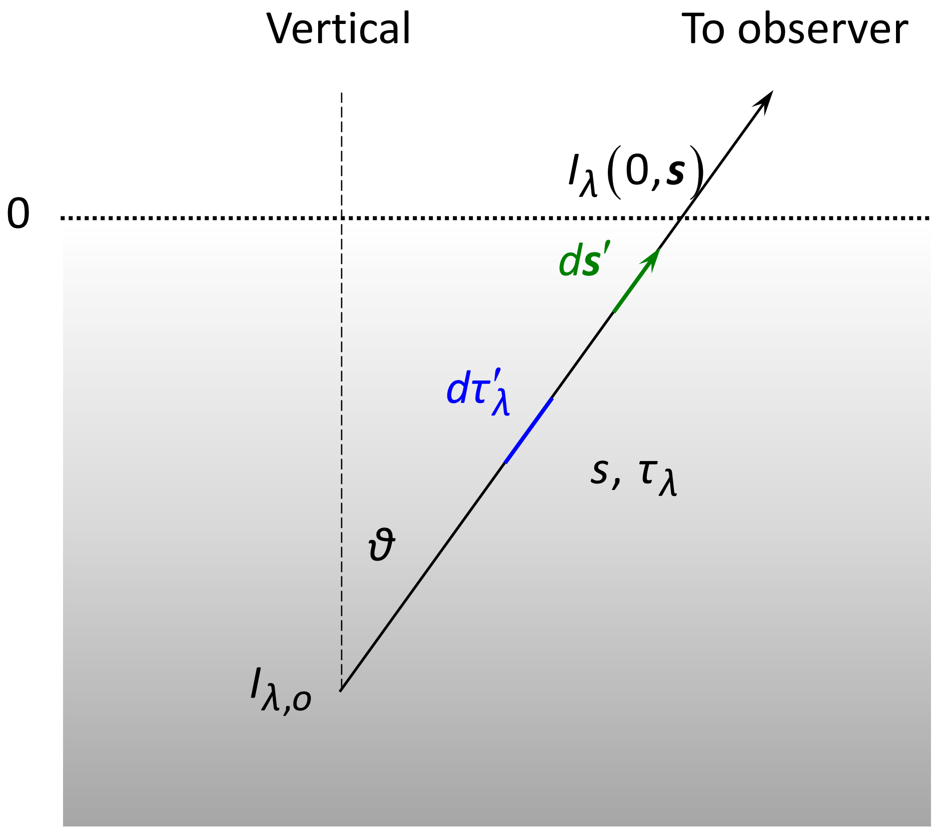
$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda \quad , \quad \text{where} \quad \tau_\lambda = \int_{\tau_\lambda}^{0} d\tau'_\lambda = \int_{s}^{0} \rho\kappa_\lambda ds'$$

where 0 is the boundary of the absorbing medium and the angle from the vertical is included in the prescription of the integration path.

- Solution, as we also know:

$$I_\lambda(0,\boldsymbol{s}) = I_{\lambda,o}e^{-\tau_\lambda} - \int_{\tau_\lambda}^{0} d\tau'_\lambda S_\lambda(\boldsymbol{s}')e^{-\tau'_\lambda}$$

which depends on the entire path of light propagation, rather than just the angle between the path and the vertical.



Solutions to the radiative transfer equation (continued)

- This is the beginning of a hard problem, as the source function $S_\lambda(\boldsymbol{s}')$ is in general a function of τ'_λ through $\boldsymbol{s}'$ and $\kappa_\lambda(\boldsymbol{s}')$.

- When valid approximations can't simplify the integral significantly, one normally solves this **inverse problem** by

  - constructs a model of the absorbing medium – density, temperature, composition;

  - outfits it with a constitutive relation for opacity (e.g. Kramers);

  - integrates $\int_0^{\tau_\lambda} d\tau'_\lambda S_\lambda(\boldsymbol{s}') e^{-\tau'_\lambda}$, often at great computational cost;

  - compares with constraints (e.g. observations);

  - repeats this sequence of steps until the constraints are satisfied within a desired tolerance.

- This would be very time and resource intensive for modern stellar models and high-resolution spectroscopic observations.

Examples of practical, valid approximations in radiative transfer

- For expanding stellar atmospheres, such as are found in red giants and asymptotic-giant-branch stars, the **Sobolev approximation**, the **escape-probability formalism**, and the **large velocity gradient (LVG) approximation**.
  - Simplifications which apply in a wide variety of astrophysical situations, not just stars.
  - Still too complicated to spend much time on in ASTR 241 but it pays to be familiar with the principles.
- For limb darkening: power series expansion and fitting of the source function.
  - Universally used in interpretation of eclipsing binaries and exoplanet transits.

The Sobolev, escape-probability, and LVG approximations

Sobolev (1963), who was trying to understand atomic spectral lines in expanding giant-star atmospheres, came up with all these.

1. What if $S_\lambda(\boldsymbol{s}')$ is independent of τ'_λ? Then it comes out of the integral:

$$I_\lambda(0,\boldsymbol{s}) = I_{\lambda,o}e^{-\tau_\lambda} - \int_{\tau_\lambda}^{0} d\tau'_\lambda S_\lambda(\boldsymbol{s}')e^{-\tau'_\lambda} = I_{\lambda,o}e^{-\tau_\lambda} - S_\lambda(\boldsymbol{s})\int_{\tau_\lambda}^{0} d\tau'_\lambda e^{-\tau'_\lambda}$$

$$= I_{\lambda,o}e^{-\tau_\lambda} + S_\lambda(\boldsymbol{s})\left(1-e^{-\tau_\lambda}\right)$$. Easy.

Sobolev approximation

2. In this case, writing the spectral-line intensities as $I_\lambda\varphi(\lambda)$, with $\int_{-\infty}^{\infty}\varphi(\lambda)d\lambda = \delta\lambda$, it is straightforward to show (see the supplement) that the radiation field's energy density u_λ is

$$u_\lambda = \frac{4\pi I_\lambda(0)}{c}\beta + \frac{4\pi S_\lambda(0)}{c}(1-\beta)$$

where $\beta = \dfrac{1}{4\pi\delta\lambda}\displaystyle\int_{4\pi} d\Omega \int_{-\infty}^{\infty} d\lambda\,\varphi(\lambda)e^{-\tau_\lambda}$. Also easy.

Here β is called the **escape probability.**

The Sobolev, escape-probability, and LVG approximations (continued)

3. Under what conditions might this apply? A wide variety, but Sobolev's favorite was the LVG approximation.

   - The idea: if the emission region exhibits total doppler shifts larger than the width $\delta\lambda$ of the spectral-line profiles, then the region's opacity is much reduced…

   - it will behave **almost** as though it is **optically thin** (zero opacity), the only modification being that emitted photons have probability β of escaping the region without being absorbed, instead of probability 1 (optically thin) or 0 (opaque; **optically thick**).

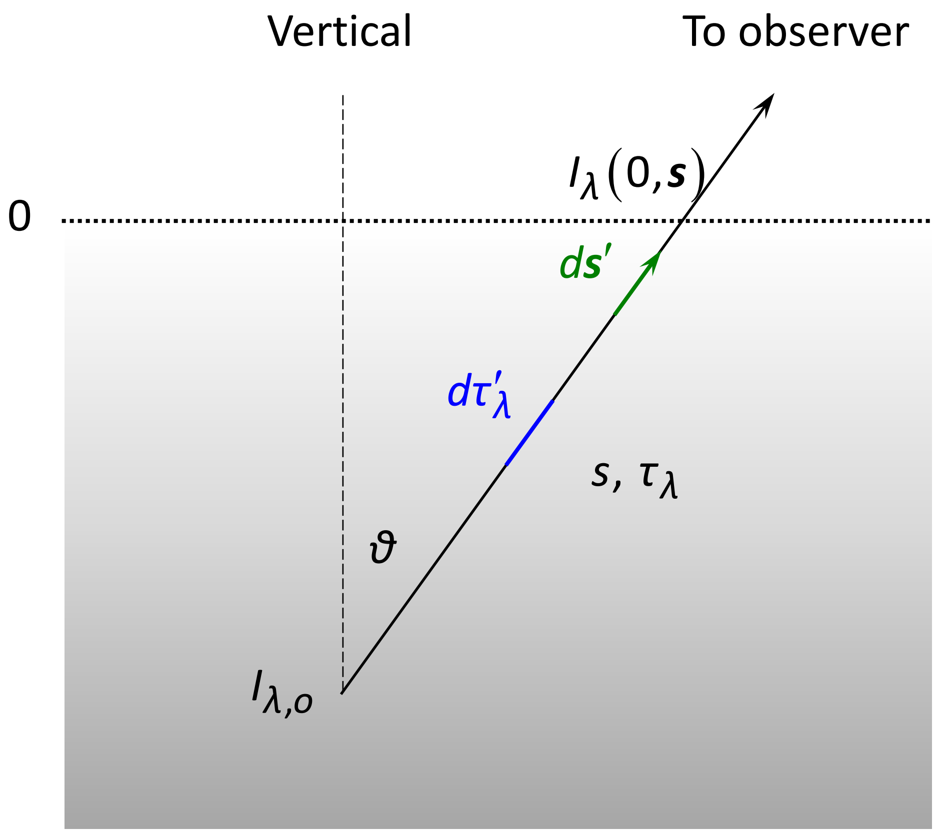
   - In this case, for a spectral line between states *j* and *i* (see the supplement for a derivation),

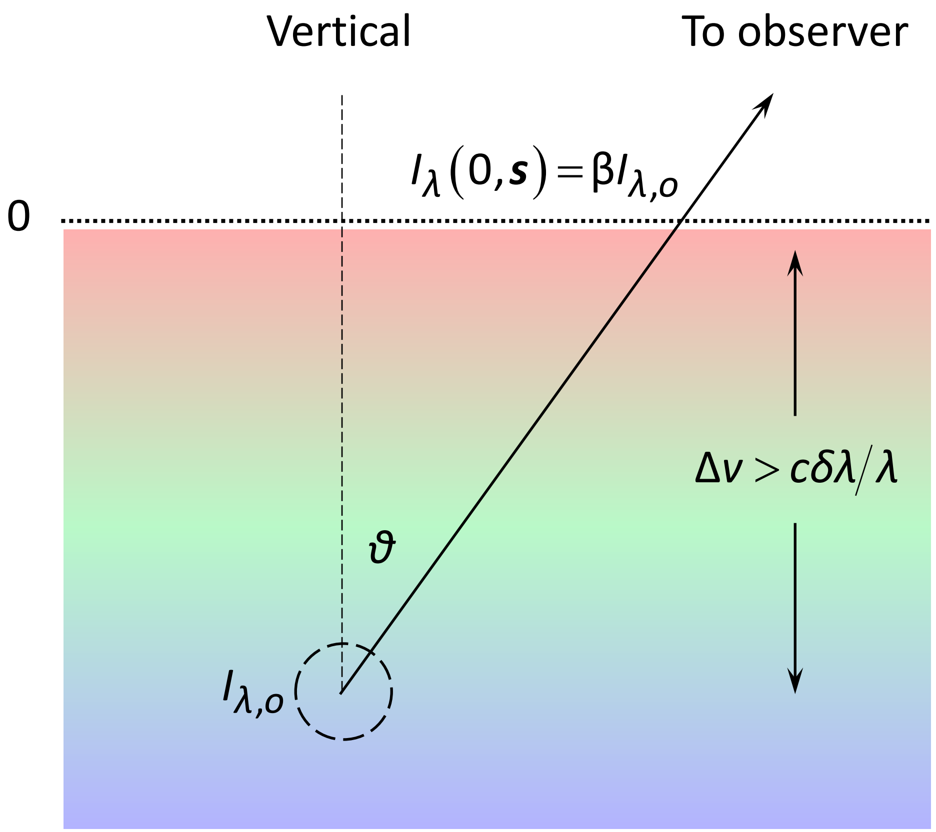
$$\beta_{ji} = \frac{1-e^{-3\tau_{ij}}}{3\tau_{ij}} \quad , \quad \text{where}$$

LVG escape probability

$$\tau_{ij} = \frac{\lambda\rho\kappa_{\lambda,ij}}{dv/dz} = \frac{\lambda^3 A_{ji}}{8\pi(dv/dz)}\left(\frac{g_j n_i}{g_i n} - \frac{n_j}{n}\right) \quad . \quad \text{Again, easy.}$$

Sobolev and LVG





Polynomial limb darkening

Another useful way to avoid time-consuming integrals is to suppose that the source function S_λ has a simple functional form, such as a series of easily integrated factors.

- Start with $$I_\lambda(0,\mathbf{s}) = I_{\lambda,o} e^{-\tau_\lambda} - \int_{\tau_\lambda}^{0} d\tau'_\lambda S_\lambda(\mathbf{s}') e^{-\tau'_\lambda} \quad \text{Write } \tau_\lambda = \tau_{\lambda,V} \sec\vartheta :$$

$$= I_{\lambda,o} e^{-\tau_{\lambda,V}\sec\vartheta} - \int_{\tau_{\lambda,V}\sec\vartheta}^{0} d\tau_{\lambda,V} \sec\vartheta S_\lambda(\mathbf{s}') e^{-\tau_{\lambda,V}\sec\vartheta} \quad ,$$

- Extend the integration bounds very deep in the atmosphere, so that all layers of the atmosphere are accounted for:

$$\tau_{\lambda,V} \to \infty \quad \Rightarrow \quad I_\lambda(0,\mathbf{s}) = -\int_{\infty}^{0} d\tau_{\lambda,V} \sec\vartheta S_\lambda(\mathbf{s}') e^{-\tau_{\lambda,V}\sec\vartheta} \quad .$$

Polynomial limb darkening (continued)

- And now suppose that the source function S_λ can be represented by a power series in the vertical optical depth:

$$S_\lambda = I_\lambda(0,V)\sum_{n=0}^{N} a_{\lambda,n}\tau_{\lambda,V}^{n} \quad ,$$

where is the intensity that emerges in the vertical direction.

- Then

$$\frac{I_\lambda(0,\boldsymbol{s})}{I_\lambda(0,V)} = \sum_{n=0}^{N} a_{\lambda,n} n! \cos^n \vartheta \quad .$$

The N = 1 and 2 cases are usually called **linear** and **quadratic limb darkening**, respectively. The latter is very widely applied in eclipsing-binary and exoplanet-transit studies, and we will use it in this week's homework.

Today’s in-class problems

D. Derive this last result. Compare it to equation 9.17 in C&O.

E. Compare the linear form of this result to that which we get for the gray atmosphere in the Eddington approximation. What are the coefficients $a_{\lambda,n}$ in this case?

F. Plot $I_\lambda(0,\boldsymbol{s})/I_\lambda(0,V)$ for linear and quadratic limb darkening, as functions of θ from 0 to 90°.

Hints for the last set of in-class problems

1. (9.17) Just integrate, starting from equations 9.3, 9.8, and 9.9:

$$\left\langle I \right\rangle = \frac{1}{4\pi}\left(\int_0^{2\pi} d\varphi \left[\int_0^{\pi/2} I_{\text{out}} \sin\vartheta d\vartheta + \int_{\pi/2}^{\pi} I_{\text{in}} \sin\vartheta d\vartheta \right] \right)$$
$$= \frac{1}{2}\left(I_{\text{out}} \left[-\cos\vartheta \right]_0^{\pi/2} + I_{\text{in}} \left[-\cos\vartheta \right]_{\pi/2}^{\pi} \right) = \frac{1}{2}\left(I_{\text{out}} + I_{\text{in}} \right)$$

$$F_{\text{rad}} = \int_0^{2\pi} d\varphi \left[\int_0^{\pi/2} I_{\text{out}} \cos\vartheta \sin\vartheta d\vartheta + \int_{\pi/2}^{\pi} I_{\text{in}} \cos\vartheta \sin\vartheta d\vartheta \right]$$
$$= 2\pi \left(I_{\text{out}} \int_0^1 u du + I_{\text{in}} \int_1^0 u du \right) = 2\pi \left(I_{\text{out}} \frac{1}{2} + I_{\text{in}} \left[-\frac{1}{2} \right] \right) = \pi \left(I_{\text{out}} - I_{\text{in}} \right)$$

$$P_{\text{rad}} = \frac{1}{c}\left(\int_0^{2\pi} d\varphi \left[\int_0^{\pi/2} I_{\text{out}} \cos^2\vartheta \sin\vartheta d\vartheta + \int_{\pi/2}^{\pi} I_{\text{in}} \cos^2\vartheta \sin\vartheta d\vartheta \right] \right)$$
$$= \frac{2\pi}{c}\left(-I_{\text{out}} \int_1^0 v^2 dv - I_{\text{in}} \int_0^{-1} v^2 dv \right) = \frac{2\pi}{c}\left(I_{\text{out}} \frac{1}{3} + I_{\text{in}} \frac{1}{3} \right) = \frac{2\pi}{3c}\left(I_{\text{out}} + I_{\text{in}} \right) = \frac{4\pi}{3c}\left\langle I \right\rangle$$