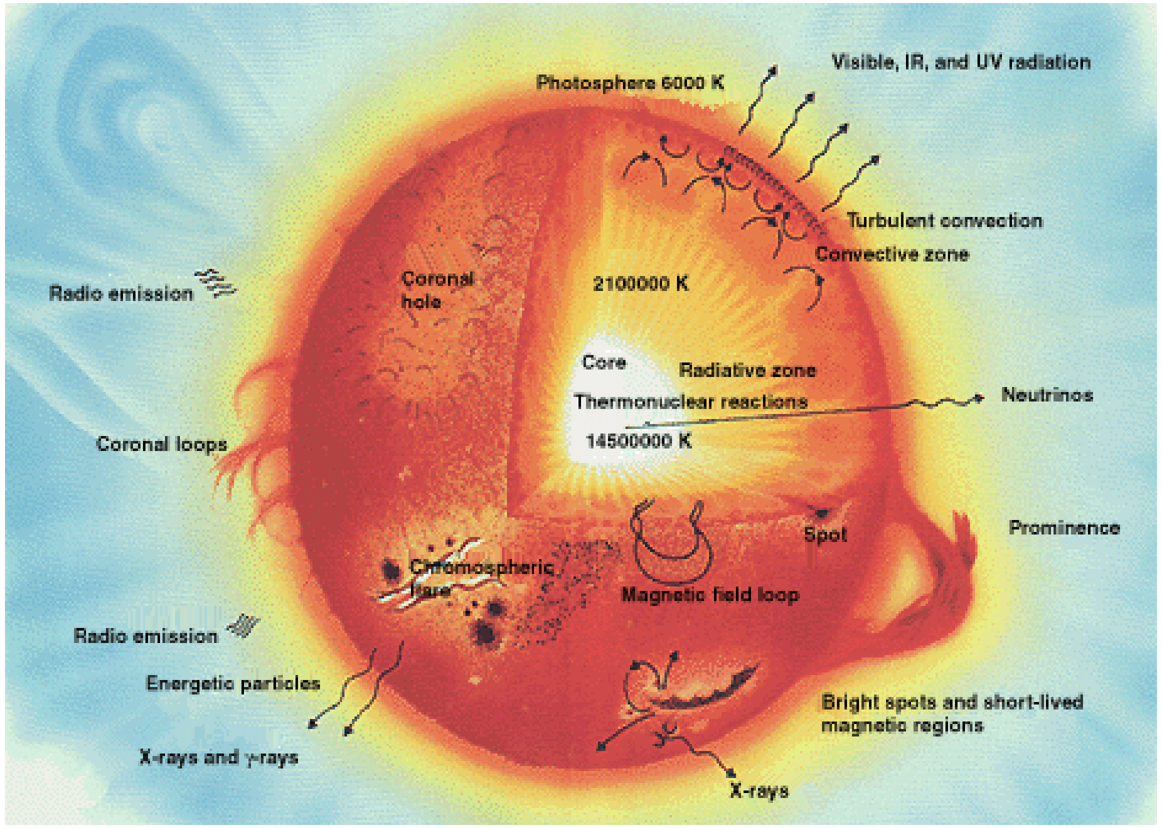
Today in Astronomy 241:

- Equations of state
- Nuclear energy generation in stars
- The luminosity equation
- **Reading**: C&O pp. 296-311



Hydrostatic equilibrium and gravity

- Again, the equation of hydrostatic equilibrium in gravity, 3-D, spherical symmetry:

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}$$
$$= -\rho g_r$$

P: pressure; r : mass density;

M_r: mass contained in sphere of radius r;

$g_r = GM_r \big/ r^2$: local gravitational acceleration.

- Mass conservation in 3-D, spherical symmetry:

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

$$M_r = 4\pi \int_0^r \rho r'^2 dr'$$

- These are two members of the network of differential equations of stellar structure.

The pressure integral and equations of state

Equations of state are, for our current purposes, relations among *P, ρ,* and *T.*

- Given a system of particles with momentum distribution function n_p, its pressure is

$$P = \frac{1}{3}\int_0^\infty n_p p v \, dp$$

p: momentum; v: velocity;

$n_p dp$: number of particles with momentum between p and $p + dp$

From this, and statistical mechanics, any equation of state can be derived.

- Good for massive or massless particles, any speed.
- Nonrelativistic limit: $p = mv$, so

$$P = \frac{m}{3}\int_0^\infty n_v v^2 \, dv$$

Note that $n_p dp = n_v dv$ for non-relativistic particles, since $p \propto v$.

The Sun is nuclear powered.

As you are presumed to know, from AST 111 and 142:

- The solar system is known to be about 4.5×10^9 years old, from isotope dating of meteorites and of the oldest rocks found on the Moon. The Sun is at least as old as the solar system.
- Thus the Sun cannot be gravity powered; this would only last of order 10^7 years (example 10.3).
- Thus the Sun cannot be burning in the normal sense (chemical-energy powered); this would only last of order 10^5 years (problem 10.3).
- Thus the Sun is nuclear powered, by fusion of the nuclei of light elements (mostly hydrogen); this can last at least 10^{10} years (example 10.4).

Nuclear reaction cross sections

- The quantity that is needed in nuclear reaction calculations, and is measurable in the laboratory:

$$\sigma(E) = \frac{\text{number of reactions per target per unit time}}{\text{flux of incident particles per unit time}}$$

- Cross section for fusion of two nuclei (charges Z_1 and Z_2):

$$\sigma(E) = \frac{S(E)}{E} e^{-bE^{-1/2}}$$

where $S(E)$ varies little with varying energy, and where

$$b \equiv \frac{2^{3/2} \pi^2 \mu_m^{1/2} Z_1 Z_2 e^2}{h}$$

$\mu_m =$ reduced mass of nuclei

Nuclear reaction rates

- Reaction rate coefficient (reactions per unit time per unit volume) for particles *i* and target *x*:

$$r_{ix} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_i n_x}{(\mu_m \pi)^{1/2}} \int_0^\infty S(E) e^{-bE^{-1/2}} e^{-E/kT} dE$$

- The integrand, apart from *S(E)*, is sharply peaked at energy

$$E_0 = \left(\frac{bkT}{2}\right)^{2/3} \quad ,$$

which is called the **Gamow peak**; see problems 10.8-9 in C&O, or better yet [Lecture 6, ASTR 142].

Nuclear energy generation in stars

- *S(E)* usually expanded in a power series; result of integral parameterized as

$$r_{ix} \cong r_0 X_i X_x \rho^{\alpha'} T^{\beta}$$

where α' is usually 2 (two-body collisions) and β can vary over a wide range. The energy generation rate for a given reaction is

$$\varepsilon_{ix} = \left(\frac{\varepsilon_0}{\rho}\right) \cong \varepsilon_0' X_i X_x \rho^{\alpha} T^{\beta}$$

where $\alpha = \alpha'$-1, and the **total energy generation rate ε** is the sum of these expressions for all reactions.

The luminosity equation

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon$$

- where L_r is the luminosity generated within the star between the center and radius r, and ρ and ε are the mass density and total energy generation rate at radius r.
- This is the fourth of the important equations that must be solved simultaneously to determine the interior structure of a star, along with hydrostatic equilibrium, mass conservation, and the equation of state.

Today's in-class problems

1. 10.5: in which one shows how the simplest equation of state – the ideal gas law – arises from the pressure integral and the Maxwell-Boltzmann distribution, $n_v dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} 4\pi v^2 dv$.

   - Hint: to integrate, over *v,* powers of *v* times gaussians, consider instead the square of the same integral, breaking the square into two integrals in which *v* is called (say) *x* in one and *y* in the other:

$$\left(\int_0^\infty v^n e^{-av^2} dv\right)^2 = \int_0^\infty \int_0^\infty x^n y^n e^{-a\left(x^2+y^2\right)} dxdy \quad .$$

   Then change to polar coordinates from *x* and *y.*

Today’s in-class problems (continued)

2. Recall that $p = h\nu/c = h/\lambda$. Show that the momentum distribution function for photons emitted by a blackbody is

$$n_p dp = \frac{8\pi p^2}{h^3}\frac{1}{e^{pc/kT}-1}dp \quad ,$$

   and from this and the pressure integral re-derive the radiation pressure from a blackbody.

3. Review the derivation of the energy of the Gamow peak from the ASTR 142 notes, bringing up to speed your colleagues who haven’t taken that class.

Hints for the last set of in-class problems

1. If gas pressure were all there is, then the pressure should decrease to zero at the edge of the star; thus in this case $C = 0$. If there's radiation pressure, the total pressure should approach the radiation pressure at the edge of the star. This in turn is given in the Eddington approximation as

$$C = P_{rad}\left(\text{edge}\right) = \frac{2\pi}{3c}\left(I_{out} + I_{in}\right) = \frac{2\pi}{3c}\left(\frac{\sigma T_e^4}{\pi} + 0\right) = \frac{2\sigma T_e^4}{3c} \quad .$$

2. As you know from, e.g., ASTR 111,

$$\frac{dP}{dz} = -\rho g = -\frac{\mu m P}{kT} g \quad \Rightarrow \quad \int_{P_0}^{P} \frac{dP'}{P'} = -\frac{\mu m g}{kT} \int_0^z dz' \quad \text{, or}$$

$$\ln\left(\frac{P}{P_0}\right) = -\frac{\mu m g z}{kT} \quad \Rightarrow \quad P = P_0 \exp\left(-\frac{\mu m g z}{kT}\right) = P_0 \exp\left(-\frac{z}{z_0}\right).$$