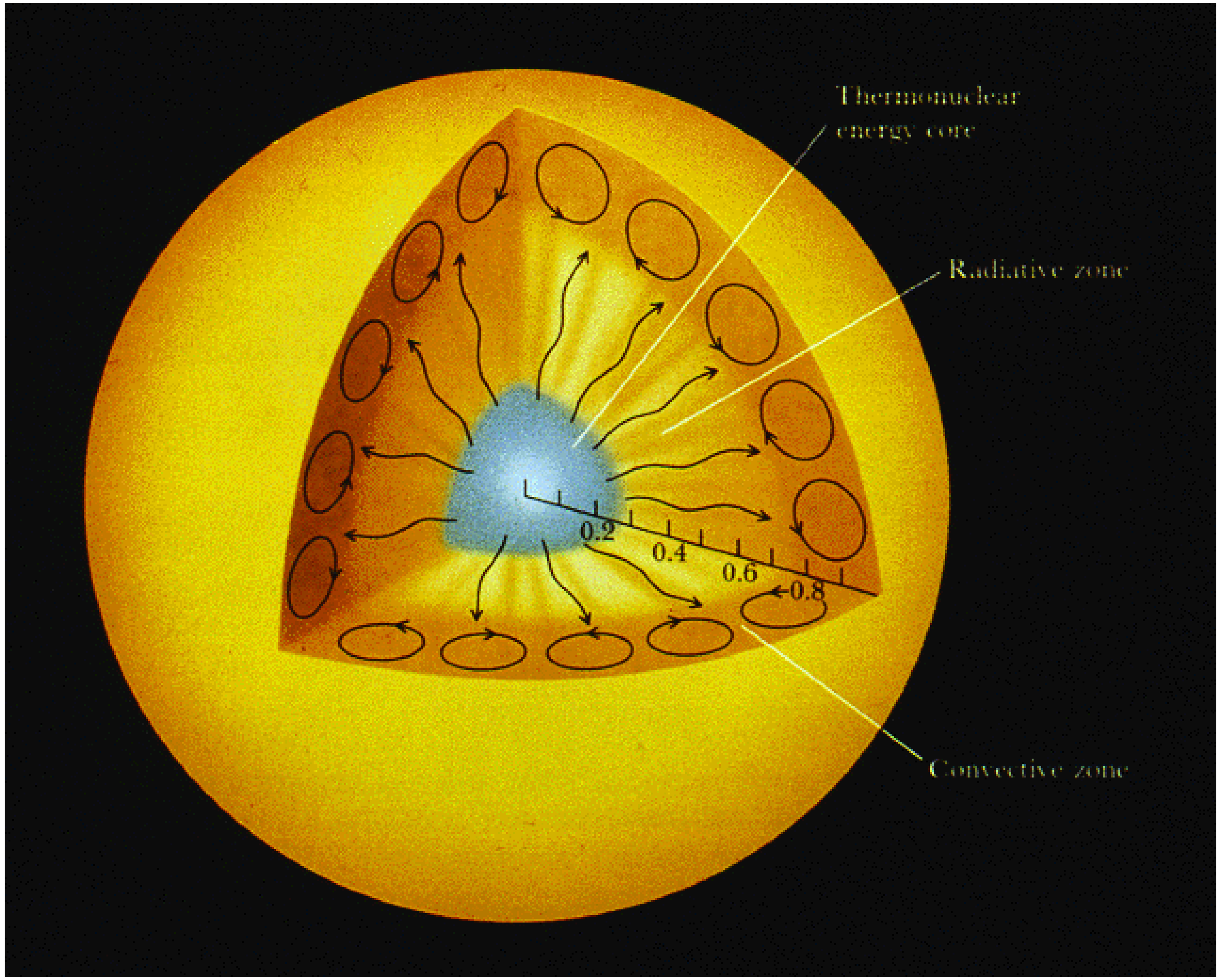
Today in Astronomy 241: energy transport by radiation

- One thing I forgot last time about normal fusion processes
- Energy transport by radiation
- **Reading**: C&O pp. 311-317



Catalytic fusion processes: the CNO Bi-Cycle

$$^{12}_{6}\mathrm{C}+^{1}_{1}\mathrm{H}\rightarrow ^{13}_{7}\mathrm{N}+\gamma$$

$$^{13}_{7}\mathrm{N}\rightarrow ^{13}_{6}\mathrm{C}+e^{+}+\nu_{e}$$

$$^{13}_{6}\mathrm{C}+^{1}_{1}\mathrm{H}\rightarrow ^{14}_{7}\mathrm{N}+\gamma$$

$$^{14}_{7}\mathrm{N}+^{1}_{1}\mathrm{H}\rightarrow ^{15}_{8}\mathrm{O}+\gamma$$

$$^{15}_{8}\mathrm{O}\rightarrow ^{15}_{7}\mathrm{N}+e^{+}+\nu_{e}$$

$$^{15}_{7}\mathrm{N}+^{1}_{1}\mathrm{H}\rightarrow ^{12}_{6}\mathrm{C}+^{4}_{2}\mathrm{He}\quad (99.96\%)$$

$$\rightarrow ^{16}_{8}\mathrm{O}+\gamma\quad (0.04\%)$$

$$^{16}_{8}\mathrm{O}+^{1}_{1}\mathrm{H}\rightarrow ^{17}_{9}\mathrm{F}+\gamma$$

$$^{17}_{9}\mathrm{F}\rightarrow ^{17}_{8}\mathrm{O}+e^{+}+\nu_{e}$$

$$^{17}_{8}\mathrm{O}+^{1}_{1}\mathrm{H}\rightarrow ^{14}_{7}\mathrm{N}+^{4}_{2}\mathrm{He}$$

$$\varepsilon_{CNO}\cong\rho X X_{CNO} T_6^{19.9}$$

$$\times 8.24\times 10^{-24}\ \mathrm{erg\ cm^{3}\ gm^{-2}\ s^{-1}}$$

where X_{CNO} is the mass fraction of *C*, *N* and *O* combined, and T_6 = $T/10^6$ K.

Note how steep the temperature dependence is.

Energy transport by radiation

The fifth of the network of equations to solve for a star's structure: the radiative energy transport equation.

$$\frac{dT}{dr} = -\frac{3}{16\sigma}\frac{\bar{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2}$$

L_r: luminosity produced between the center and radius r;

$\bar{\kappa}$: Rosseland mean opacity.

- Radiation and convection (discussed next time) are the most important transport mechanisms in stars.
- Heat conduction by diffusion of atoms can be accommodated in this expression, by including another mean free path like the one whose inverse is the Rosseland mean opacity, but it is usually unnecessary to account for this process.

Today's in-class problems

1. (A) The flux received by an observer from a gray atmosphere is $F_{\nu,gray} = \pi B_\nu\left(T\left(\bar{\tau}\right)\right)$, where $T^4\left(\tau\right) = \frac{3}{4}T_e^4\left(\tau + \bar{\tau}\right)$, and $\bar{\tau} = 2/3$.

   Show that if the opacity at frequency ν, κ_ν, differs little from the frequency-averaged mean opacity $\bar{\kappa}$, and if both are independent of stellar radius, the flux received at frequency ν is

$$F_\nu \cong \pi B_\nu\left(T\left(\bar{\tau}\right)\right) + \frac{\pi T_e}{8}\left(\frac{\bar{\kappa}}{\kappa_\nu} - 1\right)\frac{dB_\nu}{dT}(T\left(\bar{\tau}\right)) \quad .$$

   Hint: expand in a Taylor series, and use the chain rule.

Today's in-class problems (continued)

2. (B) The mean opacity $\overline{\kappa}$ deep in a stellar interior should obey

$$F = \int_0^\infty F_\nu d\nu = \int_0^\infty B_\nu\left(T\left(\tau_\nu\right)\right) d\nu = \int_0^\infty B_\nu\left(T\left(\overline{\tau}\right)\right) d\nu \quad .$$

Show from this, and the result of problem A, that the appropriate (Rosseland) mean opacity is given by

$$\frac{1}{\overline{\kappa}} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} d\nu}{\int_0^\infty \frac{dB_\nu}{dT} d\nu} \quad .$$

Hints for the last set of in-class problems

1. We're supposed to derive the ideal gas law from the pressure integral and the Maxwell-Boltzmann distribution. Ignoring relativity, we write the pressure integral as

$$P = \frac{m}{3}\int_0^\infty n_v v^2 dv = \frac{4\pi n m}{3}\left(\frac{m}{2\pi kT}\right)^{3/2}\int_0^\infty v^4 e^{-mv^2/2kT} dv \quad .$$

Abbreviate $a = m/2kT$, and instead of considering the integral $I = \int_0^\infty v^4 e^{-av^2} dv$ directly, calculate

$$I^2 = \left(\int_0^\infty v^4 e^{-av^2} dv\right)^2 = \int_0^\infty\int_0^\infty x^4 y^4 e^{-a\left(x^2+y^2\right)} dx dy \quad .$$

Hints for the last set of in-class problems (continued)

Do this by changing to polar coordinates:

$$x = r\cos\varphi,\ \ y = \rho\sin\varphi,\ \ dxdy = \rho d\rho d\varphi;$$

$$x,\, y = 0 - \infty,\quad 0 \le r < \infty,\ \ 0 \le \varphi \le \pi/2:$$

$$I^2 = \int_0^{\pi/2} d\varphi \cos^4\varphi \sin^4\varphi \int_0^{\infty} r^9 e^{-ar^2} dr \quad ,$$

which has the great virtue of an odd number of factors of *r* in the integrand. This enables another substitution:

$$u = ar^2,\ \ d\rho = 2ardr,\ \ 0 \le u < \infty;$$

$$\cos\varphi\sin\varphi = \frac{1}{2}\sin 2\varphi,\ \ \vartheta = 2\varphi,\ \ d\vartheta = 2d\varphi,\ \ 0 \le \vartheta < \pi;$$

$$I^2 = \frac{1}{2^5}\int_0^{\pi} d\vartheta \sin^4\vartheta \ \ \frac{1}{2a^5}\int_0^{\infty} u^4 e^{-u} du \quad .$$

Hints for the last set of in-class problems (continued)

You have done the *u* integral many times; integrate it by parts four times, and what's left is

$$I^2 = \frac{4!}{2^6 a^5}\int_0^\pi d\vartheta \sin^4\vartheta \quad .$$

π is half a cycle for both sin and cos, so

$$\int_0^\pi d\vartheta \sin^4\vartheta = \int_0^\pi d\vartheta \cos^4\vartheta = \frac{1}{2}\int_0^\pi d\vartheta\left(\sin^4\vartheta + \cos^4\vartheta\right)$$

$$= \frac{1}{2}\int_0^\pi d\vartheta\left(\sin^4\vartheta + 2\sin^2\vartheta\cos^2\vartheta + \cos^4\vartheta - 2\sin^2\vartheta\cos^2\vartheta\right)$$

$$= \frac{1}{2}\int_0^\pi d\vartheta\left(\left[\sin^2\vartheta + \cos^2\vartheta\right]^2 - 2\sin^2\vartheta\cos^2\vartheta\right)$$

$$= \frac{\pi}{2} - \frac{1}{4}\int_0^\pi d\vartheta \sin^2 2\vartheta = \frac{\pi}{2} - \frac{1}{8}\int_0^{2\pi} d\psi \sin^2\psi$$

$$= \frac{\pi}{2} - \frac{1}{16}\int_0^{2\pi} d\psi\left(\sin^2\psi + \cos^2\psi\right) = \frac{3\pi}{8} \quad .$$

Hints for the last set of in-class problems (continued)

Thus,

$$I^2 = \frac{4!}{2^6 a^5}\frac{3\pi}{8} = \frac{3^2}{2^6 a^4}\frac{\pi}{a}$$

$$I = \frac{3}{8a^2}\sqrt{\frac{\pi}{a}} = \frac{3}{8}\left(\frac{2kT}{m}\right)^2\sqrt{\frac{2\pi kT}{m}} = \frac{3}{8\pi^2}\left(\frac{2\pi kT}{m}\right)^{5/2}$$, and

$$P = \frac{4\pi nm}{3}\left(\frac{m}{2\pi kT}\right)^{3/2}\frac{3}{8\pi^2}\left(\frac{2\pi kT}{m}\right)^{5/2}$$

$$= \frac{nm}{2\pi}\frac{2\pi kT}{m} = nkT$$, q.e.d.

I never noticed before, but this problem is also in the book (10.5).

Hints for the last set of in-class problems (continued)

2. To derive blackbody radiation pressure from the pressure integral and the momentum distribution of photons in blackbody radiation, start with the fully-relativistic form of the integral and take the speed of the particle to be *c*:

$$P = \frac{c}{3}\int_0^\infty n_p p\,dp = \frac{8\pi c}{3h^3}\int_0^\infty \frac{p^3}{e^{pc/kT}-1}dp$$

$$= \frac{8\pi c}{3h^3}\left(\frac{kT}{c}\right)^4 \int_0^\infty \frac{u^3}{e^u-1}du = \frac{8\pi c}{3h^3}\left(\frac{kT}{c}\right)^4 \frac{\pi^4}{15}$$

$$= \frac{4}{3c}\left(\frac{2\pi^5 k^4}{15c^2h^3}\right)T^4 = \frac{4\sigma}{3c}T^4 \quad \text{, q.e.d.}$$