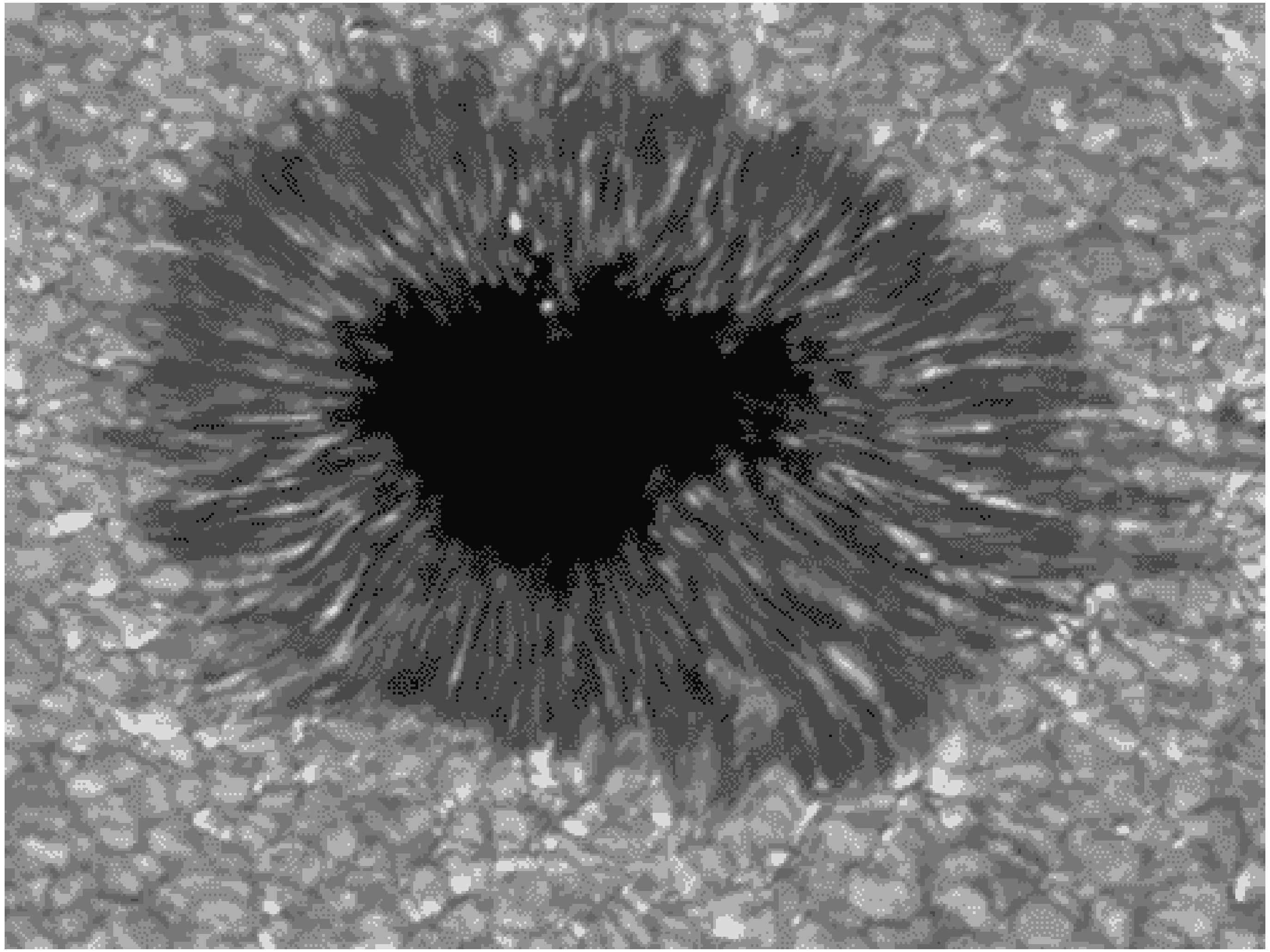
Today in Astronomy 241: convection

- Quick thermodynamics
- Adiabatic processes
- Convective instability and convective energy transport
- **Reading**: C&O pp. 317-329



Peter Sütterlin, U. Utrecht/Dutch Open Telescope.

Quick thermodynamics

$$dU = dQ - dW$$

$$= dQ - PdV$$

First law of thermodynamics (energy is conserved); dU = internal energy change, dQ = heat flowing into system, dW = mechanical work done by system on its surroundings.

$$U = \frac{3}{2}NkT = \frac{3}{2}nRT$$

Internal energy of system with N monoatomic particles (n moles) of mass m.

$$C_V = \left.\frac{dQ}{dT}\right|_V = \frac{3}{2}Nk \qquad C_P = \left.\frac{dQ}{dT}\right|_P = C_V + Nk$$

Heat capacity at
constant volume or pressure.

$$c_V = \frac{1}{N}C_V = \frac{3}{2}k \quad c_P = \frac{1}{N}C_P = \frac{5}{2}k \quad \gamma \equiv \frac{c_P}{c_V} = \frac{5}{3}$$

Specific heats and their ratio.

Quick thermodynamics (continued)

For adiabatic processes ($dQ = 0$),

$$PV^{\gamma} = K = \text{constant}$$

$$PT^{-\frac{\gamma}{\gamma\text{-}1}} = K' = \text{constant}$$

$$v_s = \sqrt{\frac{\gamma P}{\rho}}$$

Adiabatic sound speed

Convection

$$\nabla_{ad} \equiv \left.\frac{dT}{dr}\right|_{ad} = \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$

$$= -\left(1-\frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2} = -\frac{g}{C_P}$$

Adiabatic temperature gradient

$$\frac{dT}{dr} > \nabla_{ad} = \left.\frac{dT}{dr}\right|_{ad}$$

or, equivalently,

$$\frac{d(\ln P)}{d(\ln T)} = \frac{T}{P}\frac{dP}{dT} < \frac{\gamma}{\gamma - 1}$$

Condition for convective instability: if the actual temperature gradient is superadiabatic, even slightly, convection will occur.

Convective energy flux and mixing length

The idea of convective energy transport:

- Due to a random density fluctuation, a bubble of gas finds itself warmer and less dense than its surroundings, though in pressure balance with its surroundings. Buoyancy pushes it upward.

- **The temperature gradient in the surroundings is such that this rise makes the bubble expand against the ambient pressure and become even more buoyant.** (This is the convective instability.)

- As buoyancy is a force, the upward motion of the bubble accelerates. Very quickly it can rise a long way, expanding against the (decreasing) ambient pressure, until it merges with its surroundings and its heat is deposited at the upper level.

- So the bubble's heat is transported to larger radius, **much** faster than diffusion (heat conduction) would do.

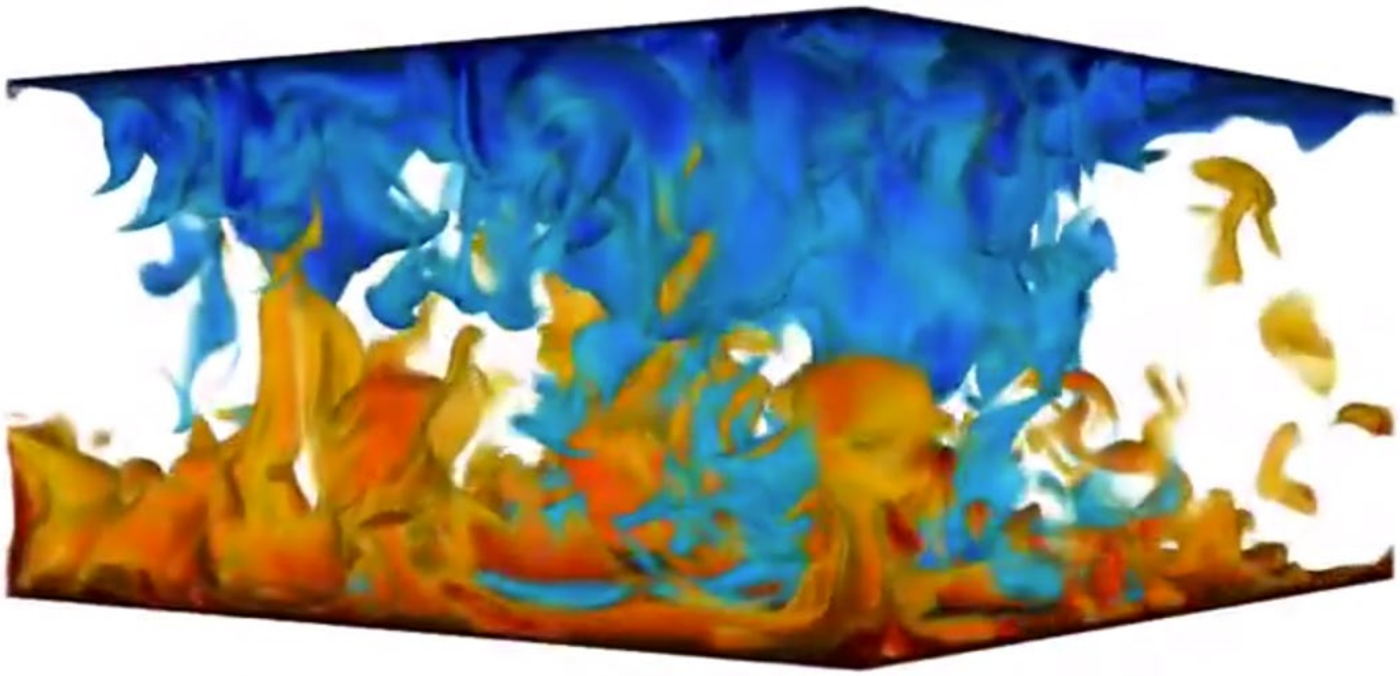
- The distance ℓ which it travels in this rise from formation to dissipation, called the **mixing length**, is usually treated as a fitting parameter, as there is no good way to calculate it.

  - Many people use the local value of the isothermal pressure scale height as an initial guess for mixing length:

Convective energy flux and mixing length (continued)

Simulation of convection between two uniform-temperature planes, the lower one warmer. From the [Physics of Fluids group, U. Twente](#).

- The planes are evidently separated by approximately one mixing length.
- Contrast this, and the case of stellar and planetary interiors, to your own experience with convection based on boiling pots of water.



Convective energy flux and mixing length (continued)

- Frequently-encountered terminology: δ represents a difference between a quantity in a bubble and its surroundings.
  - e.g. $\delta\rho = \rho_{\text{bubble}} - \rho_{\text{surroundings}} \equiv \rho_{\text{ad}} - \rho_{\text{act}} \quad < 0$ for rising bubbles,

    $\delta\left(\dfrac{dT}{dr}\right) = \nabla_{\text{ad}} - \left.\dfrac{dT}{dr}\right|_{\text{act}} \quad > 0$ for rising bubbles.

- Mixing length often written as $\ell \approx \alpha H_P = \dfrac{\alpha P}{\rho g} = \dfrac{\alpha k T}{\mu m_H g}$, α a dimensionless scale factor.
- Average kinetic energy of the bubble often written as $\dfrac{K}{\rho V} = \beta v^2$, $0 < \beta < 1$.
- In these terms, involving details you are about to work through,

$$F_c \cong \alpha^2 C_P \rho \left(\frac{k}{\mu m_{\text{H}}}\right)^2 \left(\frac{T}{g}\right)^{\frac{3}{2}} \beta^{\frac{1}{2}} \left(\delta\left(\frac{dT}{dr}\right)\right)^{\frac{3}{2}}$$

Convective heat flux

Today’s in-class problems

1. Starting with the first law of thermodynamics, fill in the details of the derivation of Equations 10.81 and 10.82 (see page 320).

2. Fill in the details in the derivation of Equation 10.99 from 10.97 and 10.98 (see pp. 327-328).

Hints for the last set of in-class problems

1. One can start either with the expression for flux or that for temperature, and expand in a Taylor series in about the point $\tau_\nu = \bar{\tau}$; both paths lead to the same place.

$$F_\nu = \pi B_\nu(T)\Big|_{T(\bar{\tau})} + \pi \frac{dB_\nu}{d\tau_\nu}\Bigg|_{T(\bar{\tau})} (\bar{\tau} - \tau_\nu) + O\left((\bar{\tau} - \tau_\nu)^2\right)$$

$$\cong \pi B_\nu(T)\Big|_{T(\bar{\tau})} + \pi \frac{dB_\nu}{dT}\frac{dT}{d\tau_\nu}\Bigg|_{T(\bar{\tau})} (\bar{\tau} - \tau_\nu) \quad .$$

Assume that the opacities are uniform – i.e. independent of depth – so that

$$\frac{\tau_\nu}{\bar{\tau}} = \frac{\kappa_\nu}{\bar{\kappa}} \quad \text{, and}$$

$$(\bar{\tau} - \tau_\nu) = \tau_\nu\left(\frac{\bar{\kappa}}{\kappa_\nu} - 1\right) = \frac{2}{3}\left(\frac{\bar{\kappa}}{\kappa_\nu} - 1\right) \quad \text{at the photosphere.}$$

Hints for the last set of in-class problems (continued)

The derivative of the gray temperature distribution comes out to

$$\frac{dT}{d\tau} = T_e \left(\frac{3}{4}\right)^{1/4} \frac{1}{4} \left(\tau + \overline{\tau}\right)^{-3/4}$$

$$\left.\frac{dT}{d\tau}\right|_{T(\overline{\tau})} = T_e \left(\frac{3}{4}\right)^{1/4} \frac{1}{4} \left(2\overline{\tau}\right)^{-3/4} = T_e \left(\frac{3}{4}\right)^{1/4} \frac{1}{4} \left(\frac{3}{4}\right)^{3/4}$$

$$= \frac{3T_e}{16} \quad \text{at the photosphere.}$$

Thus

$$F_\nu \cong \pi B_\nu\left(T\left(\overline{\tau}\right)\right) + \frac{\pi T_e}{8} \left(\frac{\overline{\kappa}}{\kappa_\nu} - 1\right) \frac{dB_\nu}{dT} \left(T\left(\overline{\tau}\right)\right) \quad \text{, q.e.d.}$$

Hints for the last set of in-class problems (continued)

2. The total flux, integrated over all frequencies and making the usual assumption of local thermal equilibrium, is

$$F = \int_0^\infty F_\nu d\nu = \pi \int_0^\infty B_\nu\left(T\left(\tau_\nu\right)\right) d\nu$$

By definition the frequency-averaged mean optical depth $\bar{\tau}$ is such that

$$\int_0^\infty B_\nu\left(T\left(\tau_\nu\right)\right) d\nu = \int_0^\infty B_\nu\left(T\left(\bar{\tau}\right)\right) d\nu$$

Thus, per problem 1,

$$\cancel{\int_0^\infty B_\nu\left(T\left(\tau_\nu\right)\right) d\nu} = \cancel{\int_0^\infty B_\nu\left(T\left(\bar{\tau}\right)\right) d\nu} + \frac{T_e}{8} \int_0^\infty \left(\frac{\bar{\kappa}}{\kappa_\nu} - 1 \right) \frac{dB_\nu}{dT} (T(\bar{\tau})) d\nu$$

Hints for the last set of in-class problems (continued)

or

$$\int_0^\infty \frac{\overline{\kappa}}{\kappa_\nu} \frac{dB_\nu}{dT} d\nu = \int_0^\infty \frac{dB_\nu}{dT} d\nu \quad ,$$

or even

$$\frac{1}{\overline{\kappa}} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} d\nu}{\int_0^\infty \frac{dB_\nu}{dT} d\nu} \quad , \quad \text{q.e.d.}$$