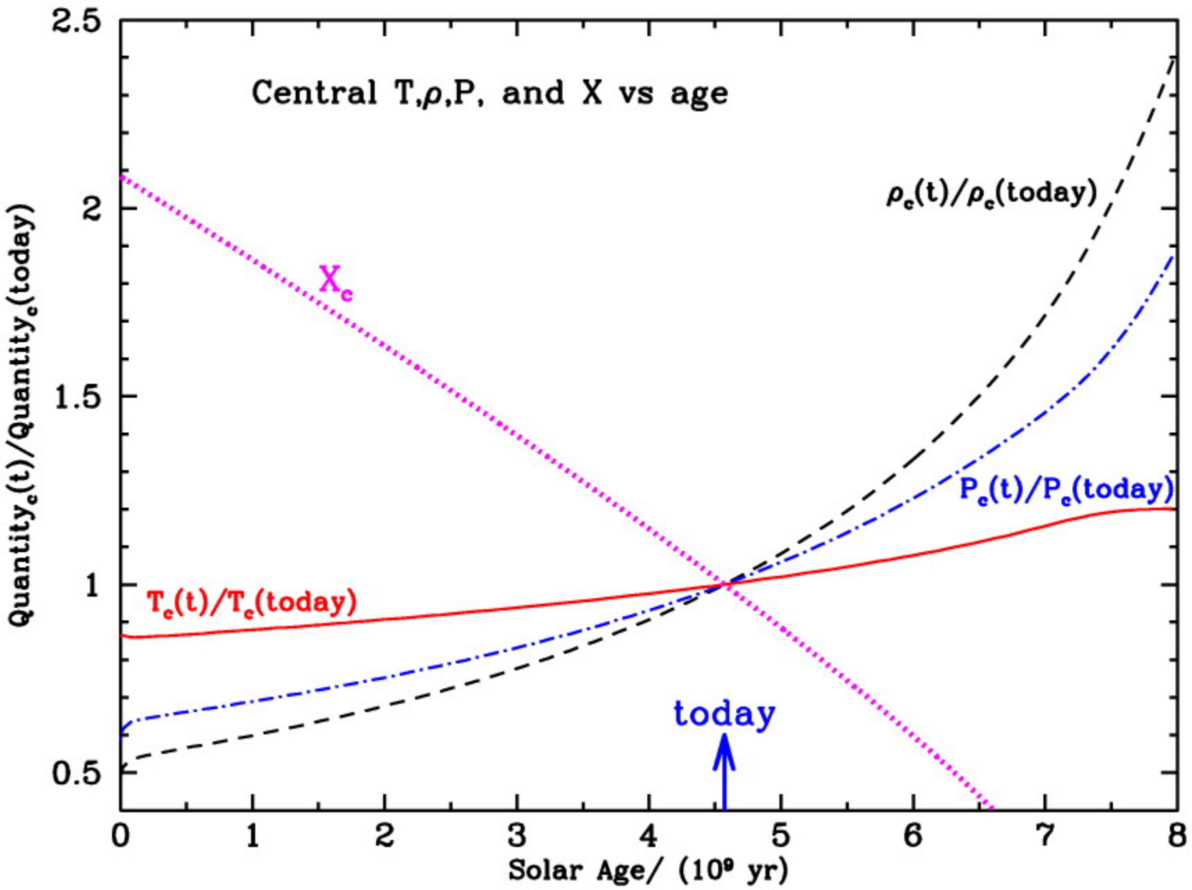
Today in Astronomy 241: model stellar interiors

- Summary of the equations of stellar structure
- There are no analytical solutions to the network of equations
- Methods of solution of the equations to produce a stellar structure:
  - Cheat on one equation
  - Cheat more blatantly but more productively: polytropes
  - MESA
- **Reading**: C&O pp. 329-335



Time-dependent structural models: Bahcall+2001

The equations of stellar structure, and the problem to solve

Hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}$$

Mass conservation:

$$\frac{dM_r}{dr} = 4\pi r^2\rho$$

Energy generation:

$$\frac{dL_r}{dr} = 4\pi r^2\rho\varepsilon$$

Radiative energy transport:

$$\frac{dT}{dr} = -\frac{3}{16\sigma}\frac{\overline{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2}$$

Adiabatic temperature gradient:

$$\left(\frac{dT}{dr}\right)_{\mathrm{ad}} = -\left(1-\frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2}$$

Convective instability if:

$$\frac{T}{P}\frac{dP}{dT} < \frac{\gamma}{\gamma-1}$$

Given M and initial composition (X, Y, Z), solve for $P(r)$, $\rho(r)$, $T(r)$, R_{star} and L_{star}, as functions of time if desired.

Constitutive relations

Pressure equation of state:

$$P = P(\rho, T, \text{composition}) \text{ in general}$$

$$= \frac{\rho k T}{\mu m_H} + \frac{4\sigma T^4}{3c} \quad \text{throughout most normal stars}$$

Rosseland mean opacity:

$$\bar{\kappa} = \bar{\kappa}(\rho, T, \text{composition}) \text{ in general}$$

$$= 0.2(1+X) \text{ cm}^2 \text{ gm}^{-1} \quad \text{electron scattering}$$

$$= 3.68 \times 10^{22} g_{ff} (1-Z)(1+X) \frac{\rho}{T^{3.5}} \text{ cm}^2 \text{ gm}^{-1} \quad \text{free-free}$$

$$= 4.34 \times 10^{25} \frac{g_{bf}}{t} Z(1+X) \frac{\rho}{T^{3.5}} \text{ cm}^2 \text{ gm}^{-1} \quad \text{bound-free}$$

Constitutive relations (continued)

Nuclear energy generation: $\varepsilon = \varepsilon\,(\rho, T, \text{composition})$ in general

$$\cong 2.38 \times 10^6 \rho X^2 T_6^{-2/3} e^{-33.80 T_6^{-1/3}} \text{ erg gm}^{-1}\text{ s}^{-1} \qquad \text{p-p}$$

$$\cong 8.67 \times 10^{27} \rho X X_{CNO} T_6^{-2/3} e^{-152.28 T_6^{-1/3}} \text{ erg gm}^{-1}\text{ s}^{-1} \qquad \text{CNO}$$

$$\cong 5.09 \times 10^{11} \rho^2 Y^2 T_8^{-3} e^{-44.027 T_8^{-1}} \text{ erg gm}^{-1}\text{ s}^{-1} \qquad 3\alpha$$

Convective energy flux, in mixing-length theory:

$$F_c = \alpha^2 \rho C_P \left(\frac{k}{\mu m_{\mathrm{H}}} \right)^2 \left(\frac{T}{g} \right)^{3/2} \beta^{1/2} \left[\delta \left(\frac{dT}{dr} \right) \right]^{3/2}$$

Boundary conditions

$$\left.\begin{array}{l} M_r \to 0 \\ L_r \to 0 \end{array}\right\} \text{ as } r \to 0$$

No singularities.

$$\left.\begin{array}{l} T \to 0 \\ P \to 0 \\ \rho \to 0 \end{array}\right\} \text{ as } r \to R_{\text{star}} \ \ldots$$

… or match to solution for atmosphere.

Usually bookkept with the boundary conditions is the **Vogt-Russell “theorem:”** the mass and composition of a star uniquely determine its radius, luminosity, internal structure, and subsequent evolution.

- Not a theorem, as it’s never been proven, and most of us think it can’t be proven. But it’s often assumed.
- Maybe better to think of it as a wish, or a hope.

There are no general, analytical solutions to these equations.

But that doesn't mean we get to go home now. There are three tactics which can be used to build your modelling skills:

- **Cheat**. Assume a solution to one of the equations; there are choices which yield analytical solutions for the rest of the network.
    - We will explore one of these in the in-class problems today.
- Cheat even worse, but more productively: the **polytrope**.
    - Most of the trouble is the link between the energy-generation and energy-transport equations.
    - If we ignore them, and instead suppose an adiabatic equation of state $-\ P = K\rho^{\gamma} = K\rho^{1+1/n}\ -$ then we get a network with analytical solutions for three values of n – 0, 1, and 5 – and straightforward numerical solutions for other values. Such models are called **polytropes**, after the Greek for **multifaceted**, or **versatile**.
        - In the Odyssey, Odysseus is normally referred to as *polytropos Odysseus*.
    - We will explore polytropes in the homework this week and next.

There are no general, analytical solutions to these equations (continued).

- Give up, and calculate stellar structures by **numerical integration** of the network.
  - Because of mixed boundary conditions, start either from the star's center or edge, and integrate to the other bound; adjust and iterate til the boundary conditions are satisfied.
    - This is generally called a **shooting method**.
  - We will take this up next week and keep it up for most of the rest of the semester, while learning simpler concepts on the side.
  - C&O offer a simple program called **StatStar** which does this, and which was used in previous editions of ASTR 241.
    - In FORTRAN here or there, in python 2 here, in python 3 over here. "Here" is where Michael Richmond (RIT) has gathered these resources, for RIT's PHYS 370.
  - Instead we will use the Modules for Experiments in Stellar Astrophysics (**MESA**), mostly via the MESA-Web implementation by Rich Townsend (University of Wisconsin at Madison).

Today's in-class problems

Assume that the run of density in a star is $\rho(r) = \rho_0 \left(1 - r/R\right)$.

A. Integrate the equation of mass conservation, thereby obtaining expressions for mass as a function of radius, M_r, and the central density, ρ_0.

   It will be convenient to leave ρ_0 in the equations, and to use $x = r/R$ as the independent variable.

B. Then integrate the equation of hydrostatic equilibrium, to get the pressure as a function of radius, and the central pressure.

C. Then use the ideal gas law to get the temperature as a function of radius, and the central temperature. Note that

$$5 - 24x^2 + 28x^3 - 9x^4 = (1-x)^2 \left(5 + 10x - 9x^2\right) \quad .$$

D. Assume that radiative energy transport dominates, and that a Kramers law can be used for the opacity, to obtain an expression for dT/dr. Evaluate it at $r = R/2$.

Today's in-class problems (continued)

E. Then produce an expression for dT/dr from the result of problem C, similarly evaluated at $r = R/2$.

F. Set equal the results from problems D and E, and obtain an expression for the luminosity generated within $r = R/2$ – which, because of the strong temperature dependence of energy generation, should be equal to the total luminosity of the star.

Hints for the last set of in-class problems

1. Done in class in ASTR 111, e.g. on 31 October 2023. The differential equation can be integrated directly – both sides integrate to natural logs – and all you need to do is choose a convenient integration constant. I called mine ln(*K*).

2. Mostly this is relatively simple algebra and substitution; the only feature by which people seemed confused is the introduction of β:

$$\overline{v}_c^2 = \beta v_f^2 = \beta \frac{2\langle f_{\mathrm{net}}\rangle \ell}{\rho} \quad .$$