Today in Astronomy 241: polytropes

- Adiabatic EOS
- The Lane-Emden equation
- Solving the LE equation analytically
- Solving the LE equation numerically
- **Reading**: C&O pp. 334-340

ἄνδρα μοι ἔννεπε, μοῦσα, **πολύτροπον**, ὃς μάλα πολλὰ
πλάγχθη, ἐπεὶ Τροίης ἱερὸν πτολίεθρον ἔπερσεν:
πολλῶν δ᾽ ἀνθρώπων ἴδεν ἄστεα καὶ νόον ἔγνω,
πολλὰ δ᾽ ὅ γ᾽ ἐν πόντῳ πάθεν ἄλγεα ὃν κατὰ θυμόν,
ἀρνύμενος ἥν τε ψυχὴν καὶ νόστον ἑταίρων.

Sing, goddess, **of that resourceful man**, who was driven
far journeys after he had sacked Troy's sacred citadel.
Many were they whose cities he saw, whose minds he learned of;
many were the pains he suffered in his spirit on the wide sea,
struggling for his own life and the homecoming of his companions.

The first five lines of Homer's *Odyssey*

Adiabatic equation of state

As we have seen, pressure, density, and temperature are related for adiabatic processes by

$$PV^{\gamma} = \text{constant}$$

$$\text{or} \quad P\rho^{-\gamma} = \text{constant}$$

$$\text{or } PT^{\gamma/(\gamma-1)} = \text{constant}$$

where
$$\gamma = \frac{C_P}{C_V} = \frac{n+1}{n} = 1 + \frac{1}{n}$$

- n is called the order number of the polytrope whose equation of state is one of the above expressions. Usually we will use $P = K\rho^{1+1/n}$, $K = \text{constant}$.

- Ideal monoatomic gases have n = 3/2; adiabatic processes in ideal monoatomic gases follow $P = K\rho^{5/3}$.

The Lane-Emden equation

The equations of hydrostatic equilibrium and mass conservation can be made into an equation reminiscent of the Poisson equation:

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2} \quad \Rightarrow \quad \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dM_r}{dr} = -G4\pi r^2\rho$$

$$P = K\rho^\gamma \quad \Rightarrow \quad \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = K\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d\rho}{dr}\frac{d}{d\rho}\rho^\gamma\right) = \gamma K\frac{d}{dr}\left(r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right)$$

$$\Rightarrow \quad \frac{n+1}{n}\frac{K}{r^2}\frac{d}{dr}\left(r^2\rho^{(1-n)/n}\frac{d\rho}{dr}\right) = -4\pi G\rho$$

Lane-Emden equation in real units

It is usually best, especially in numerical solutions, to express this in dimensionless form. Define dimensionless density and radius, D_n and ξ:

$$\frac{\rho(r)}{\rho_0} = \left(D_n(r)\right)^n \quad , \; 0 \le D_n \le 1 \qquad\qquad r = \xi\sqrt{\frac{(n+1)K\rho_C^{(1-n)/n}}{4\pi G}}$$

The Lane-Emden equation (continued)

and one gets, by substitution,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\left(D_n\right)}{d\xi}\right)=-\left(D_n\right)^n \quad .$$

Lane-Emden equation

- Boundary conditions: $D_n\left(0\right)=1,\ D_n'\left(0\right)=\frac{dD_n}{d\xi}\left(0\right)=0.$
- The LE equation yields analytical solutions for three polytrope orders: *n* = 0, 1, and 5.
- None of these are particularly interesting, physically, though we will solve them in class for practice.

Solving the LE equation numerically

- The physically interesting polytropes begin with
  - the monoatomic ideal gas and the nonrelativistic degenerate gas, both of which have $n = 3/2$, and
  - the ultrarelativistic degenerate gas, which has $n = 3$,

  both of which require numerical solution of LE.

The first thing to do to solve the LE equation is to take the second-order equation apart into two first-order equations, as the best differential-equation solvers, like the **Runge-Kutta** method, work only on first-order differential equations. Define a two-element vector δ, with elements

$$\delta_0 = D_n, \quad \delta_1 = \xi^2 \delta_0' \quad ,$$

whereupon we get the dimensionless forms of the hydrostatic equilibrium and mass-conservation equations:

$$\delta_0' = \frac{1}{\xi^2}\delta_1 \quad \text{and} \quad \delta_1' = -\xi^2 \left(\delta_0\right)^n \quad .$$

Solving the LE equation numerically (continued)

- The boundary conditions on δ descend directly from those on *D*: $\delta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at $\xi = 0$.
- These are the ingredients of common Runge-Kutta solvers, which take as their inputs the ξ = 0 values; the maximum value of ξ to explored, at which one expects $\delta_0 = 0$; the number of steps *N* between start and maximum value; and the vector of derivatives,

$$\delta' = \begin{pmatrix} \delta_1 / \xi^2 \\ -\xi^2 (\delta_0)^n \end{pmatrix} \quad .$$

- Most RK solvers return a matrix for which the columns are ξ, δ_0, and $\delta_1 = \xi^2 \, d\delta_0 / d\xi$, and the rows are the *N* steps in the integration between start and maximum.
- The shooting-method aspect of this type of solution is that one doesn't know the appropriate maximum value *a priori,* and to go past $\delta_0 = 0$ yields complex-valued solutions.
- You get to try this in this week's team homework.

Today's in-class problems: leftovers from last time (gray = completed)

Assume that the run of density in a star is $\rho(r) = \rho_0 \left(1 - r/R\right)$.

A. Integrate the equation of mass conservation, thereby obtaining expressions for mass as a function of radius, M_r, and the central density, ρ_0.

   It will be convenient to leave ρ_0 in the equations, and to use $x = r/R$ as the independent variable.

B. Then integrate the equation of hydrostatic equilibrium, to get the pressure as a function of radius, and the central pressure.

C. Then use the ideal gas law to get the temperature as a function of radius, and the central temperature. Note that

$$5 - 24x^2 + 28x^3 - 9x^4 = (1-x)^2\left(5 + 10x - 9x^2\right) \quad .$$

D. Assume that radiative energy transport dominates, and that a Kramers law can be used for the opacity, to obtain an expression for dT/dr. Evaluate it at $r = R/2$.

Today's in-class problems (continued)

E. Then produce an expression for dT/dr from the result of problem C, similarly evaluated at $r = R/2$.

F. Set equal the results from problems D and E, and obtain an expression for the luminosity generated within $r = R/2$ – which, because of the strong temperature dependence of energy generation, should be equal to the total luminosity of the star.

Hints for the last set of in-class problems

Answers, in convenient units:

A. $$\rho_C = 5.64\left(\frac{M}{1M_\odot}\right)\left(\frac{R_\odot}{R}\right)^3 \text{ gm cm}^{-3}$$

B. $$P = \frac{\pi}{36} G\rho_C^2 R^2 \left[5 - 24\left(\frac{r}{R}\right)^2 + 28\left(\frac{r}{R}\right)^3 - 9\left(\frac{r}{R}\right)^4\right]$$

$$P_C = 4.44\times10^{15}\left(\frac{M}{1M_\odot}\right)^2\left(\frac{R_\odot}{R}\right)^4 \text{ dyne cm}^{-2}$$

C. $$T = \frac{\pi}{36}\frac{G\mu m_\text{H}}{k}\rho_C R^2\left(1-\frac{r}{R}\right)\left[5 + 10\frac{r}{R} - 9\left(\frac{r}{R}\right)^2\right]$$

$$T_C = 9.62\times10^6\left(\frac{M}{1M_\odot}\right)\left(\frac{R_\odot}{R}\right)\text{K}$$

D. $$\left.\frac{dT}{dr}\right|_{r=R/2} = -\frac{3}{16\sigma}\frac{\kappa_0\rho_C^2}{4\pi R^2}\left(\frac{31\pi}{288}\frac{G\mu m_\text{H}}{k}\rho_C R^2\right)^{-6.5} L_{R/2}$$

E. $$\left.\frac{dT}{dr}\right|_{r=R/2} = -\frac{29\pi}{144}\frac{G\mu m_\text{H}}{k}\rho_C R$$

F. $$L_{R/2} = \frac{116\pi^3}{81}\left(\frac{31}{96}\right)^{6.5}\frac{\sigma}{\kappa_0}\left(\frac{Gm_\text{H}}{k}\right)^{7.5}\frac{\mu^{7.5}M^{5.5}}{R^{0.5}} = L$$

$$L = 1.2L_\odot\left(\frac{M}{M_\odot}\right)^{5.5}\left(\frac{R_\odot}{R}\right)^{0.5}$$