Today in Astronomy 241: degenerate stars I

- White dwarfs
- Degeneracy pressure
- The equation of state of degenerate electrons
- **Reading**: C&O chapter 16, pp. 557-572.

Color image of 40 Eridani B, the second white dwarf discovered, after Sirius B, and the first in an easily-resolved visual binary. By Dr. Robert Fealey of North Memorial Health Hospital and the Rochester (Minnesota) Astronomy Club. This is the best image of the 40 Eri system I've ever seen. I think I'll challenge next year's ASTR 244 class to match it, early some evening. Note 40 Eri B's blue color.

White dwarf stars

- Prototype: Sirius B. Mass = 1 $M_\odot$. It is hotter (27000 K vs. 9900 K), yet much less luminous (0.03 $L_\odot$ vs. 24 $L_\odot$), than its A1 V companion Sirius A, and therefore must be vastly smaller (0.008 $R_\odot$, smaller than the Earth).
- They come in all colors, not just white. Spectral types:
  - DA - only pressure-broadened hydrogen absorption lines in spectrum. Most numerous type. (Sirius B = DA.)
    - ZZ Ceti (or DAV) stars: DA white dwarfs on continuation of instability strip, executing nonradial pulsation.
  - DB - no hydrogen lines, only helium lines.
  - DC - no lines at all, just a featureless continuum.

White dwarf stars (continued)

- Estimate of central pressure and temperature of Sirius B, using equations of stellar structure and electron-scattering opacity from its observed size and mass: $P = 4\times10^{23}$ dyne cm^{-2}, $T = 7.6\times10^{7}$ K.

- Thus there cannot be any hydrogen inside - it would undergo rapid p-p chain and CNO fusion at this temperature and produce a much larger luminosity.

  - With no heat generation in the core, gas pressure cannot be holding it up. But as victims of ASTR 142 will recall, …

  - **degeneracy pressure** from electrons can support the weight.

Degenerate electron gas, as (most of) you have derived in PHYS 227

$$n_e = \frac{Z}{A}\frac{\rho}{m_H}$$

Electron density

$$\varepsilon_F = \frac{\hbar^2}{2m_e}\left(3\pi^2 \frac{Z}{A}\frac{\rho}{m_H}\right)^{2/3}$$

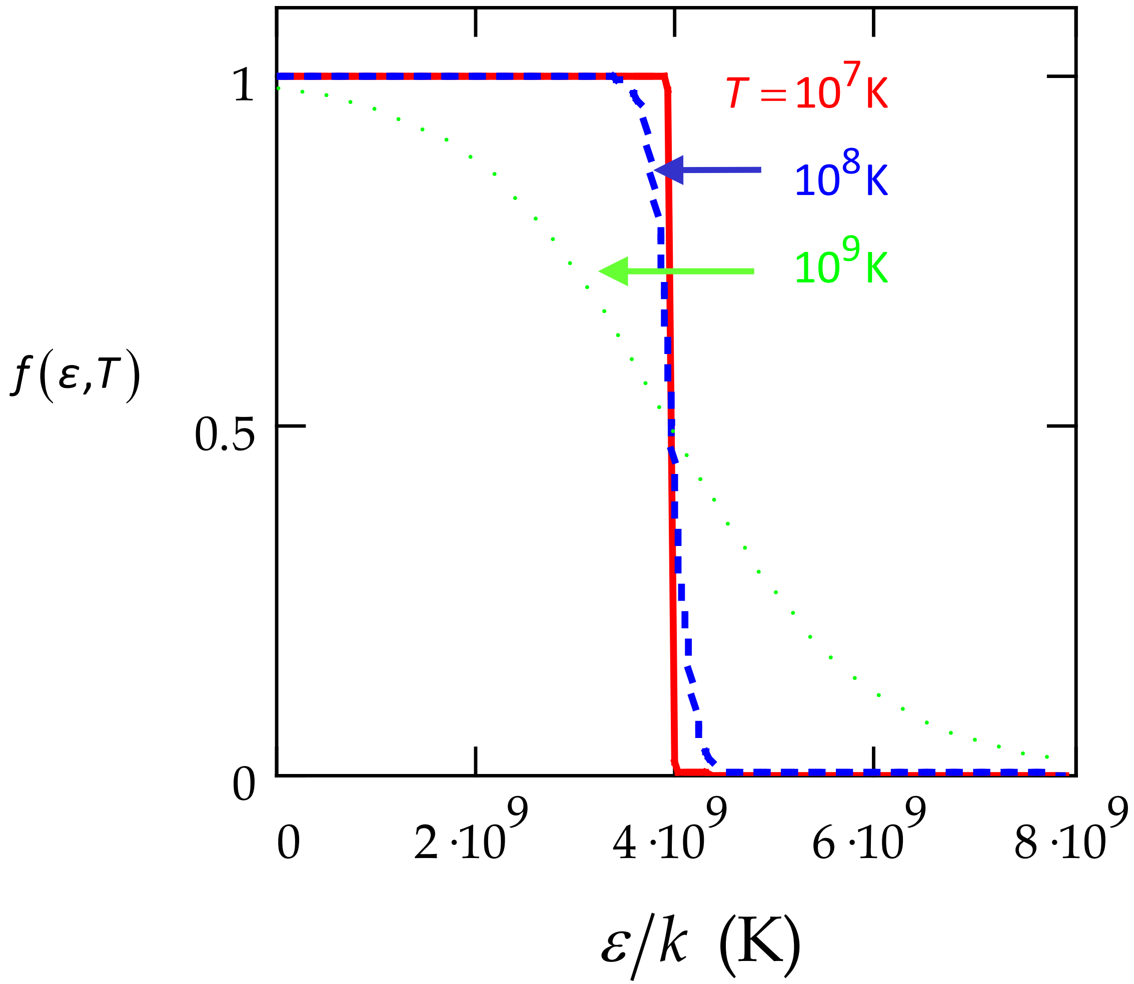
Fermi energy

$$f(\varepsilon, T) = \frac{1}{e^{(\varepsilon - \varepsilon_F)/kT} + 1}$$

Fraction of available states occupied
(Fermi distribution)

The farther T is below the value of ε_F / k, the more degenerate the gas.

Degenerate electron gas, as derived in PHY 227 (continued)



Fermi distribution for $Z/A = 0.5$, $\rho = 3 \times 10^6$ gm cm^{-3} ($\varepsilon_F / k = 3.9 \times 10^9$ K). Note that it can be quite hot and still have $T \ll \varepsilon_F / k$, so that

$$f(\varepsilon, T) \cong \begin{cases} 1 & \varepsilon \leq \varepsilon_F \\ 0 & \varepsilon > \varepsilon_F \end{cases}$$

Degeneracy pressure

- Electrons obey the Pauli exclusion principle: no two can occupy the same quantum state.
  - When they are strongly degenerate, they cannot occupy the same space.
  - Thus in a gas of *N* electrons, each is confined to a "box" with volume 1/*N* of the total volume.
  - Confinement to dimensions comparable to or smaller than the de Broglie wavelength leads to a large uncertainty in momentum for the electrons. Some will have very large momenta.
  - Therefore the electrons push on the walls of their "boxes," harder the smaller are the boxes.

    (This is beginning to sound familiar to ASTR 142 victims, but it gets better.)

Electron degeneracy pressure

- Nonrelativistic case (Fowler 1926)

$$P = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}{n_e}^{5/3} = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}\left(\frac{Z}{A}\frac{\rho}{m_H}\right)^{5/3}$$

as you are about to show.

$$R \approx \frac{(18\pi)^{2/3}}{10}\frac{\hbar^2}{G m_e M^{1/3}}\left(\frac{Z}{A}\frac{\rho}{m_H}\right)^{5/3}$$

taking, unnecessarily crudely, a uniform density. But wait til homework #9…

$$MV \approx \text{constant}$$

More massive white dwarfs are smaller. Sound familiar, polytrope fans?

- Relativity turns out to be important, though. In Sirius B $\left(M = 1M_\odot\right)$,

$$v_e = \frac{\hbar}{m_e}\left(\frac{Z}{A}\frac{\rho}{m_H}\right)^{5/3} = 1.1\times 10^{10}\ \text{cm s}^{-1}$$

The Stoner-Anderson-Chandrasekhar limit for white-dwarf masses

- The smaller the confinement boxes, the faster the electrons hit the walls - until electron speeds approach the speed of light.

- Thus relativity places an upper limit on the pressure a degenerate electron gas can exert, and an upper limit on the mass of white dwarfs.

- Ultrarelativistic limits (Chandrasekhar 1931):

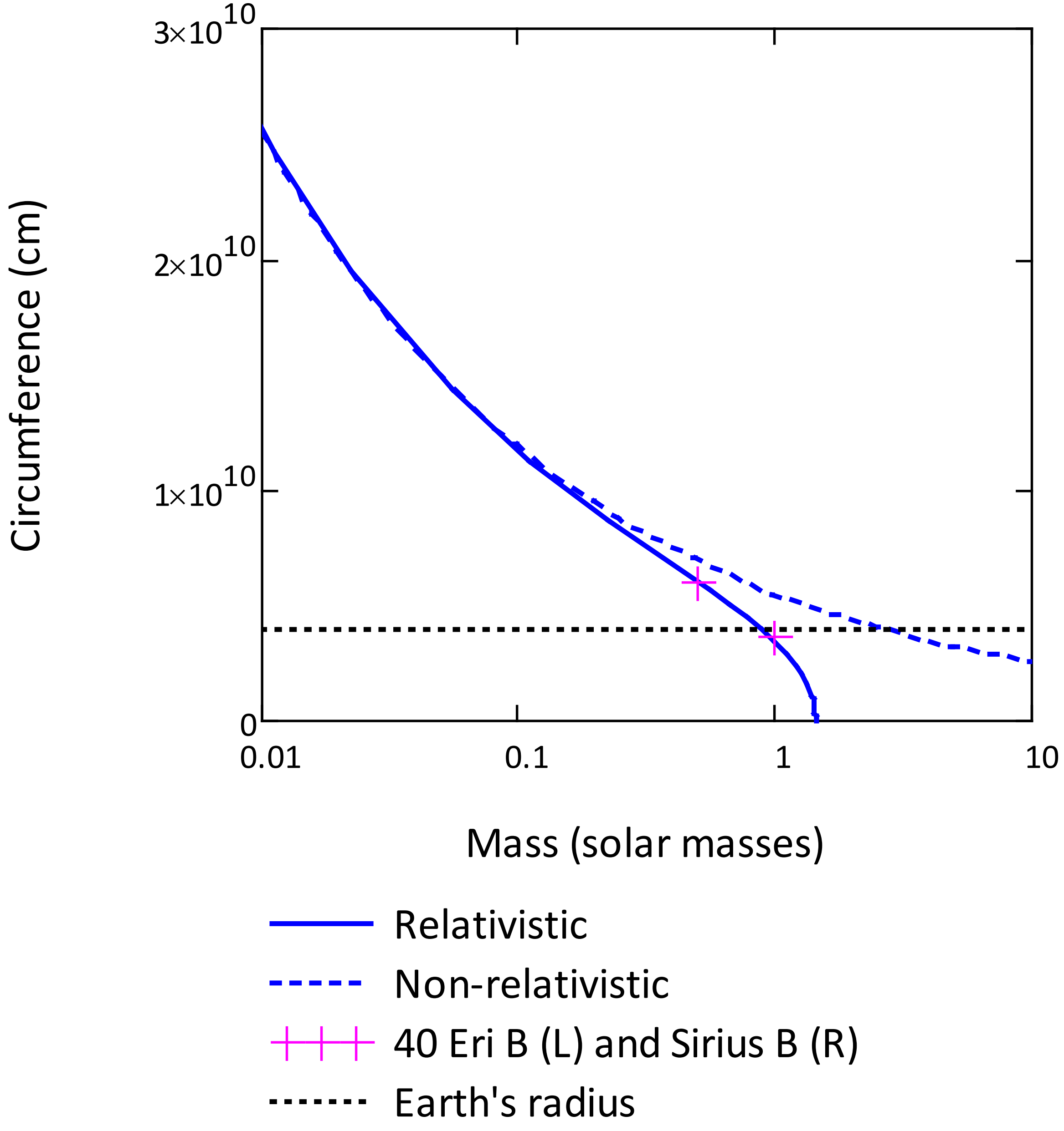
$$P = \frac{\left(3\pi^2\right)^{1/3}}{4}\hbar c\left(\frac{Z}{A}\frac{\rho}{m_H}\right)^{4/3}$$

Note: dynamically unstable.

$$M \approx \frac{9\sqrt{2\pi}}{8}\left(\frac{\hbar c}{G}\right)^{3/2}\left(\frac{Z}{A}\frac{1}{m_H}\right)^2$$

The Stoner-Anderson-Chandrasekhar white dwarf theories

The exact relativistic solution will be dealt with next class.



Today's in-class problems

1. Following the graph on page 5, the momentum distribution for degenerate fermions is

$$n_p dp = \begin{cases} \frac{8\pi}{h^3} p^2 dp & p \leq p_F \\ 0 & p > p_F \end{cases}$$

Use this distribution function to calculate the number density of electrons, n_e, and then use it again in the pressure integral,

$$P = \frac{1}{3}\int_0^\infty n_p p v dp \quad ,$$

in the nonrelativistic limit, to derive the pressure equation of state for nonrelativistic degenerate electrons, given on page 7, and as equation 16.12 in C&O.

Hint (applies to this problem and the next one): you don't need to know the value of p_F, the momentum that corresponds to the Fermi energy.

Today's in-class problems (continued)

2. Repeat the procedure in problem 1 in the extreme relativistic limit $(p^2c^2 \gg m^2c^4)$, and derive thereby the pressure equation of state for extreme relativistic degenerate electrons, given on page 8 (equation 15.14 in C&O).

Hints for the last set of in-class problems

1. (14.9) Simple, do not overthink: $U(r) = U(0) + r\frac{dU}{dr}(0) + \frac{1}{2}r^2\frac{d^2U}{dr^2}(0) + \ldots$

   If $U(0)$ is a stable equilibrium point, then $\frac{dU}{dr}(0) = 0$ and $\frac{d^2U}{dr^2}(0) > 0$

   so the force on the particle is $\boldsymbol{F} = -\hat{\boldsymbol{r}}\frac{dU}{dr}(r) \cong -r\hat{\boldsymbol{r}}\frac{d^2U}{dr^2}(0) = -\boldsymbol{r}\frac{d^2U}{dr^2}(0)$,

   like Hooke's law with spring constant $\left. d^2U/dr^2 \right|_0$.

2. (14.11). $A > 0$ means $\frac{1}{\rho}\frac{d\rho}{dr} > \frac{1}{\gamma P}\frac{dP}{dr}$. Differentiate the ideal gas law with respect to r for constant μ and substitute,

   and you get $\left(\frac{\gamma - 1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} > \frac{dT}{dr}$, whence the chain rule leads to the convective instability criterion, $\frac{T}{P}\frac{dP}{dT} - \frac{\gamma}{\gamma - 1} > 0.$