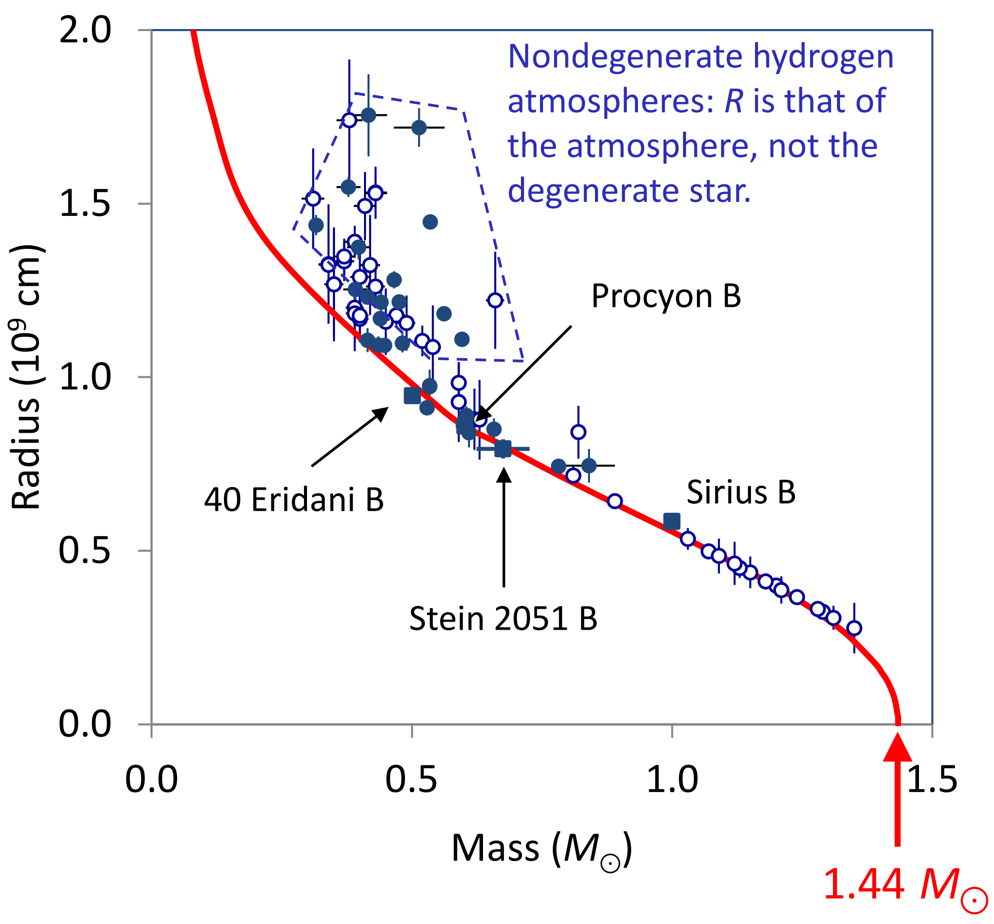
Today in Astronomy 241: degenerate stars II

- Toward the exact white-dwarf interior solution under conditions of complete degeneracy $\left(T \ll \varepsilon_F / k\right)$.
- No reading today



Exact solution of the white-dwarf interior, compared to observations of white dwarfs in visual or eclipsing binary systems. From [ASTR 142 lecture 8](), where it says, “You’ll learn how to do the calculation (—) in ASTR 241.”

It is now time.

The equation of state for relativistic degenerate electrons

- Last class you derived expressions for the pressure equation of state of degenerate fermions in the two limiting cases $v \ll c$ and $v \to c$.

- In Homework #9 next week, you will derive the pressure equation of state for the **general** case, again starting from the pressure integral. This will be your result:

$$P = \frac{\pi}{3}\frac{m_e^4 c^5}{h^3} f\left(x\right) \equiv af\left(x\right) \quad ,$$

where
$$f\left(x\right) = x\left(x^2+1\right)^{1/2}\left(2x^2-3\right) + 3\operatorname{arcsinh}\left(x\right) \quad ,$$

and
$$x = \frac{p_F}{m_e c} \quad .$$

- Now we can show (more easily) that $\rho = \frac{m_H A}{Z}\frac{8\pi m_e^3 c^3}{3h^3}x^3 \equiv bx^3 \quad .$

The equation of state for relativistic degenerate electrons (continued)

- Thus the equation of state is expressed **parametrically**: both pressure and density are given completely in terms of $x = p_F / m_e c$.
- This is a feature, not a bug: we can express the equations of stellar structure in terms of *x* instead of *P* and ρ, and solve for it.
- This is what you'll do in Homework #9, as you complete the calculation. The next few in-class problems will help complete the setup.

Today’s in-class problems

1. Show that the mass density is indeed given in terms of x by

$$\rho = \frac{m_H A}{Z} \frac{8\pi m_e^3 c^3}{3h^3} x^3 \equiv bx^3 \quad .$$

2. The radial gradient of pressure shows up in the stellar-structure equations, so it will be useful to show that

$$\frac{df}{dx} = \frac{8x^4}{\left(1+x^2\right)^{1/2}} \quad .$$

   Do so.

3. Take the limit $x \ll 1$, and show that in this limit the nonrelativistic degenerate equation of state, which you derived last class, is produced.

Hints for the last set of in-class problems

1. If the medium is completely degenerate $\left(T \ll \varepsilon_F / k\right)$ and nonrelativistic $\left(v \ll c\right)$, then

$$n_e = \int_0^\infty n_p dp = \frac{8\pi}{h^3}\int_0^{p_F} p^2 dp = \frac{8\pi}{3h^3} p_F^3 = \frac{Z}{A}\frac{\rho}{m_{\mathrm{H}}}$$

$$\Rightarrow \quad p_F^3 = \frac{3h^3}{8\pi} n_e = \frac{3h^3}{8\pi}\frac{Z}{A}\frac{\rho}{m_{\mathrm{H}}}$$

$$P_e = \frac{1}{3}\int_0^\infty p v n_p dp = \frac{8\pi}{3h^3}\frac{1}{m_{\mathrm{H}}}\int_0^{p_F} p^4 dp = \frac{1}{5}\frac{8\pi}{3h^3}\frac{1}{m_{\mathrm{H}}} p_F^5 = \frac{1}{5}\frac{8\pi}{3h^3}\frac{1}{m_{\mathrm{H}}}\left(\frac{3h^3}{8\pi} n_e\right)^{5/3}$$

$$= \frac{1}{5m_{\mathrm{H}}}\left(\frac{3h^3}{8\pi}\right)^{2/3} n_e^{5/3} = \frac{1}{5m_{\mathrm{H}}}\left(\frac{3h^3}{8\pi}\right)^{2/3}\left(\frac{Z}{A}\frac{\rho}{m_{\mathrm{H}}}\right)^{5/3} ,$$

which is the same as C&O equation 16.12.

Hints for the last set of in-class problems

2. Still completely degenerate but now ultrarelativistic $(v \to c)$:

$$P_e = \frac{1}{3}\int_0^{\infty} p v n_p dp = \frac{8\pi}{3h^3}\frac{c}{m_{\mathrm{H}}}\int_0^{p_F} p^3 dp = \frac{1}{4}\frac{8\pi}{3h^3}\frac{c}{m_{\mathrm{H}}} p_F^4 = \frac{c}{4m_{\mathrm{H}}}\frac{8\pi}{3h^3}\left(\frac{3h^3}{8\pi}n_e\right)^{4/3}$$

$$= \frac{c}{4m_{\mathrm{H}}}\left(\frac{3h^3}{8\pi}\right)^{1/3} n_e^{4/3} = \frac{c}{4m_{\mathrm{H}}}\left(\frac{3h^3}{8\pi}\right)^{1/3}\left(\frac{Z}{A}\frac{\rho}{m_{\mathrm{H}}}\right)^{4/3} \quad ,$$

which is the same as C&O equation 16.15.

Hints for this set of in-class problems

1. As in two problems done last class, use

$$n_e = \int_0^{p_F} n_p dp = \frac{Z}{A}\frac{\rho}{m_H} \quad .$$

2. This simply requires straightforward differentiation, as long as you remember that

$$\frac{d}{dx}\text{arcsinh}(x) = \frac{1}{\left(1+x^2\right)^{1/2}} \quad .$$

Today's in-class problems (continued)

3. This turns out to be the lengthy one. Expand the binomial and the arcsinh(x) in Taylor series and multiply it out. It turns out that all terms in x and x^3 cancel out, so one must keep fifth order terms in the expansions:

$$\left(1+x^2\right)^{1/2} = 1+\frac{1}{2}x^2-\frac{1}{8}x^4+O\left(x^6\right)$$
$$\operatorname{arcsinh}\left(x\right) = x-\frac{1}{6}x^3+\frac{3}{40}x^5+O\left(x^7\right) \quad ,$$

to obtain

$$f\left(x\right) = x\left(1+\frac{1}{2}x^2-\frac{1}{8}x^4+O\left(x^6\right)\right)\left(2x^2-3\right)$$
$$+x-\frac{1}{6}x^3+\frac{3}{40}x^5+O\left(x^7\right)$$

Today's in-class problems (continued)

or

$$f(x) = 2x^3 + x^5 - \frac{x^7}{4} - 3x - \frac{3x^3}{2} + \frac{3x^5}{8} + 3x - \frac{x^3}{2} + \frac{9x^5}{40} + O\left(x^6\right)$$

$$= \left(1 + \frac{15}{40} + \frac{9}{40}\right)x^5 + O\left(x^6\right)$$

$$\xrightarrow[x \ll 1]{} \frac{8}{5}x^5 \quad .$$

Thus

$$P \cong a\frac{8}{5}x^5 = \frac{8}{5}a\left(\frac{\rho}{b}\right)^{5/3} = K\rho^{5/3} \quad ,$$

the nonrelativistic form, q.e.d.