Today in Astronomy 241: accretion disks III

- Viscosity and viscous disks
- The α prescription of viscosity in accretion disks
- **Reading**: C&O chapter 18, pp 673-705
- Also recommended: [Hartmann 2009](), chapter 7, and [Armitage 2020](), chapter 3.

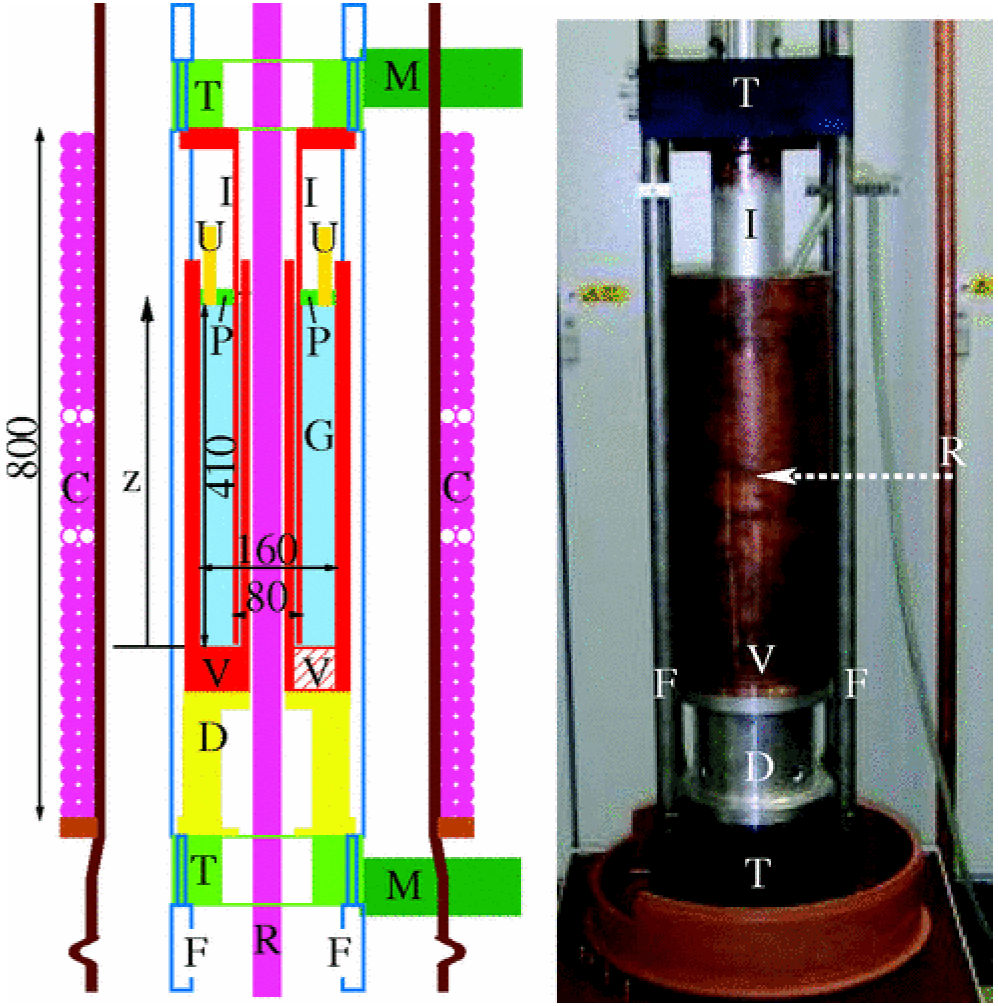
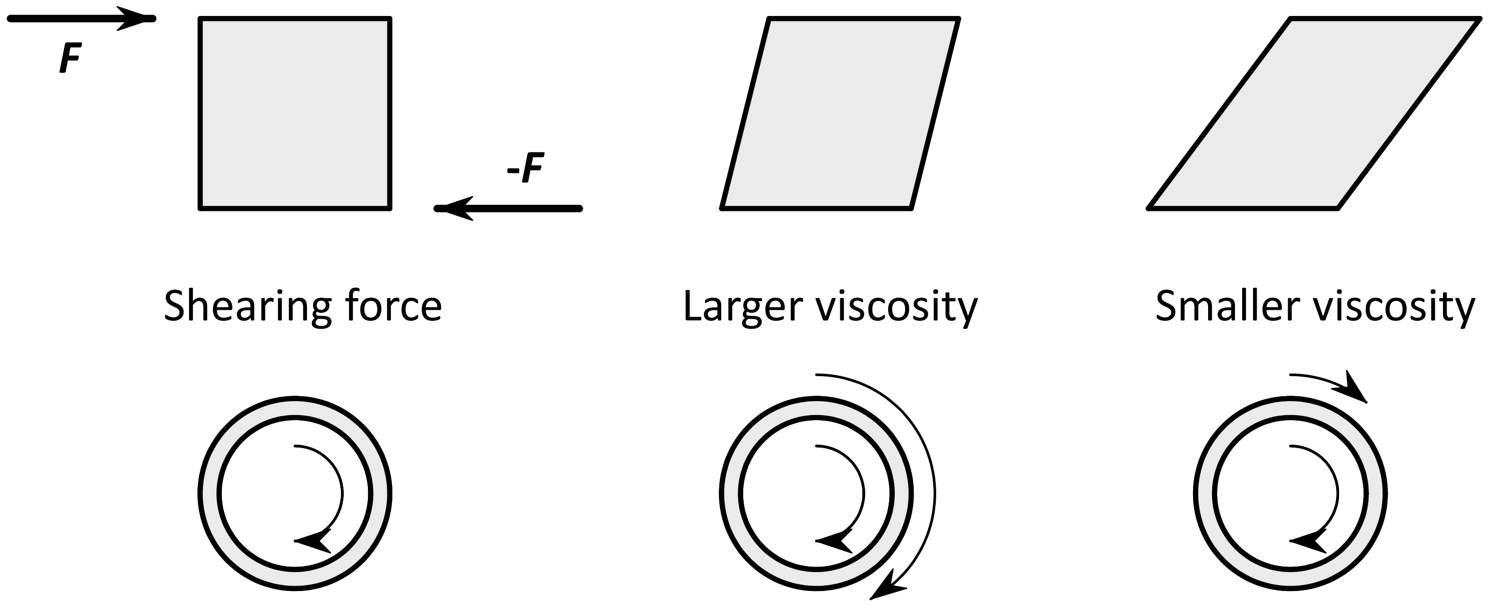


Diagram and image of a magnetic Couette-flow system for viscosity measurement of fluids subjected to helical magnetic fields, from [Stefani+2006]().

Viscosity

Viscosity is the resistance of a fluid to **shear**: that is, to a gradient in displacement or speed across an object.



Shearing force | Larger viscosity | Smaller viscosity

Shearing spin **(Couette flow)** | Larger viscosity | Smaller viscosity

Viscosity (continued)

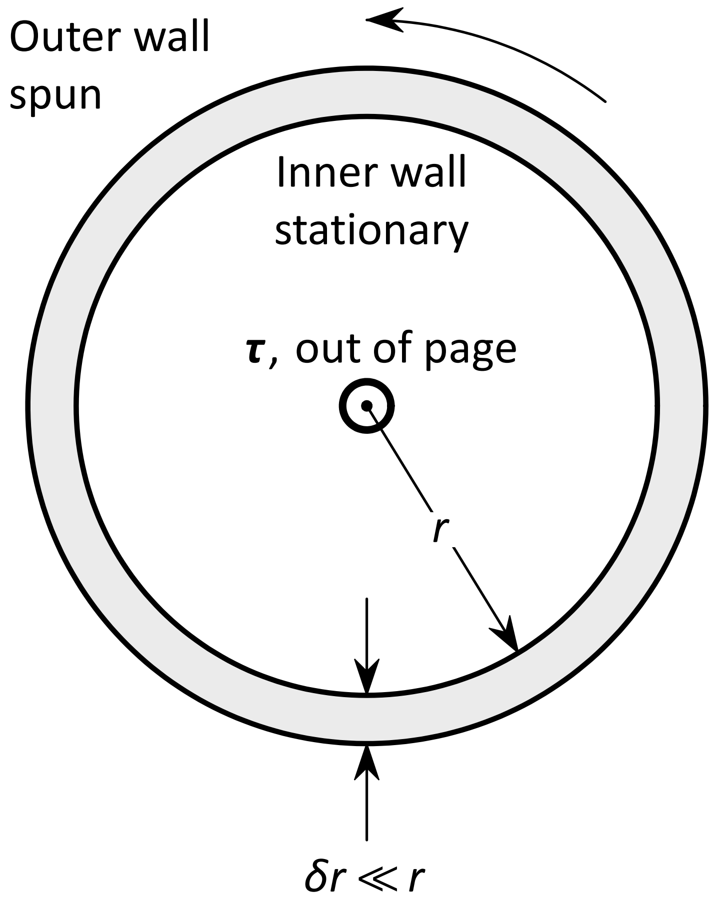
Viscosity is usually defined, and measured, in cylindrical geometry. Which is good for us, because that's the geometry of disks.

- Holding the inner ring still and torquing the outer one, the fluid in between exhibits a gradient of angular velocity which defines the fluid's viscosity:

$$\tau = rF = \mu_V A r \frac{r\delta\Omega}{\delta r} \xrightarrow[\delta r \to 0]{} \mu_V A r^2 \frac{d\Omega}{dr}$$

  where A is the area of the inner wall and μ_V is the **dynamic viscosity**, with units dyne sec cm^{-2}.

- Sorry for yet another μ in your life, but we won't use it long.



Viscous accretion disks

- Take the disk to be thin $\left(H(r) \ll r\right)$, and define its **surface density** $\Sigma(r) = \int_{-\infty}^{\infty} \rho(r,z)\,dz$, which has units of mass per unit disk area.

- The disk is rotating differentially, with angular speed $\Omega(r)$. Define the **specific angular momentum**, $h(r)$, which is the local value of the angular momentum per unit disk area: $h(r) = r^2\Sigma(r)\Omega(r)$

- A narrow $\left(\Delta r \ll r\right)$ annulus with radius r experiences a torque with magnitude

$$\begin{aligned}\tau &= \mu_V \cdot 2\pi r H \cdot r^2 \frac{d\Omega}{dr} = \nu_V \rho H \cdot 2\pi r^3 \frac{d\Omega}{dr} \\ &= 2\pi r^3 \nu_V \Sigma \frac{d\Omega}{dr} \quad ,\end{aligned}$$

where $\nu_V \equiv \mu_V / \rho$, with units $\mathrm{cm}^2\ \mathrm{sec}^{-1}$, is called the **kinematic viscosity**. This is the one we'll use today.

Viscous accretion disks (continued)

- Suppose that, along with rotational motion, material in the disk moves also moves radially, with magnitude $u_r(r)$. Mass conservation dictates that any increase of the annulus's mass with time balances the flow in from and out to the neighboring annuli. To first order in Δ*r*,

$$\frac{\partial}{\partial t}(2\pi r\Delta r\Sigma) = u_r(r,t)2\pi r\Sigma(r,t) - u_r(r+\Delta r,t)2\pi(r+\Delta r)\Sigma(r+\Delta r,t)$$

$$\cong u_r 2\pi r\Sigma - \left[u_r 2\pi r\Sigma + \frac{\partial u_r}{\partial r}\Delta r 2\pi r\Sigma + u_r 2\pi\Delta r\Sigma + u_r 2\pi r\frac{\partial\Sigma}{\partial r}\right]$$

$$= -2\pi\Delta r\left[\frac{\partial u_r}{\partial r}r\Sigma + u_r\Sigma + u_r r\frac{\partial\Sigma}{\partial r}\right] = -2\pi\Delta r\frac{\partial}{\partial r}(r\Sigma u_r)$$

$$\therefore \quad r\frac{\partial\Sigma}{\partial t} + \frac{\partial}{\partial r}(r\Sigma u_r) = 0 \quad .$$

Viscous accretion disks (continued)

- Similarly, and as you will show today for practice, angular momentum conservation dictates

$$\frac{\partial}{\partial t}(2\pi r\Delta r h) = u_r(r,t)2\pi r h(r,t) + \tau(r,t) - u_r(r+\Delta r,t)2\pi(r+\Delta r)h(r+\Delta r,t) - \tau(r+\Delta r,t)$$

$$\vdots$$

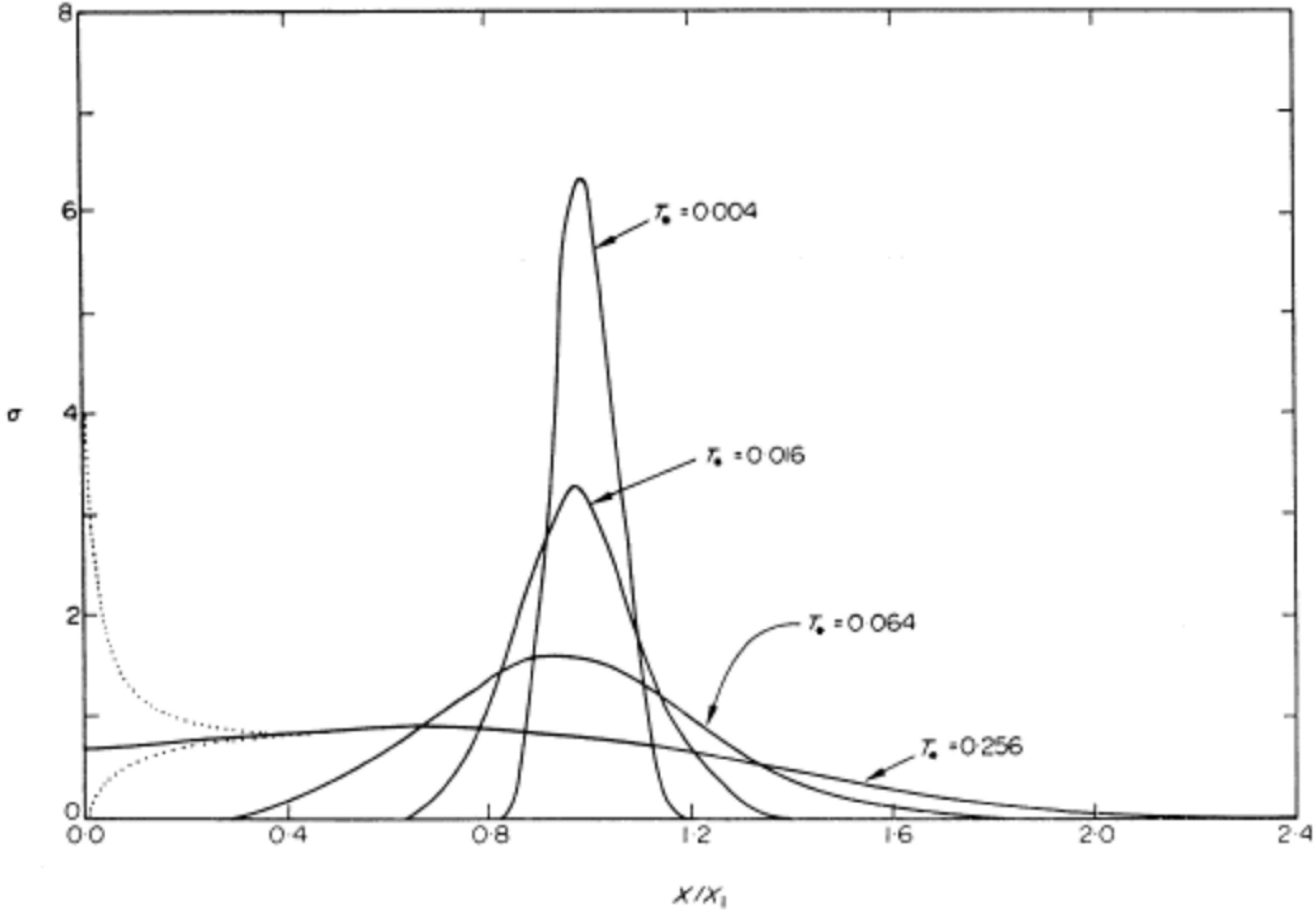
$$r\frac{\partial}{\partial t}\left(\Sigma r^2\Omega\right) + \frac{\partial}{\partial r}\left(u_r\Sigma r^3\Omega\right) = -\frac{\partial}{\partial r}\left(v_V\Sigma r^3\frac{d\Omega}{dr}\right) \quad .$$

- This, plus mass conservation, is a hard system of equations to solve, but the solution is useful to consider, before we proceed to simplify it.
  - First as a **similarity solution** in a classic paper by Lynden-Bell & Pringle (1974).
- Solution upshot: **disk viscosity leads to outward angular momentum transport and inward mass transport.**

Viscous accretion disks (continued)

Development over time of an originally infinitesimally-narrow annulus.

- Note the broadening of the annulus, and the shift of matter to smaller radii, as time goes on.

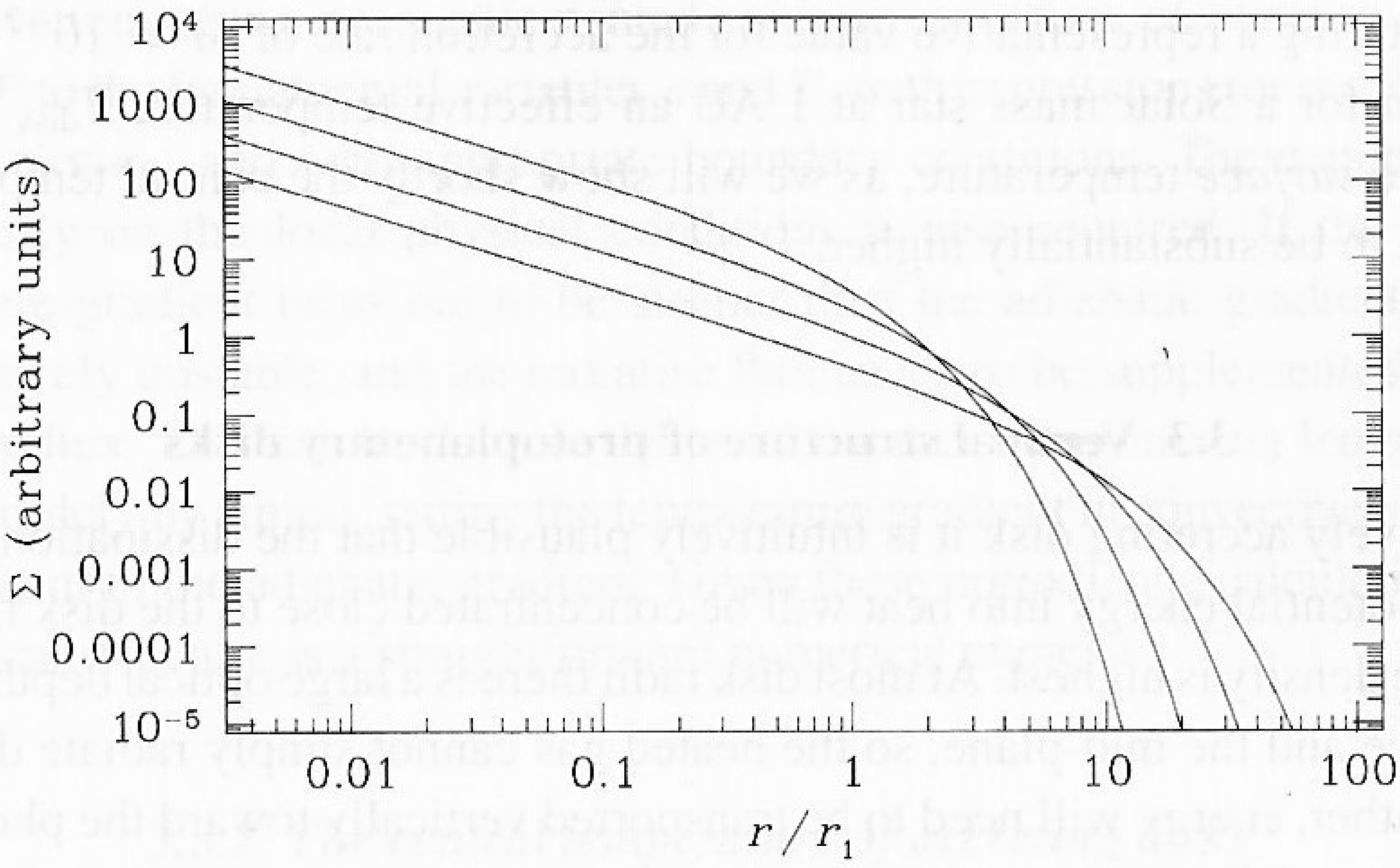


[Lynden-Bell & Pringle 1974]

Viscous accretion disks (continued)

Development over time of surface density Σ(r) in a disk with viscosity $\nu_V(r) \propto r$. The curves are for (dimensionless) times 1, 2, 4, and 8, top to bottom for their y-intercepts.

- Note that the total mass decreases with increasing time – it's being accreted by the central star.
- Note also that the mass at the largest radii, which has the lion's share of the angular momentum, increases with increasing time.



Armitage 2020

Steady, viscous accretion disks

By steady, we mean that no quantity depends explicitly on time, so all the time-derivative terms above are zero.

- Thus the mass-conservation equation becomes

$$\frac{\partial}{\partial r}\left(r\Sigma u_r\right) = 0 \quad \Rightarrow \quad 2\pi r\Sigma u_r = \dot{M} = \text{constant} \quad .$$

- And the angular-momentum conservation equation becomes

$$\frac{\partial}{\partial r}\left(u_r\Sigma r^3\Omega\right) = -\frac{\partial}{\partial r}\left(\nu_V\Sigma r^3\frac{d\Omega}{dr}\right) \quad \Rightarrow \quad \frac{\dot{M}\Omega}{r} = -2\pi\nu_V\Sigma\frac{d\Omega}{dr} + C$$

Since the disks we're interested in this week are magnetically truncated far from the stellar surface, we can neglect the integration constant *C*.

Steady, viscous accretion disks

- Take the disk's rotation profile to be Keplerian, so that

$$\Omega = r^{-3/2}\sqrt{GM_1} \quad \text{, and} \quad \frac{d\Omega}{dr} = -\frac{3}{2}\sqrt{GM_1}\,r^{-5/2} \quad ;$$

$$v_V \Sigma = \frac{\dot{M}}{3\pi} \quad .$$

- If only we knew what the viscosity is, we would now have a relation which can tell us the disk mass and its radial distribution, which has been obscure to us up to now.
  - We know that astrophysical viscosities are not like the viscosities measured for gases at practical laboratory pressures and densities; these extrapolate to values many orders of magnitude too small to explain accretion in, well, anything.
  - The current best ideas for the origin of disk viscosity involve magnetorotational instabilities and magnetic turbulence.

The Shakura-Sunyaev viscosity prescription

All this is why Shakura & Sunyaev (1973) created a very popular bit of dimensional analysis: take the kinematic viscosity simply to be proportional to the product of a characteristic speed and a characteristic length:

$$\nu_V = \alpha c_s H \quad ,$$

where $$H = \left(2kTr^3 \middle/ G\mu m_H M_1 \right)$$ is the pressure scale height,

$$c_s = \left(kT / \mu m_\mathrm{H} \right)^{1/2}$$ is the isothermal sound speed, and

$$\alpha \lesssim 1 \quad .$$ *α* would be about 10^{-12} for air.

Good accounts of observations are produced with $\alpha \sim 0.2$ in (largely ionized) cataclysmic-variable and active-galaxy-nucleus disks, and about ten times smaller in (largely neutral) protoplanetary disks.

- Disks with this viscosity prescription are called ***α* disks**.

The Shakura-Sunyaev viscosity prescription (continued)

- This leaves us with the expression you need for getting the surface density of the optically-thick disk in this week's homework:

$$\Sigma(r) = \frac{\dot{M}}{3\pi\nu_V} = \frac{\dot{M}}{3\pi\alpha c_S(r)H(r)} \quad ,$$

  where the dependences on *r* of the sound speed and pressure scale height come from their dependence on *T.*

Today’s in-class problems

1. Complete the derivation on page 6.

2. C&O problem 18.18 (b)

Answers

1. Same first order approximation as in mass conservation:

$$\frac{\partial}{\partial t}(2\pi r\Delta r h) = u_r(r,t)2\pi r h(r,t) + \tau(r,t) - u_r(r+\Delta r,t)2\pi(r+\Delta r)h(r+\Delta r,t) - \tau(r+\Delta r,t)$$

$$= -2\pi\frac{\partial}{\partial r}(rhu_r)\Delta r - \frac{\partial\tau}{\partial r}\Delta r$$

$$\frac{\partial}{\partial t}\left(2\pi r\Delta r r^2\Sigma\Omega\right) = -2\pi\frac{\partial}{\partial r}\left(rr^2\Sigma\Omega u_r\right)\Delta r - \frac{\partial}{\partial r}\left(2\pi r^3 v_V\Sigma\frac{d\Omega}{dr}\right)\Delta r$$

$$r\frac{\partial}{\partial t}\left(\Sigma r^2\Omega\right) + \frac{\partial}{\partial r}\left(u_r\Sigma r^3\Omega\right) = -\frac{\partial}{\partial r}\left(v_V\Sigma r^3\frac{d\Omega}{dr}\right) \quad .$$

Answers (continued)

2. From the neutron star: $$\frac{dL}{dt} = I\frac{d\omega}{dt} = I\frac{d}{dt}\left(\frac{2\pi}{P}\right) = -2\pi I\frac{\dot{P}}{P^2}$$

   Via the lines of ***B***, to the disk: $$\frac{dL}{dt} = \dot{I}\Omega = \dot{M}r_d^2\Omega = \dot{M}\sqrt{GM_1 r_d}$$

   $$\Rightarrow \quad \frac{\dot{P}}{P} = -\frac{P\dot{M}}{2\pi I}\sqrt{GM_1 r_d} = -\frac{P\dot{M}}{2\pi I}\left(\frac{GM_1}{2}\right)^{\frac{1}{2}}\left(\frac{m_B^4}{2GM_1\dot{M}^2}\right)^{1/14}$$