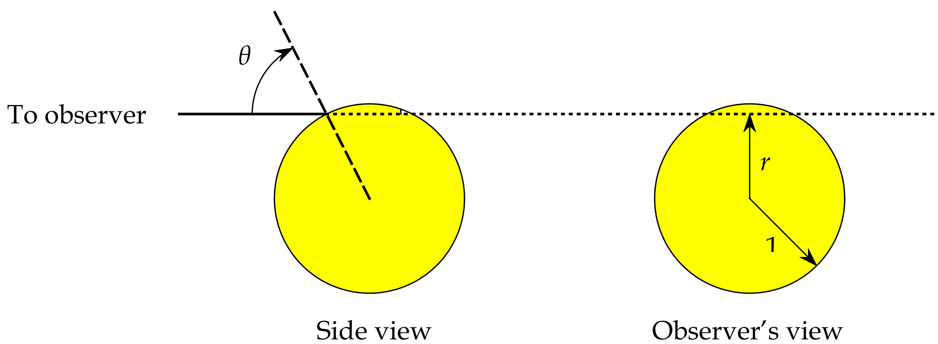
Astronomy 241 problem set #3: solutions

17 February 2024

The integrals in the following all turn out to be analytically tractable (see Mandel & Agol 2002), but I recommend evaluating them numerically when using them in problem G.

A. *A ray of light emerges from a point on the meridian of a star, in an observer's viewpoint. Take the star to have radius 1. (That is, use the star's radius as the unit of distance.) The point lies north of center by a fraction r of the star's radius in the observer's viewpoint, as shown in the diagram below. Show that the ray leaves the photosphere at angle given by* $\cos\theta = \sqrt{1-r^2}$.



The observer is distant, so the lines of sight through the emission point and the star's center are parallel. Therefore, from the diagram: $\theta = \arcsin r$, and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-r^2}$, q.e.d.

B. *The star exhibits quadratic limb darkening; its intensity in the vertical direction is 1. (That is, use the vertical intensity as our unit of intensity.) Show that its flux* F_0 *as viewed by an observer is*

$$F_0 = 2\pi\int_0^1 I(r)\, rdr \quad \text{, where}$$

$$I(r) = \sum_{n=0}^{2} a_n n!\left(1-r^2\right)^{n/2} \quad \text{, and where}$$

$$\sum_{n=0}^{2} a_n n! = 1 \quad .$$

The last relation means that the three a_ns *are not independent of one another.*

To save writing I have omitted the wavelength subscripts, but keep in mind that the measurements are made at a certain wavelength or relatively narrow range of wavelengths, rather than integrated over all wavelengths.

As we saw on 6 February 2024, the emergent intensity is

$$I(0,\theta) = -\int_\infty^0 d\tau_V \sec\theta e^{-\tau_V \sec\theta} I(0,V)\sum_{n=0}^{N} a_n \tau_V^n = I(0,V)\int_0^\infty du e^{-u}\sum_{n=0}^{N} a_n u^n \cos^n\theta \quad ,$$

 All rights reserved

whence, as we have reminded ourselves several times, a few integrations by parts gives

$$I(a_n,0,\theta)=I(0,V)\sum_{n=0}^{N}a_n n!\cos^n\theta \quad .$$

From the definition of flux (25 January 2024) and the high opacity of the star – only the front side of the star, $\theta=0-\pi/2$, contributing to observed flux – we get

$$F_0(a_n)=\int_0^{2\pi}d\varphi\int_0^{\pi/2}d\theta I(a_n,0,\theta)\cos\theta\sin\theta \quad .$$

The integral over φ is trivial. For the other one, note again from the diagram that an apt substitution is $r=\sin\theta$, $dr=\cos\theta d\theta$, and $r=0-1$ as $\theta=0-\pi/2$:

$$F_0=2\pi\int_0^1 I(a_n,r)rdr \quad \text{, where}$$

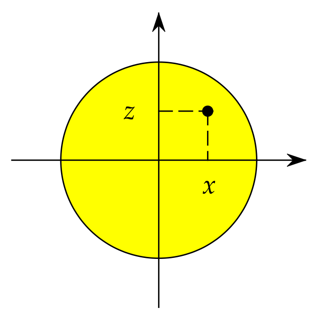
$$I(a_n,0,r)=I(0,V)\sum_{n=0}^{N}a_n n!\left(1-r^2\right)^{n/2} \quad \text{, q.p.d.}$$

Now note that $I(a_n,0,0)=I(0,V)$, so, trivially,

$$\sum_{n=0}^{N}a_n n!=1 \quad \text{, q.e.d.}$$

Since the star is specified to exhibit quadratic limb darkening, N = 2, so this last result implies $a_0=1-a_0-2a_2$.

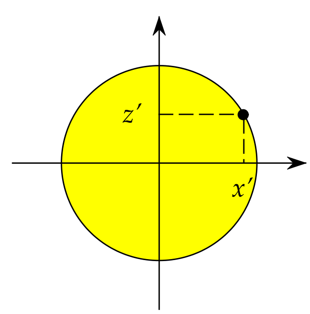
C. *A planet with radius $p\ll 1$ has moved so that its shadow lies completely within the star's disk, at projected coordinates x, z from the center of the star, in the observer's view. Show that the star's flux is now*



$$F(a_n,p,r)=F_0(a_n)-\pi p^2 I(a_n,0,r) \quad , \quad r=\sqrt{x^2+z^2}\le 1-p.$$

If the shadow lies completely within the star's disk, it can be no further from the center than $r=\sqrt{x^2+z^2}=1-p$. If the planet is opaque and its shadow much smaller than the star, then the stellar intensity varies little over the shadow's position and can be taken to be simply the intensity at the shadow's center. The stellar flux absorbed by the planet is therefore $F_{\text{abs}}=\pi p^2 I(a_n,0,r)$, and the flux which reaches the observer is $F(a_n,p,r)=F_0(a_n)-\pi p^2 I(a_n,0,r)$, q.e.d.

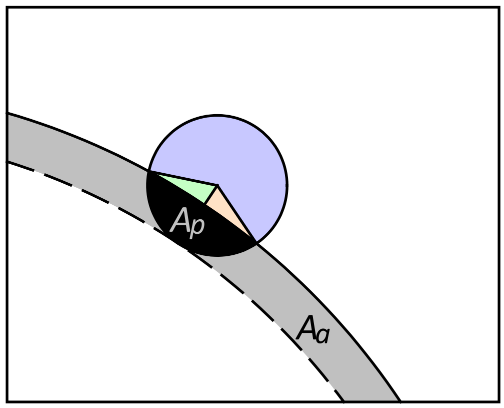
D. *In the short time it took to solve that problem, its orbital motion has taken the planet to the limb of the star, part of its shadow past the limb, at position x',z', a distance $r'=\sqrt{x'^2+z'^2}$ from the direction toward the star's center. Show the area of shadow remaining within the stellar disk is $A_p=p^2\arccos\left([r'-1]/p\right)$, that the area of the annulus of the stellar disk*



within which it lies is $A_a = \pi\left[1-(r'-p)^2\right]$*, and that the flux from the star is now* approximately

$$F'(a_n, p, x', z') = F_0 - \frac{2}{1-(r'-p)^2}\left[p^2 \arccos\left(\frac{r'-1}{p}\right) - (r'-1)\sqrt{p^2-(r'-1)^2}\right]\int_{r'-p}^{1} I(a_n, r)\, r dr \quad ,$$
$$1-p \le r' \le 1+p.$$

Zoom in on the planet in the diagram above:



Since the planet is much smaller than the star, the edge of the star can be considered to be a straight line perpendicular to the line between the centers of star and shadow. This permits the part of the shadow outside the star to be decomposed into a circular sector – in blue, above – and two right triangles (green and red). The triangles' common side has length $r'-1$ and they both have hypoteneuse length p, so their adjacent angles are $\arccos([r'-1]/p)$. This makes the area of the sector $\pi p^2\left[\left(\pi - \arccos([r'-1]/p)\right)/\pi\right]$, and the total area of the two triangles $(r'-1)\sqrt{p^2-(r'-1)^2}$. Thus

$$\begin{aligned} A_p &= \pi p^2 - \pi p^2\left[\left(\pi - \arccos([r'-1]/p)\right)/\pi\right] - (r'-1)\sqrt{p^2-(r'-1)^2} \\ &= p^2 \arccos([r'-1]/p) - (r'-1)\sqrt{p^2-(r'-1)^2} \quad . \end{aligned}$$

The radius of the inner edge of the annulus is $1-\left[p-(r'-1)\right] = r'-p$. Neglecting the small shadow, the area of the annulus is $A_a = \pi - \pi(r'-p)^2$, and it produces flux given by

$$F_a(a_n) = 2\pi\int_{r'-p}^{1} I(a_n, r)\, r dr \quad .$$

Again because the shadow is small, we can take the intensity to be uniform across A_p to first approximation, so the shadow blocks a fraction A_p/A_a of the annulus' flux. The total flux received by the observer is therefore

© 2024, University of Rochester All rights reserved

$$F'\left(a_n,p,x',z'\right)=F_0\left(a_n\right)-\frac{A_p}{A_a}F_a\left(a_n\right)$$
$$=F_0\left(a_n\right)-\frac{2}{1-\left(r'-p\right)^2}\left[p^2\arccos\left(\frac{r'-1}{p}\right)-\left(r'-1\right)\sqrt{p^2-\left(r'-1\right)^2}\right]\int_{r'-p}^{1}I\left(a_n,r\right)rdr \quad \text{, q.e.d.}$$

You will hopefully notice that this is the same as the small-planet-transit result by Mandel & Agol (2002; equation 8; note that they refer to our r' as z).

E. *In the literature, one often sees limb darkening described by*

$$\frac{I_\lambda\left(0,s\right)}{I_\lambda\left(0,V\right)}=1+\sum_{n=1}^{N}b_{\lambda,n}\left(1-\cos\theta\right)^n \quad .$$

Show that this formula is the same as that derived in class for quadratic limb darkening, and give the equations relating the $a_{\lambda,n}$ in our formulation with the $b_{\lambda,n}$ in this one.

(One also often sees the definition $\mu=\cos\theta$ used in such articles; this is common enough that lots of authors just use μ in place of $\cos\theta$ from the beginning without ever reminding the reader of the definition.)

$$\frac{I\left(0,\theta\right)}{I\left(0,V\right)}=\sum_{n=0}^{N}a_n n!\cos^n\theta=1+\sum_{n=1}^{N}b_{\lambda,n}\left(1-\cos\theta\right)^n \quad ,$$

and with $N = 2$,

$$1-a_1-2a_2+a_1\cos\theta+2a_2\cos^2\theta=1+b_1+b_2-b_1\cos\theta-2b_2\cos\theta+b_2\cos^2\theta \quad .$$

For this equality to hold at all values of θ, the coefficients of the powers of $\cos\theta$ have to be equal separately:

$$a_1=-b_1-2b_2 \quad ,$$
$$a_2=b_2/2 \quad .$$

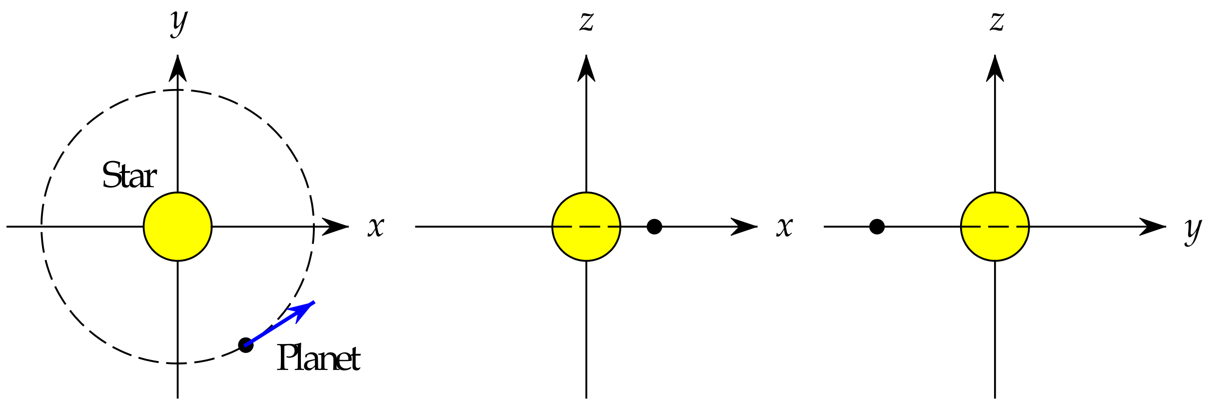
Note that this also makes $1-a_1-2a_2=1+b_1+b_2$.

F. *An exoplanet orbits a star with zero eccentricity, orbital radius a, and axis inclination i with respect to the line of sight. The system rotation/revolution axis lies along z in projection. The planet's orbital period is P. Derive expressions for $z\left(t\right)$ and $x\left(t\right)$ which apply as the planet transits the star.*

Here are three views of planet and star at an arbitrary nontransiting time, starting with the orbital plane in *x*-*y*, and an edge-on view with the observer's line of sight along *y*. The observer is located at $y\to-\infty$; $t=0$ at mid-transit, when the centers of planet and star align. Keep in mind that we can't resolve the dimensions in observations; the whole scene appears starlike in transit observations.

© 2024, University of Rochester

All rights reserved

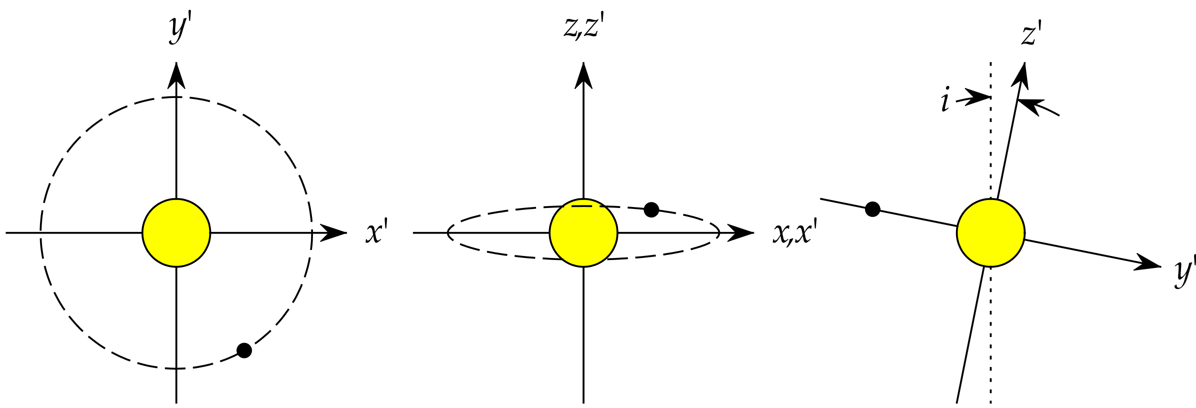


Uniform circular motion (e.g. ASTR 111, Lecture 9) implies that in the system's equatorial plane, the position would be $x(t) = a\sin\omega t = a\sin(2\pi t/P) = a\sin\varphi$, and similarly $y = a\cos\varphi$, where $\varphi = \varphi(t)$ is the orbital phase in radians. If the observer were to view the system edge on, and if the planet and star were aligned at $t = \varphi = 0$, it would appear in the coordinate system above as

$$x(t) = a\sin(\varphi(t)) \quad , \quad y(t) = a(\varphi(t)) \quad , \quad z(t) = 0 \quad .$$

If the orbital axis were inclined with respect to the line of sight by angle *i*, then this changes to

$$x(t) = a\sin(\varphi(t)) \quad , \quad y(t) = a\sin i\cos(\varphi(t)) \quad , \quad z(t) = a\cos i\cos(\varphi(t)) \quad .$$



Because the transit duration is a small fraction of the orbital period, *z* is essentially constant in the fit of the formulas to the transit light curve. Note that we can't tell whether the orbital axis is inclined toward, or away from, the observer, since $\cos(-x) = \cos(x)$.

G. *(**Computer problem**) Here is a time sequence of Kepler data on a main-sequence star called Kepler-6, taken during the mission's first 90-day observing campaign. Kepler-6 has a planet, Kepler-6 b, which transited the star 27 times during this campaign. These data have been corrected for a few imperfections and are flux-normalized.*

Fit the limb-darkened transit light curve model you have developed in problems A-D and F above, to these data, employing the function ***`mod()`*** *to produce a phase-folded light curve (e.g. ASTR 111, 7 December 2023); that is, with the time coordinate converted to*

$$\varphi\left(t,t_0,P\right)=\frac{2\pi}{P}\mathrm{mod}(t-t_0,P) \quad ,$$

where t_0 *is adjusted to place* $\varphi=0$ *at mid transit. Use as fitting parameters the orbital period* P*, the time offset* t_0*, the two independent values of the quadratic limb-darkening coefficients (LDCs)* a_n*, the planet's radius p, and its orbital radius and inclination a and i to produce the best fit possible. Report the parameters of the optimal fit, estimate their uncertainties, plot the best-fitting model on top of the data, and compare the LDCs to those of the Sun (ASTR 241, 8 February 2024, page 12).*

One good way to optimize is to minimize the reduced chi-squared for the model and data, χ_R^2*. With* F, $F_{\text{Kepler-6}}$, and $\Delta F_{\text{Kepler-6}}$ *as the model, flux data, and flux-uncertainty data respectively, this is given by*

$$\chi_R^2=\frac{1}{N-m-1}\sum_{j=0}^{N-1}\left(\frac{F_{\text{Kepler-6},j}-F_j}{\Delta F_{\text{Kepler-6},j}}\right)^2 \quad .$$

Adjustment of the m = 7 model parameters to reach a minimum value produces the best fit to the N data points. If the minimum value is within a factor of a few of $\chi_R^2=1$*, then an increment or decrement of a parameter which increases* χ_R^2 *by 1 is a good estimate of the uncertainty in that parameter.*

Result: $P=3.23474\pm0.00005$ days, $t_0=170.006\pm0.0003$ days (since the beginning of the mission), $a_1=2.13\pm0.18$, $2a_2=-1.06\pm0.14$, $a=5.817\pm0.015$, $i=83.77°\pm0.04°$, $p=0.096\pm0.001$; $\chi_R^2=4.3$. I restricted the fit to the $N=229$ points with folded phase $|\varphi|\le0.25$ radians. The uncertainties are increments in the model parameters that raise χ_R^2 by 1.

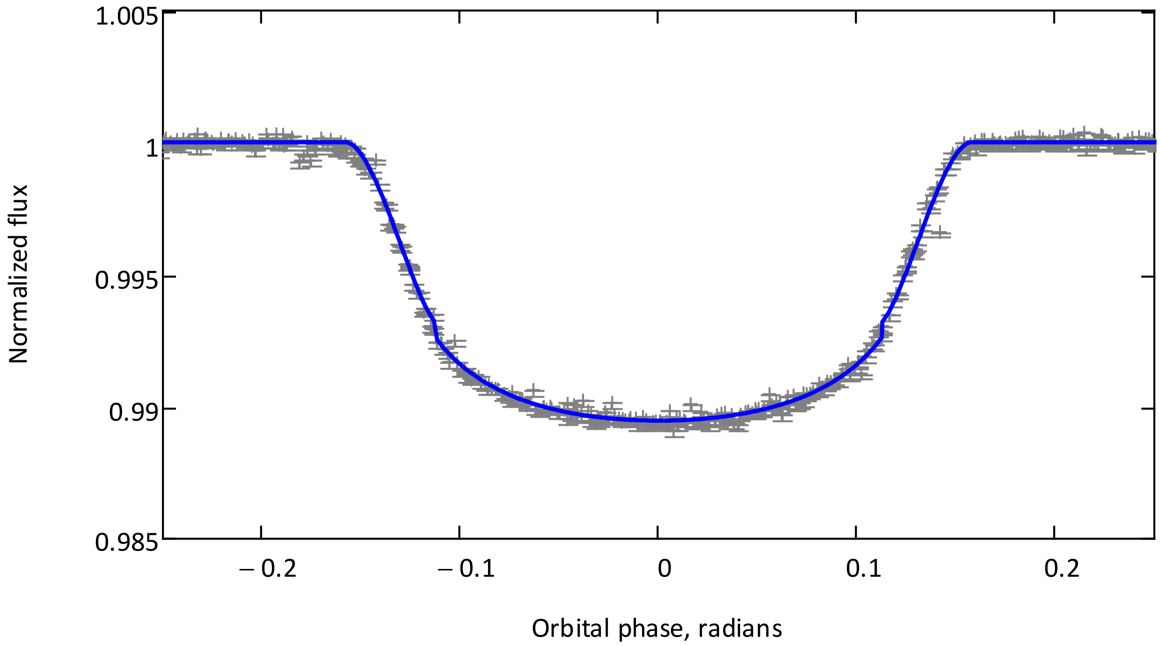
At the end I append the Mathcad code I used for the calculations.

Note that the LDCs are significantly different from the Sun's, for which we get $a_1=1.057$, $2a_2=-0.349$ at $\lambda=600$ nm (class handout, 8 February 2024, page 13). These very same Kepler data were presented in one of the mission's early papers: Dunham+2010. Dunham et al. don't say what LDCs or which formulas they used to fit the light curve; limb darkening wasn't the point of their paper. They do, however, present spectra demonstrating that Kepler-6 has significantly larger metallicity than the Sun, so it's not surprising that our Kepler-6 LDCs are significantly different from the Sun's.

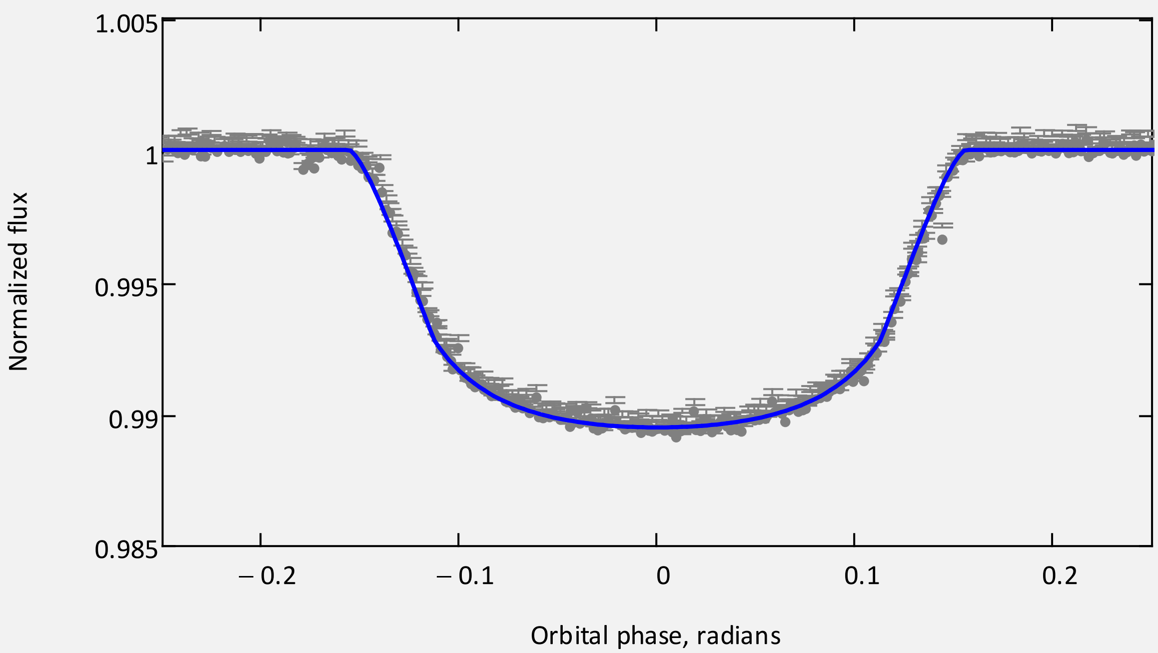
The LDCs we obtain are also a bit unphysical, which John Southworth has pointed out in modelling of these data (Southworth 2011). I'll let that slide til there are LDC measurements over a broader range of wavelengths.

© 2024, University of Rochester

All rights reserved



Note that there is a symmetrical pair of small kinks in the model curve, too small to keep the curve from fitting well, caused by p not really being *much* smaller than 1. Another Mandel & Agol result – the one I use for fitting transit curves – is accurate for larger planets, but is harder to derive. Those who are interested can find it in the Mathcad code at the end. Its use gives the kink-free plot below, resulting in $P = 3.23473 \pm 0.00005$ days, $t_0 = 170.007 \pm 0.0007$ days, $a_1 = 2.161 \pm 0.071$, $2a_2 = -1.082 \pm 0.057$, $a = 5.812 \pm 0.037$, $i = 83.76° \pm 0.09°$, $p = 0.0956 \pm 0.0008$; $\chi_R^2 = 3.9$. Practically the same uncertainties, and not that much better than the small-planet approximation, for these data.



© 2024, University of Rochester All rights reserved

Limb darkening of Kepler-6

For ASTR 241 Spring 2024 Homework #3, problems A-D and F-G.

Light curve for 27 transits of Kepler-6 b, from the first 90-day Kepler campaign ("quarter 1") of the main Kepler mission. See Dunham et al. 2010, https://ui.adsabs.harvard.edu/abs/2010ApJ...713L.136D/abstract.
Other files used: kepler_6b_ltcrv.csv, as downloaded from the Kepler archive.

Last modified 13 December 2024 DMW for consistency with models presented in ASTR 111 Fall 2024

Constants and formulas:

Reference:C:\Program Files (x86)\Mathcad\Mathcad 14\template\Dan's constants and formulas.xmcd

To skip the bits about the preparation of the phase-folded light curve, go directly to section III.

I. Read the data

The .csv file contains time in days, flux and its noise in DN sec^{-1}, extracted from the Q1 FITS data available on MAST, with rows that are all NaNs pruned out.

$M := \text{READFILE}(\text{"kepler\_6b\_ltcrv.csv"}, \text{"delimited"})$ $\qquad J := \text{last}\left(M^{\langle 0 \rangle}\right)$ $\qquad j := 0 .. J$

Take M apart into vectors with recognizable labels:

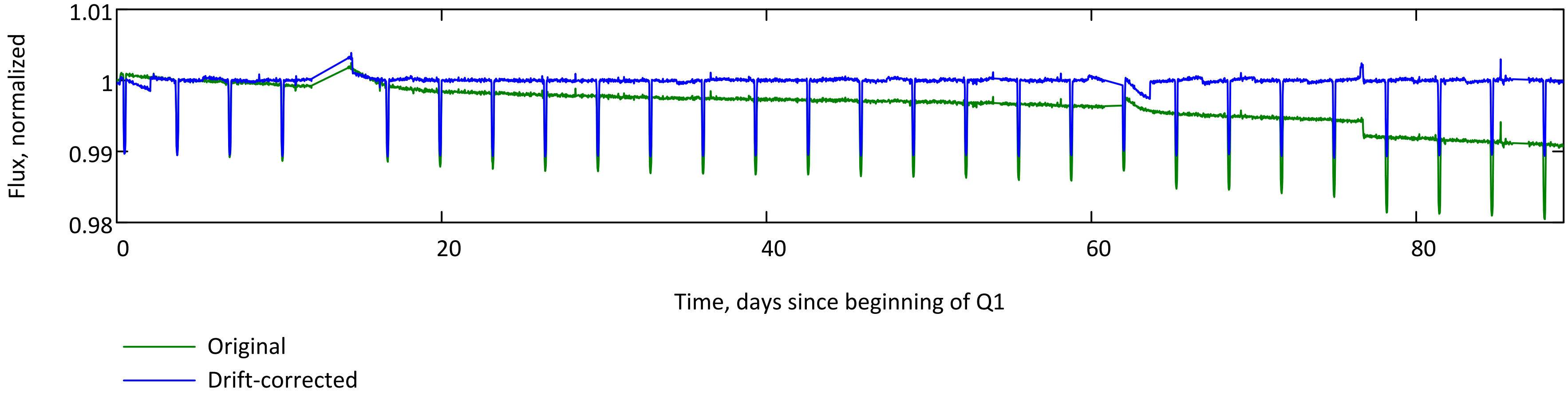
$t_j := \left(M^{\langle 0 \rangle}\right)_j$ $\qquad F := M^{\langle 1 \rangle}$ $\qquad \Delta F := M^{\langle 2 \rangle}$ $\qquad t_{q1} := \left(M^{\langle 0 \rangle}\right)_0$

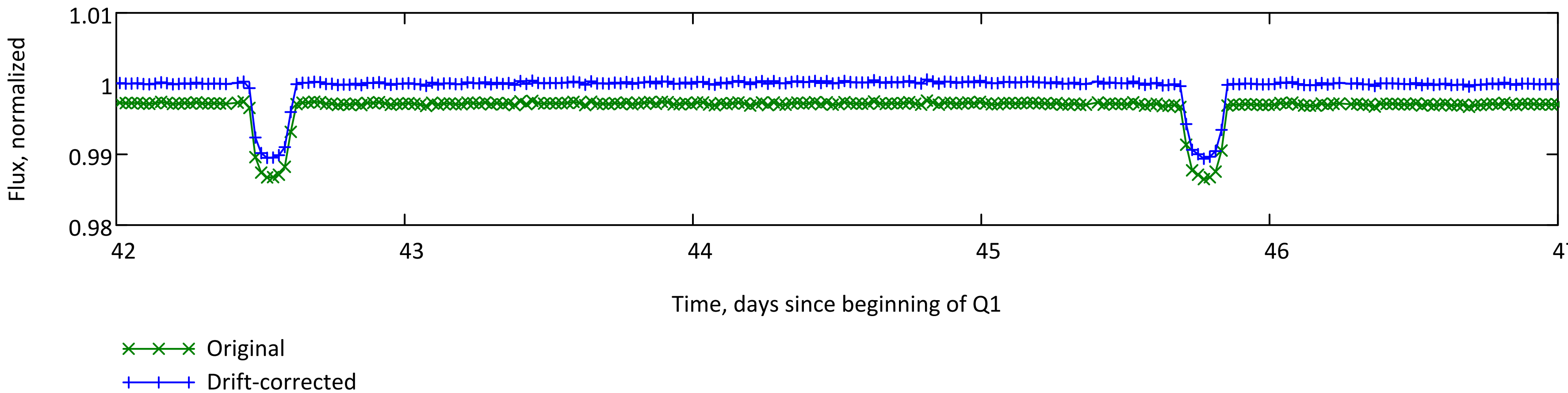
Henceforth the units of time are left in days, and those of flux will be normalized in the next step.

II. De-trend -- correct baseline drift in -- the data, and make a phase-folded normalized light curve

See at right -- or the end, if this is viewed in pdf -- for the script which does the baseline drift correction and normalization.

Full Q1 time series, **with** and **without** baseline correction





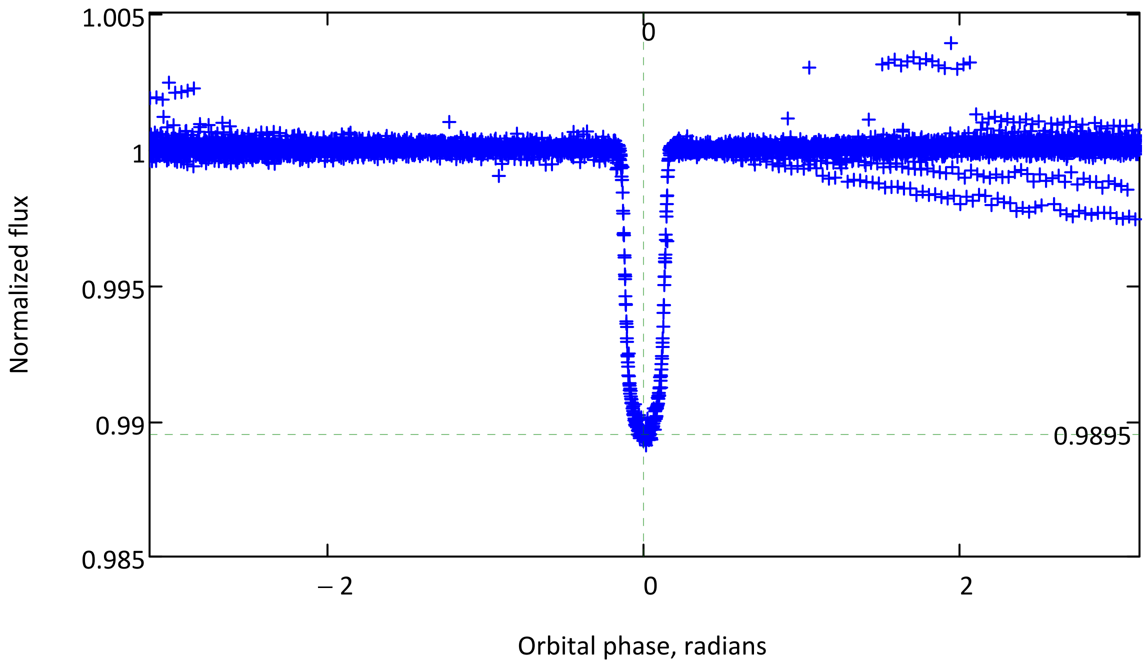
Scale the uncertainties for the normalized light curve:

$$\Delta S_j := \frac{\Delta F_j}{F_j}$$

Phase folding should, in the best case, be done below in the same stroke as all the other fitting, as it is below. This is a just quick look at the region around the transit in the folded light curve, introducing the phase function with a trap to ensure centering of zero phase:

$$P := 3.234723 \qquad t_0 := 170.006 \qquad end := \pi$$

$$\varphi(P, t, t_0) := \mathrm{if}\left(\frac{2\pi}{P}\cdot \mathrm{mod}(t - t_0, P) > \pi, \frac{2\pi}{P}\cdot \mathrm{mod}(t - t_0, P) - 2\pi, \frac{2\pi}{P}\cdot \mathrm{mod}(t - t_0, P)\right)$$



Record a normalized light curve for class use.

$M^{\langle 1\rangle} := S$ $M^{\langle 2\rangle} := \Delta S$ $\text{PRNPRECISION} := 6$ $\text{WRITEPRN}(\text{"kepler-6\_folded.prn"}) := M$

III. Fit a transit curve to the data; determine the star's quadratic limb darkening coefficients (LDCs)

Enable only one of these expressions for the star's intensity normalized to the vertical (problems A-B):

$$I(\alpha,\beta,r) := 1-\alpha-\beta+\alpha\cdot\left(1-r^2\right)^{\frac{1}{2}}+\beta\cdot\left(1-r^2\right)$$

as in C&O chapter 9, and in class on 6 February 2024, **except that $\beta = 2a_2$** here. $\alpha = a_1$ as usual.

$$I(\alpha,\beta,r) := 1+\alpha\cdot\left[1-\left(1-r^2\right)^{\frac{1}{2}}\right]+\beta\cdot\left[1-\left(1-r^2\right)^{\frac{1}{2}}\right]^2$$

as frequently in the literature, and in problem E

If the small planet's shadow lies entirely within the stellar disk in the observer's viewpoint (problem C) -- that is, if

$$r = \sqrt{x^2+z^2} \le 1-p$$

-- then the star's normalized flux is

$$S_{in}(\alpha,\beta,r,p) := 1-\pi p^2\cdot I(\alpha,\beta,r)\cdot\left(2\pi\cdot\int_0^1 I(\alpha,\beta,r)\cdot r\,dr\right)^{-1}$$

If the edge of the star lies within the planet's shadow -- that is, if $1-p < r \le 1+p$

-- then the normalized stellar flux is given by the following formulas, of which **only one should be enabled at any given time**.

First the simpler approximation, as in Problem D (compare Mandel & Agol 2002, equation 8):

$$S_{edge}(\alpha, \beta, r, p) := 1 - \frac{2}{1-(r-p)^2} \cdot \left[p^2 \cdot \mathrm{acos}\left(\frac{r-1}{p}\right) - (r-1) \cdot \sqrt{p^2 - (r-1)^2}\right] \cdot \left(\int_{r-p}^{1} I(\alpha, \beta, r') \cdot r' \, dr'\right) \cdot \left(2\pi \cdot \int_0^1 I(\alpha, \beta, r') \cdot r' \, dr'\right)^{-1}$$

and then a more accurate form, a la Mandel & Agol 2002:

$$S_{edge}(\alpha, \beta, r, p) := 1 - 2\pi \cdot \int_{r-p}^{1} I(\alpha, \beta, r') \cdot r' \, dr' \cdot \left(\frac{\pi \cdot p^2 \cdot I(\alpha, \beta, 1-p)}{2\pi \cdot \int_{1-2p}^{1} I(\alpha, \beta, r') \cdot r' \, dr'}\right) \cdot \left(2\pi \cdot \int_0^1 I(\alpha, \beta, r') \cdot r' \, dr'\right)^{-1}$$

IF-ed together:

$$S(\alpha, \beta, r, p) := \mathrm{if}\left[r > 1 + p, 1, \mathrm{if}\left[(r \le 1 + p) \wedge (r > 1 - p), S_{edge}(\alpha, \beta, r, p), \mathrm{if}\left(r \le 1 - p, S_{in}(\alpha, \beta, r, p), \mathrm{if}(r \le p - 1, 1, -999)\right)\right]\right]$$

Coordinates for orbital motion (problem F). Here and in the following, **θ is used for axis-line of sight angle instead of i**.

$$x(a, \theta, P, t, t_0) := a \cdot \sin(\theta) \sin(\varphi(P, t, t_0))$$

$$z(a, \theta, P, t, t_0) := a \cdot \cos(\theta) \cdot \cos(\varphi(P, t, t_0))$$

$$r(a, \theta, P, t, t_0) := \sqrt{x(a, \theta, P, t, t_0)^2 + z(a, \theta, P, t, t_0)^2}$$

IV. Fitting these formulas to the data

For initial guesses one may use the LDCs for the Sun at $\lambda = 600$ nm. Angular measures are left in **radians**.

$$\begin{pmatrix} P \\ t_0 \\ \alpha \\ \beta \\ a \\ \theta \\ p \end{pmatrix} := \begin{pmatrix} 3.234732 \\ 170.006711 \\ 2.161095 \\ -1.081821 \\ 5.812695 \\ 83.758753 \cdot deg \\ 0.095598 \end{pmatrix}$$

$m := 7$ Number of variables

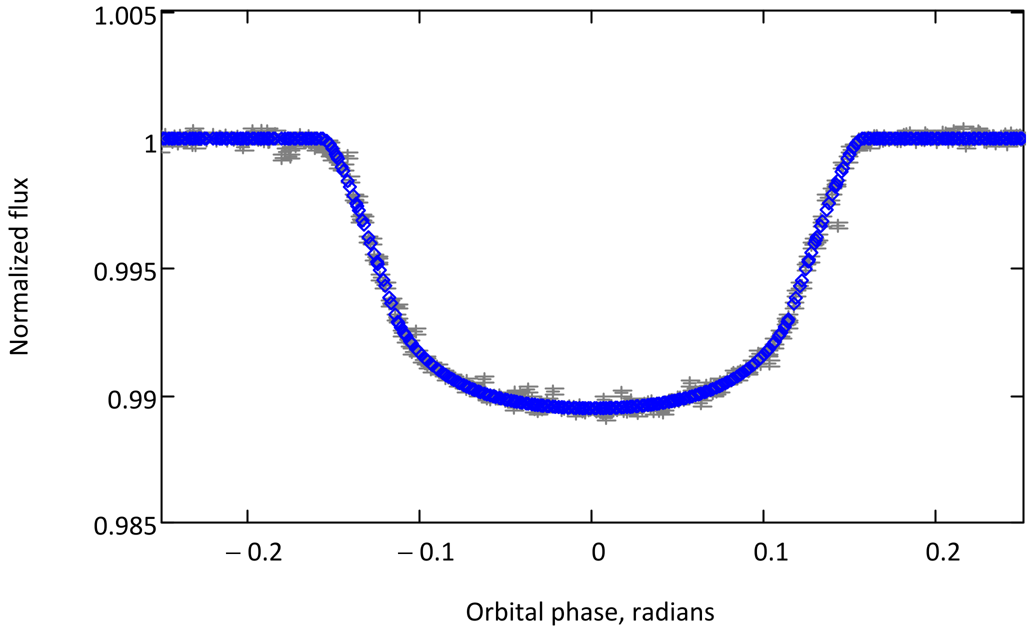
Since the out-of-transit baseline was fit accurately above, we can trim the phase range for minimizing $\chi^2{}_R$ closely around the transit; see the previous plot for guidance.

$$\begin{pmatrix} go \\ stop \end{pmatrix} := \begin{pmatrix} 0 \\ last\left(M^{\langle 0 \rangle}\right) \end{pmatrix} \qquad end := 0.25$$

$$Var(P, t_0, \alpha, \beta, a, \theta, p) := \left[\sum_{j = go}^{stop} \left[if\left[\varphi\left[P, \left(M^{\langle 0 \rangle}\right)_j, t_0\right] \geq -end \wedge \varphi\left[P, \left(M^{\langle 0 \rangle}\right)_j, t_0\right] \leq end, \left[\left(M^{\langle 1 \rangle}\right)_j - S\left[\alpha, \beta, r\left[a, \theta, P, \left(M^{\langle 0 \rangle}\right)_j, t_0\right], p\right]\right]^2 \right.\right.\right.$$

$$DOF := \sum_{j = go}^{stop} if\left[\varphi\left[P, \left(M^{\langle 0 \rangle}\right)_j, t_0\right] \geq -end \wedge \varphi\left[P, \left(M^{\langle 0 \rangle}\right)_j, t_0\right] \leq end, 1, 0\right] - m - 1$$

$$\chi(P, t_0, \alpha, \beta, a, \theta, p) := \frac{1}{DOF} \cdot Var(P, t_0, \alpha, \beta, a, \theta, p)$$



Now find the χ^2 minimum precisely. $TOL := 10^{-5}$

Given

$$0 = \chi(P, t_0, \alpha, \beta, a, \theta, p)$$

$$\begin{pmatrix} P' \\ t_{0'} \\ \alpha' \\ \beta' \\ a' \\ \theta' \\ p' \end{pmatrix} := \text{Minerr}(P, t_0, \alpha, \beta, a, \theta, p) \qquad \begin{pmatrix} P' \\ t_{0'} \\ \alpha' \\ \beta' \\ a' \\ \dfrac{\theta'}{deg} \\ p' \end{pmatrix} = \begin{pmatrix} 3.234715 \\ 170.006986 \\ 2.153419 \\ -1.088771 \\ 5.793657 \\ 83.679253 \\ 0.095807 \end{pmatrix} \qquad \chi(P', t_{0'}, \alpha', \beta', a', \theta', p') = 3.91$$

Calculate uncertainties roughly by using Find (rather than Minerr) to calculate the variable differences which increase χ^2_R to χ^2_R+1. Runs much faster if done one at a time rather than using one Solve block.

$$\begin{pmatrix} \Delta P \\ \Delta t_1 \\ \Delta\alpha \\ \Delta\beta \\ \Delta a \\ \Delta\theta \\ \Delta p \end{pmatrix} := \begin{pmatrix} -0.000049 \\ -0.000692 \\ -0.072067 \\ -0.056692 \\ -0.036811 \\ 0.001611 \\ 0.00076 \end{pmatrix}$$

Guesses. Originals are all zero; usually an iteration appears here.

Given

$$1 = \chi(P + \Delta P, t_0, \alpha, \beta, a, \theta, p) - \chi(P, t_0, \alpha, \beta, a, \theta, p)$$

$$\Delta P := \mathrm{Find}(\Delta P)$$

Given

$$1 = \chi(P, t_0, \alpha, \beta, a, \theta, p + \Delta p) - \chi(P, t_0, \alpha, \beta, a, \theta, p)$$

$$\Delta p := \mathrm{Find}(\Delta p)$$

Given

$$1 = \chi(P, t_0, \alpha, \beta, a + \Delta a, \theta, p) - \chi(P, t_0, \alpha, \beta, a, \theta, p)$$

$$\Delta a := \mathrm{Find}(\Delta a)$$

Given

$$1 = \chi(P, t_0, \alpha, \beta, a, \theta + \Delta\theta, p) - \chi(P, t_0, \alpha, \beta, a, \theta, p)$$

$$\Delta\theta := \mathrm{Find}(\Delta\theta)$$

Given

$$1 = \chi(P, t_0, \alpha + \Delta\alpha, \beta, a, \theta, p) - \chi(P, t_0, \alpha, \beta, a, \theta, p)$$

$\Delta\alpha := \mathrm{Find}(\Delta\alpha)$

Given

$$1 = \chi(P, t_0, \alpha, \beta + \Delta\beta, a, \theta, p) - \chi(P, t_0, \alpha, \beta, a, \theta, p)$$

$\Delta\beta := \mathrm{Find}(\Delta\beta)$

Given

$$1 = \chi(P, t_0 + \Delta t_1, \alpha, \beta, a, \theta, p) - \chi(P, t_0, \alpha, \beta, a, \theta, p)$$

$\Delta t_1 := \mathrm{Find}(\Delta t_1)$ $\Delta t_1 = -7.022 \times 10^{-4}$

For reasons unknown, it takes a very long runtime to Find the uncertainty in t_0 when the formula derived in Problem D (Mandel & Agol 2002 equation 8) is used, but not when the more accurate form is used.

Grand total:

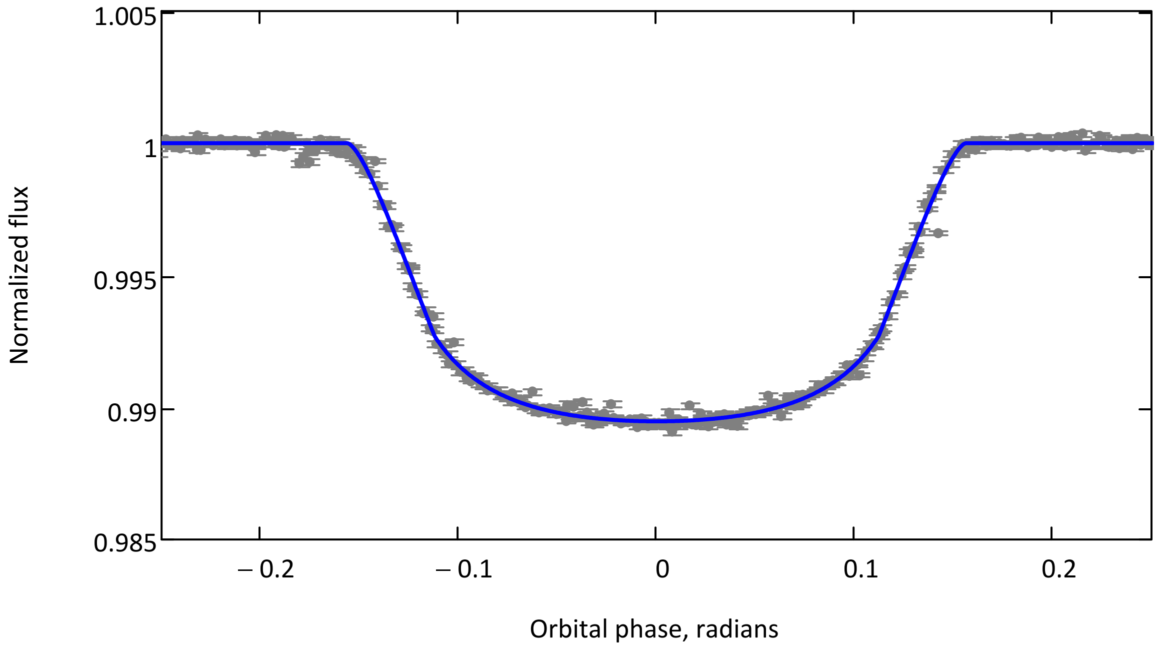
$\chi 2 := \chi(P', t_{0'}, \alpha', \beta', a', \theta', p')$

$$\begin{pmatrix} P \\ p \\ a \\ \dfrac{\theta}{deg} \\ \alpha \\ \beta \\ \chi 2 \end{pmatrix} = \begin{pmatrix} 3.23473 \\ 0.0956 \\ 5.81269 \\ 83.75875 \\ 2.16109 \\ -1.08182 \\ 3.90976 \end{pmatrix} \qquad \begin{pmatrix} |\Delta P| \\ |\Delta p| \\ |\Delta a| \\ \dfrac{|\Delta\theta|}{deg} \\ |\Delta\alpha| \\ |\Delta\beta| \\ \chi 2 + 1 \end{pmatrix} = \begin{pmatrix} 0.00005 \\ 0.00076 \\ 0.03701 \\ 0.09284 \\ 0.07247 \\ 0.057 \\ 4.90976 \end{pmatrix}$$

Plot the final result:

$Z := 500$ $\qquad i := 0 .. 2Z + 1$

$\varphi_{m_i} := \mathrm{end} \cdot \frac{(i - Z)}{Z}$ $\qquad x_{m_i} := a' \cdot \sin(\theta') \sin(\varphi_{m_i})$ $\qquad z_{m_i} := a' \cdot \cos(\theta') \cdot \cos(\varphi_{m_i})$ $\qquad r_{m_i} := \sqrt{(x_{m_i})^2 + (z_{m_i})^2}$



Has two small kinks when using the Problem D result, smooth when more accurate form is used.

$$
\begin{aligned}
S := \;& i \leftarrow 0 \\
& \text{for } j \in 0 .. J \\
& \quad \text{if } F_j < 69569 \\
& \qquad dips_i \leftarrow j \\
& \qquad i \leftarrow i + 1 \\
& m \leftarrow 0 \\
& buff \leftarrow 0 \\
& ibuff \leftarrow 0 \\
& \text{for } i \in 0 .. \operatorname{last}(dips) \\
& \quad buff \leftarrow buff + dips_i \\
& \quad ibuff \leftarrow ibuff + 1 \\
& \quad \text{if } (i < \operatorname{last}(dips)) \wedge (dips_{i+1} > dips_i + 1) \\
& \qquad mins_m \leftarrow \frac{buff}{ibuff} \\
& \qquad m \leftarrow m + 1 \\
& \qquad buff \leftarrow 0 \\
& \qquad ibuff \leftarrow 0 \\
& \quad mins_m \leftarrow \frac{buff}{ibuff} \quad \text{otherwise} \\
& mid_0 \leftarrow 0 \\
& \text{for } imins \in 0 .. \operatorname{last}(mins) - 1 \\
& \quad mid_{imins+1} \leftarrow \operatorname{trunc}\left[\frac{(mins_{imins} + mins_{imins+1})}{2}\right] \\
& mid_{\operatorname{last}(mins)+1} \leftarrow J
\end{aligned}
$$

Baseline drift correction, called **detrending** in the Kepler literature

Find flux minima.

Find midpoints in light curve between minima – thus making it impossible to look for secondary eclipse, but that's OK.

$\mathrm{last}(\mathrm{mins}) + 1$

$\mathrm{fitpts} \leftarrow 9$

$\mathrm{npoly} \leftarrow 1$

$\text{for} \quad \mathrm{imins} \in 0 .. \mathrm{last}(\mathrm{mins})$

Take a ten-point range on either side of each minimum.

$\quad \text{for} \quad \mathrm{ifitrange} \in 0 .. \mathrm{fitpts}$

$\quad\quad \mathrm{fitrange}_{\mathrm{ifitrange}} \leftarrow \mathrm{mins}_{\mathrm{imins}} - 15 + \mathrm{ifitrange}$

$\quad\quad \mathrm{fitrange}_{\mathrm{ifitrange}+1+\mathrm{fitpts}} \leftarrow \mathrm{mins}_{\mathrm{imins}} + 6 + \mathrm{ifitrange}$

$\quad \text{for} \quad \mathrm{ifitrange} \in 0 .. 2\mathrm{fitpts} + 1$

$\quad\quad \mathrm{fr} \leftarrow \mathrm{trunc}\left(\mathrm{fitrange}_{\mathrm{ifitrange}}\right)$

$\quad\quad \mathrm{tfit}_{\mathrm{ifitrange}} \leftarrow t_{\mathrm{fr}}$

$\quad\quad \mathrm{Ffit}_{\mathrm{ifitrange}} \leftarrow F_{\mathrm{fr}}$

$\quad \mathrm{vs} \leftarrow \mathrm{regress}(\mathrm{tfit}, \mathrm{Ffit}, \mathrm{npoly})$

Fit a line to those points, and inter/extra-polate for the reset of the range around each minimum.

$\quad \text{for} \quad \mathrm{istar} \in \mathrm{mid}_{\mathrm{imins}} .. \mathrm{mid}_{\mathrm{imins}+1}$

$\quad\quad \mathrm{star}_{\mathrm{istar}} \leftarrow \mathrm{interp}\left(\mathrm{vs}, \mathrm{tfit}, \mathrm{Ffit}, t_{\mathrm{istar}}\right)$

$\text{for} \quad j \in 0 .. J$

$$S_j \leftarrow \frac{F_j}{\mathrm{star}_j}$$

Normalize.

S