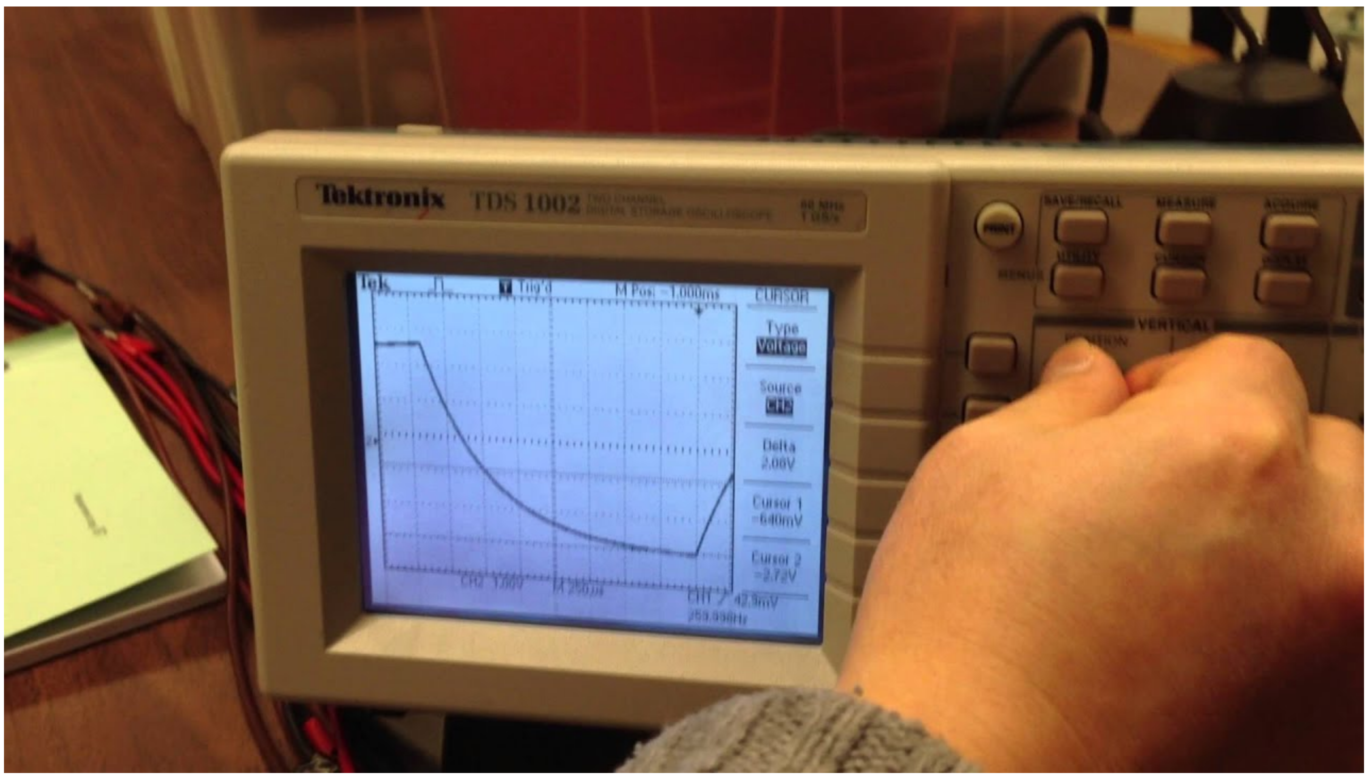
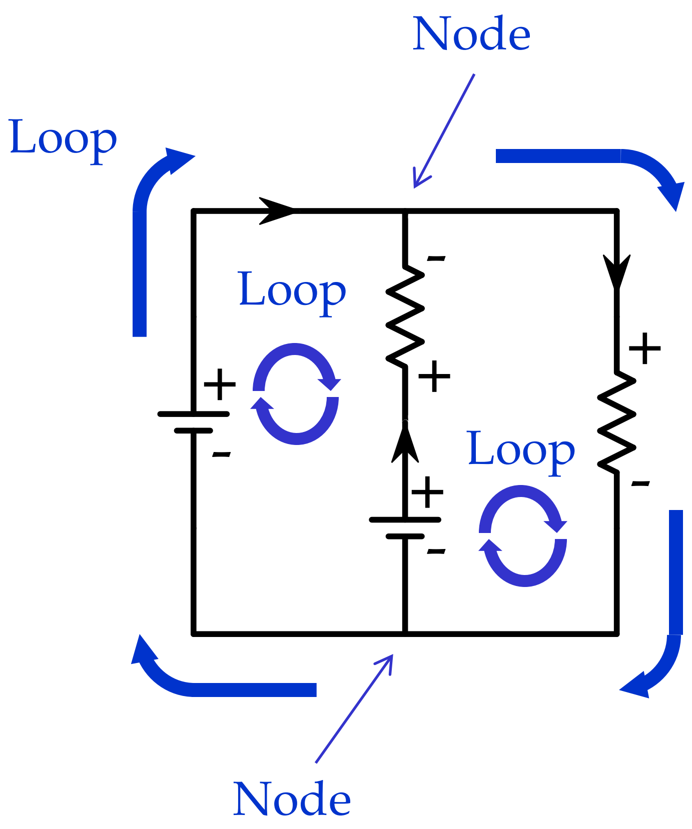
Today in Physics 122: Kirchhoff's rules and *RC* circuits

- ❑ The *RC* time constant
- ❑ Using the Kirchhoff rules in *RC* circuits
- ❑ Dielectric relaxation



Recap: Kirchhoff's Rules

- Charge conservation: the sum of the currents into any node is zero; as much current flows in as out.
- Energy conservation: the sum of the voltage drops for a complete loop through the circuit is zero.



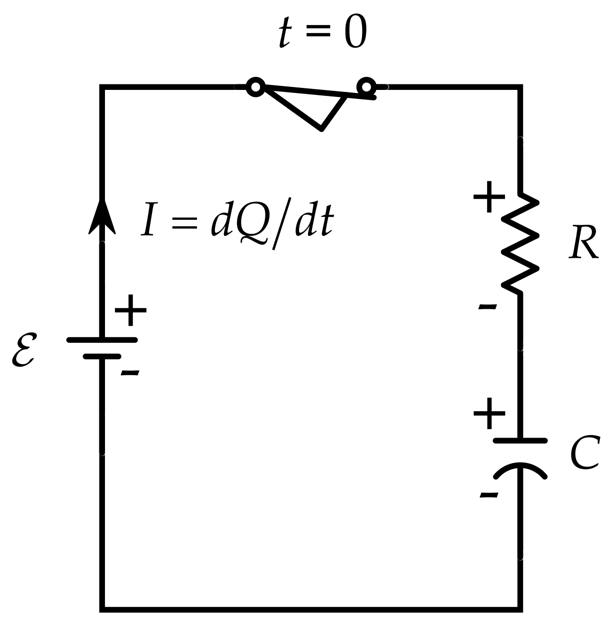
Recap: use of Kirchhoff's Rules

- ❑ Identify the unknown quantities – *N*, say – in the circuit, and count them.
- ❑ Write the node rule and/or the loop rule to generate as many relations between the voltages and currents as there are unknowns (*N*).
  - Use both the node rule and the loop rule, at least once each.
- ❑ This gives a system of *N* equations in *N* unknowns, solvable with algebra.
- ❑ Important point: it **doesn't matter whether you correctly guess the direction of each current**. If you guess wrong, your answer will just be a negative number. You will still know which way the current flows.

RC circuits and the *RC* time constant

In a steady state, ideal capacitors draw no current. But if a circuit is assembled and switched on, one will observe that it take time for the capacitor to receive its full charge.

- Key: since capacitors accumulate a charge Q, the resistors in series with them carry a current $I = dQ/dt$.
- The current in the series R should decrease to zero as C reaches its full charge.



Note: dQ/dt is presumed to have the direction that increases $+Q$.

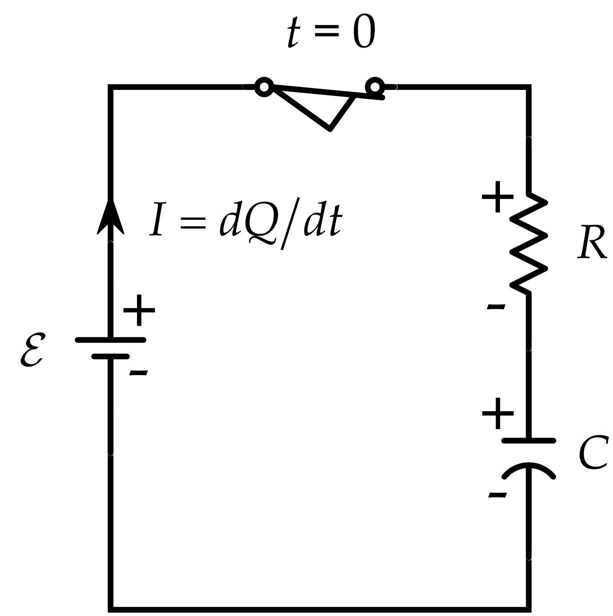
RC circuits and the *RC* time constant (continued)

Let's apply the loop rule to this simple circuit, in which the switch is closed at $t = 0$.

$$V_0 - IR - \frac{Q}{C} = 0$$

$$RC\frac{dQ}{dt} = CV_0 - Q$$

$$\int_0^Q \frac{dQ'}{CV_0 - Q'} = \frac{1}{RC}\int_0^t dt'$$



Change variables: $q = CV_0 - Q' \quad dq = -dQ'$

As $Q' = 0 \to Q,\ q = CV_0 \to CV_0 - Q.$

RC circuits and the *RC* time constant (continued)

Thus the integral is an old friend:

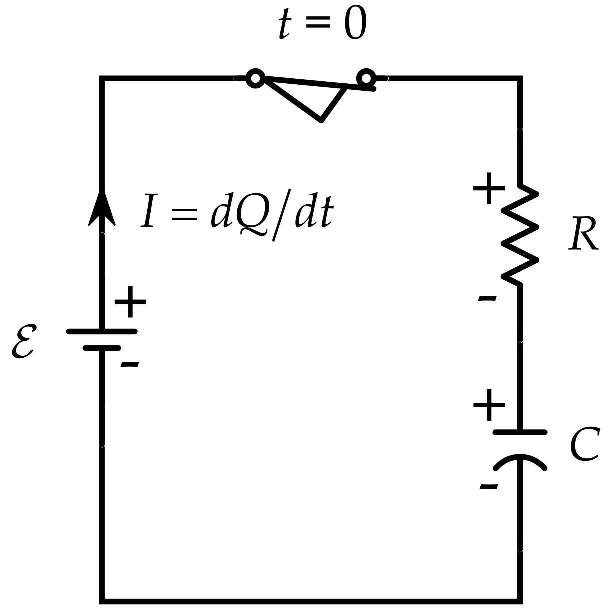
$$-\int_{CV_0}^{CV_0-Q}\frac{dq}{q}=\frac{t}{RC}$$

$$\ln q\Big|_{CV_0}^{CV_0-Q}=\ln\frac{CV_0-Q}{CV_0}=-\frac{t}{RC}$$

$$CV_0-Q=CV_0e^{-t/RC}$$

$$Q(t)=\boxed{CV_0\left(1-e^{-t/RC}\right)}$$

$$I(t)=\frac{dQ}{dt}=\boxed{\frac{CV_0}{RC}e^{-t/RC}}$$



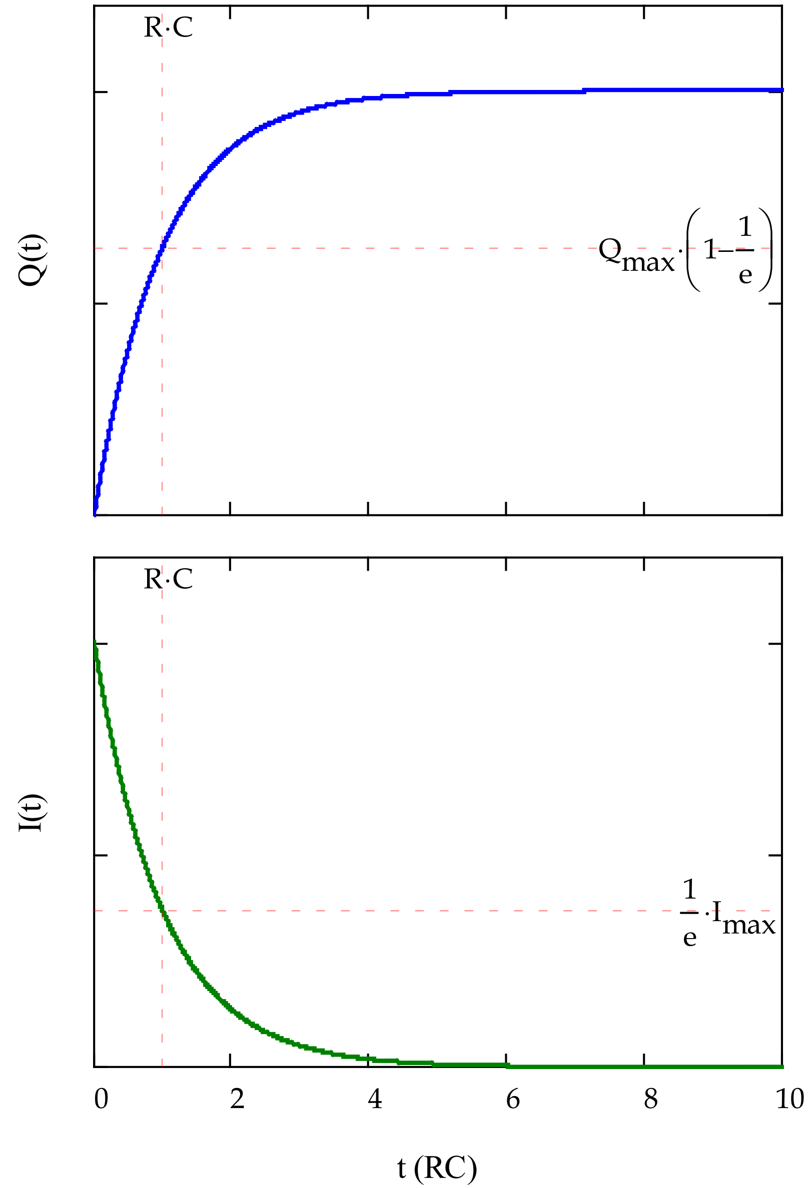
RC circuits and the *RC* time constant (continued)

The results are plotted at right.

- ❑ The current in the resistor eventually drops to zero; it reaches $1/e$ of its initial value in $t = RC$.

- ❑ The charge on the capacitor eventually reaches CV_0; it comes within $1/e$ of this value at $t = RC$.

The quantity $\tau = RC$ is called the circuit's **time constant**, and is a handy measure of how long it takes to charge the capacitor.

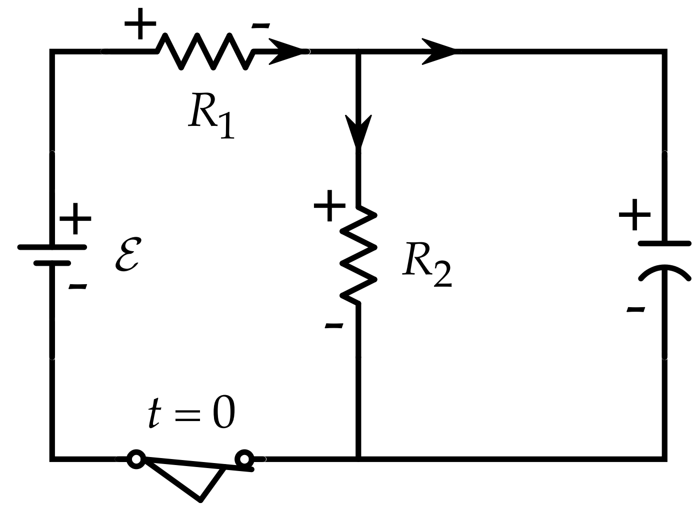


C with series and parallel R

In more complex circuits with capacitors one can still use Kirchhoff's rules confidently in much the same way as with purely resistive circuits.

- **Example 1**: *Determine the time constant, and the maximum charge on the capacitor, in this circuit.*

  [Intuitively, we can tell that the maximum charge should be that for which all the current flows through the resistors; in this case the voltage across C would be $\mathcal{E}R_1/(R_1+R_2)$, and the charge $Q = C\,\mathcal{E}R_1/(R_1+R_2)$.]

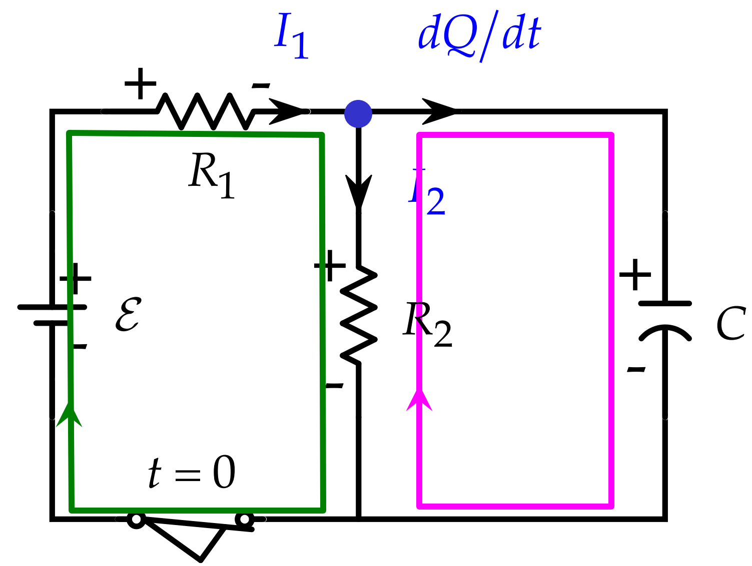


C with series and parallel R (continued)

- Like the other three-branch circuits we've dealt with, we use the node equation once and the loop equation twice:

$$I_1 - I_2 - dQ/dt = 0$$
$$V_0 - I_1 R_1 - I_2 R_2 = 0$$
$$I_2 R_2 - Q/C = 0$$

- As before, the algebra strategy is to target one variable, and substitute out the other two in its favor. This time we target Q.



Three unknown currents.

Note: dQ/dt is presumed to have the direction that increases $+Q$.

C with series and parallel R (continued)

- Second loop equation:

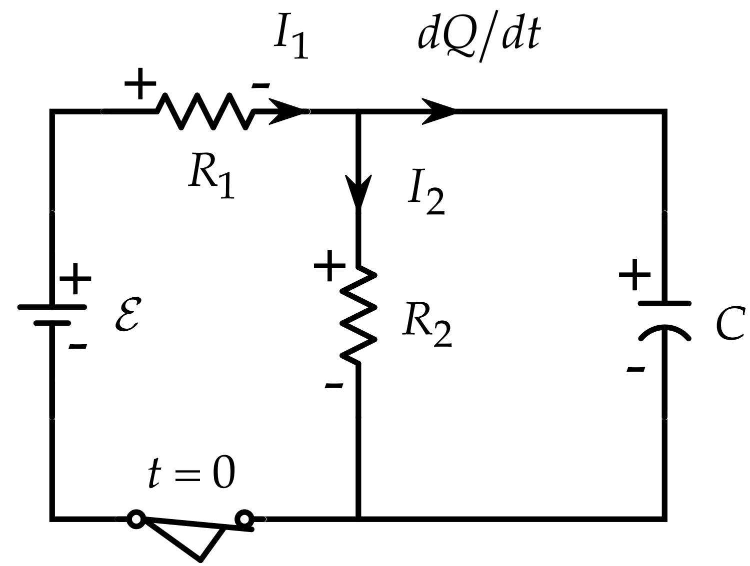
$$I_2 = Q/R_2C$$

- Node equation:

$$I_1 = I_2 + \frac{dQ}{dt} = \frac{Q}{R_2C} + \frac{dQ}{dt}$$

- Results into first loop equation, now in Q alone.

$$V_0 - R_1\left(\frac{Q}{R_2C} + \frac{dQ}{dt}\right) - \frac{Q}{C} = 0$$



C with series and parallel R (continued)

- Solve for dQ/dt:

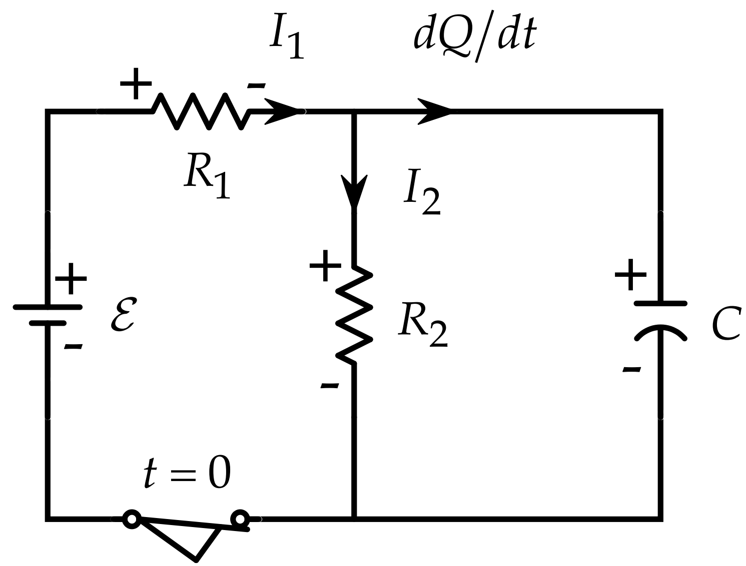
$$\frac{dQ}{dt} = \frac{V_0}{R_1} - \left(\frac{Q}{R_2 C} + \frac{Q}{R_1 C}\right)$$

$$= \frac{V_0}{R_1} - Q\left(\frac{R_1 + R_2}{R_1 R_2 C}\right)$$

$$= \frac{1}{R_1 C}\left(CV_0 - \frac{R_1 + R_2}{R_2} Q\right)$$

- Separate and integrate:

$$\int_0^Q \frac{dQ'}{CV_0 - \frac{R_1 + R_2}{R_2} Q'} = \frac{1}{R_1 C}\int_0^t dt'$$



C with series and parallel R (continued)

- Same integral as before. Change variables:

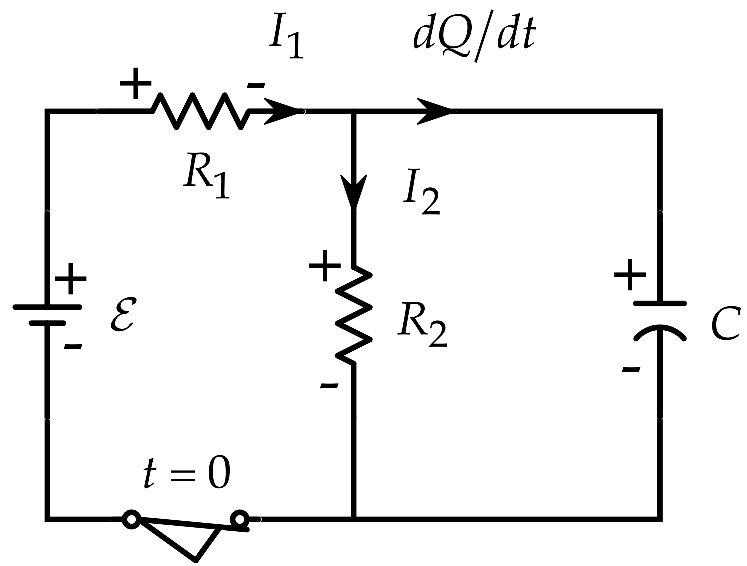
$$q = CV_0 - \frac{R_1 + R_2}{R_2} Q'$$

$$dq = -\frac{R_1 + R_2}{R_2} dQ'$$

As $Q' = 0 \rightarrow Q$,

$$q = CV_0 \rightarrow CV_0 - \frac{R_1 + R_2}{R_2} Q:$$

$$-\frac{R_2}{R_1 + R_2} \int_{CV_0}^{CV_0 - \frac{R_1+R_2}{R_2}Q} \frac{dq}{q} = \frac{t}{R_1 C}$$



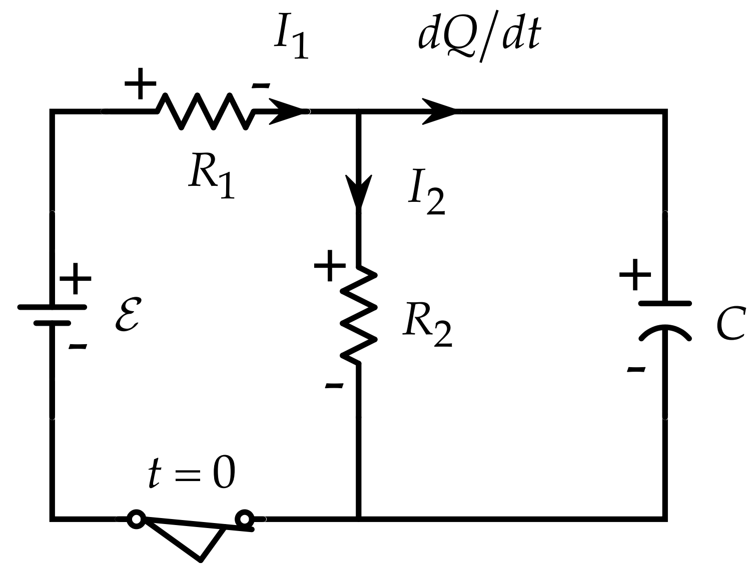
C with series and parallel R (continued)

$$\ln q\Big|_{CV_0}^{CV_0-\frac{R_1+R_2}{R_2}Q} = -\frac{R_1+R_2}{R_1R_2C}t \equiv -\frac{t}{R_{||}C}$$

$$\ln\frac{CV_0-\frac{R_1+R_2}{R_2}Q}{CV_0} = -\frac{t}{R_{||}C}$$

$$\frac{CV_0-\frac{R_1+R_2}{R_2}Q}{CV_0} = e^{-t/R_{||}C}$$

$$Q(t) = \boxed{\frac{R_2}{R_1+R_2}CV_0\left(1-e^{-t/R_{||}C}\right)}$$



C with series and parallel R (continued)

Comments on the solution:

- Indeed the maximum charge on the capacitor comes out to

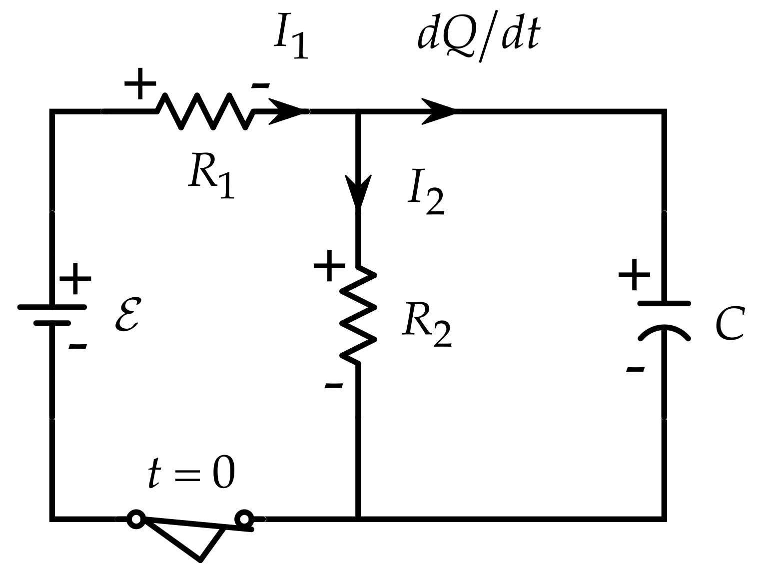
$$Q_{\max} = \lim_{t\to\infty} Q(t) = \frac{R_2}{R_1 + R_2} C V_0 \,,$$

as we expected.

- The time constant is

$$\tau = -\frac{t}{R_{\parallel} C} = \frac{R_1 + R_2}{R_1 R_2 C} t,$$

as if determined by the **parallel** combination of the Rs.

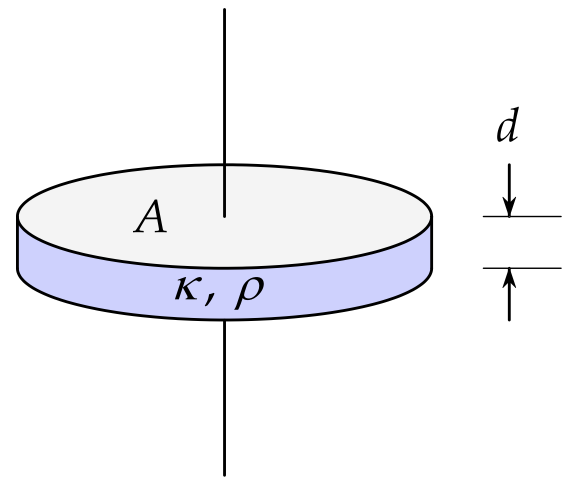


Dielectric relaxation

An **application** of *RC* circuits to the physics of imperfect conductors/insulators:

Materials that have very large – but not infinite – resistivity can exhibit significant polarization, characterized by a dielectric constant > 1.

- Even the "bad" conductors out of which resistors are made, like graphite, have resistivity small enough that polarization effects are small: $\kappa = 1$.



Consider a parallel-plate capacitor (area A, separation d) filled with such material (dielectric constant K, resistivity ρ). At $t = 0$ it has charge Q. What is the time constant with which the charge leaks away?

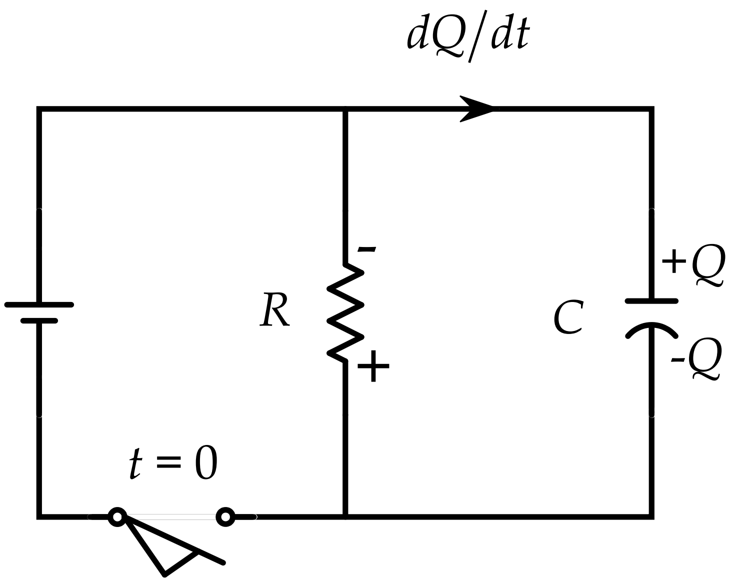
Dielectric relaxation (continued)

Solution:

- We worked out the capacitance and resistance of this arrangement before, on 25 September and 27 September respectively:

$$C = \frac{\kappa\varepsilon_0 A}{d} \quad , \quad R = \frac{\rho d}{A} \quad ,$$

and we noted that the R and C could be considered to be in parallel in the circuit-element sense.



Note: *dQ/dt* is presumed to have the direction that increases $+Q$.

Dielectric relaxation (continued)

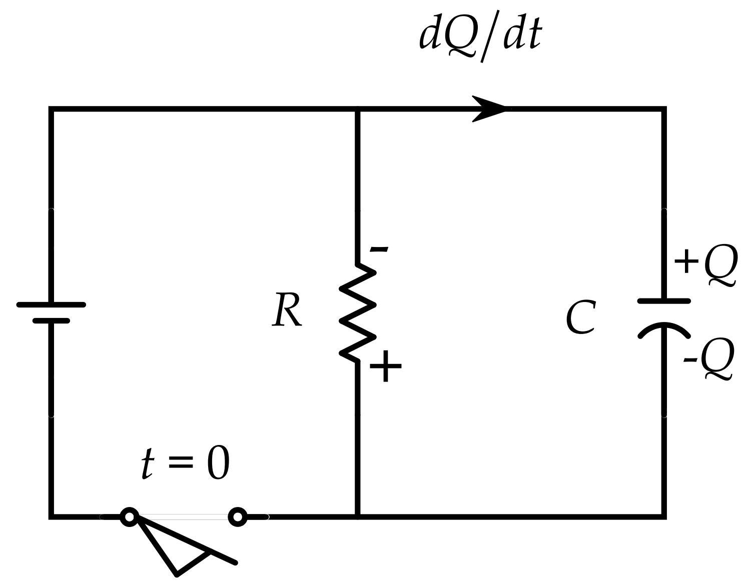
- Only one Kirchhoff Rule equation here, and it's a loop:

$$R\frac{dQ}{dt}+\frac{Q}{C}=0$$

$$\int_{Q_0}^{Q(t)}\frac{dQ'}{Q'}=-\frac{1}{RC}\int_0^t dt'$$

$$\ln\left(\frac{Q(t)}{Q_0}\right)=-\frac{t}{RC}$$

$$Q(t)=Q_0e^{-t/RC}$$



so $\tau = RC$, as we might have expected.

Dielectric relaxation (continued)

- What's interesting about this result – at least to physicists, materials scientists, and electrical engineers – is that everything having to do with the shape of the capacitor vanishes from the expression for the time constant:

$$\tau = RC = \frac{\rho d}{A}\frac{\kappa\varepsilon_0 A}{d} = \rho\kappa\varepsilon_0 .$$

cf. Exam #1, problem 1d.

- Check with cylindrical capacitor (25 September) and resistor (27 September):

$$\tau = RC = \frac{\rho}{2\pi L}\ln\left(\frac{R_2}{R_1}\right)\frac{2\pi\kappa\varepsilon_0 L}{\ln\left(R_2/R_1\right)} = \rho\kappa\varepsilon_0 .$$

- That is, this time constant is a property of the material, not of the shape: it is the time constant associated with how quickly charges can be moved around in high-resistivity material. That there is a limit to how quickly, is a phenomenon called **dielectric relaxation**.