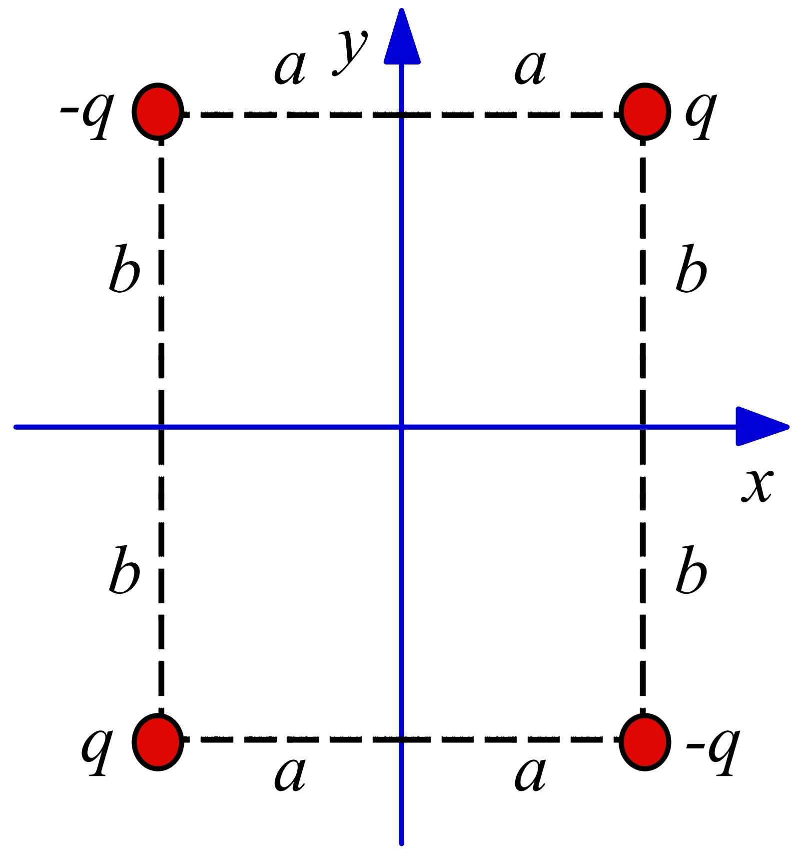
Today in Physics 217: the method of images

- ❑ Solving the Laplace and Poisson equations by sleight of hand
- ❑ Introduction to the method of images
- ❑ Caveats
- ❑ Example: a point charge and a grounded conducting sphere
- ❑ Multiple images



Solving the Laplace and Poisson equations by sleight of hand

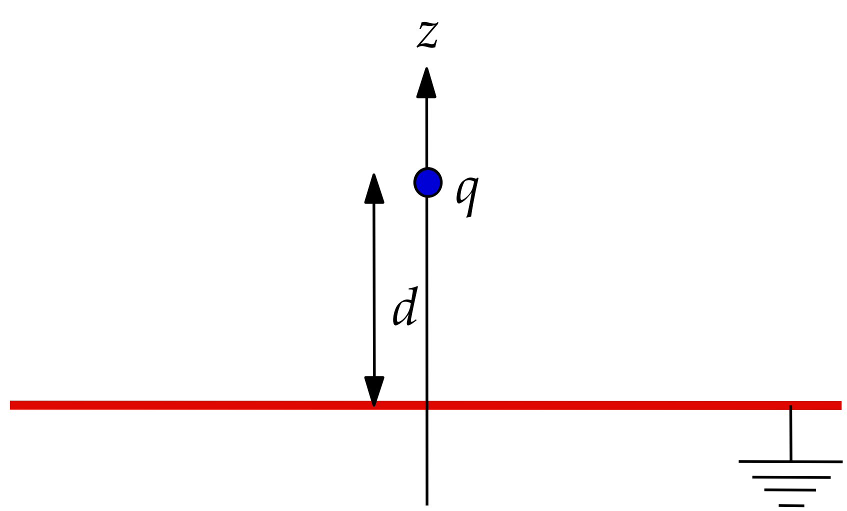
The guaranteed uniqueness of solutions has spawned several creative ways to solve the Laplace and Poisson equations for the electric potential. We will treat three of them in this class:

- ❑ **Method of images** (today).
  Very powerful technique for solving electrostatics problems involving charges and conductors.
- ❑ **Separation of variables**
  Perhaps the most useful technique for solving partial differential equations. You'll be using it frequently in quantum mechanics too.
- ❑ **Multipole expansion**
  Fermi used to say, "When in doubt, expand in a power series." This provides another fruitful way to approach problems not immediately accessible by other means.

Introduction to the method of images

A point charge lies a distance d above a infinite, conducting, grounded plane. Calculate the potential V everywhere above the plane.

- This looks like a Laplace-equation problem, and we know some boundary conditions at the plane: $V = 0$, $E_x = E_y = 0$.
- But there's charge induced on the grounded plane.The electrostatic potential can not be calculated directly without knowing the induced charge distribution.



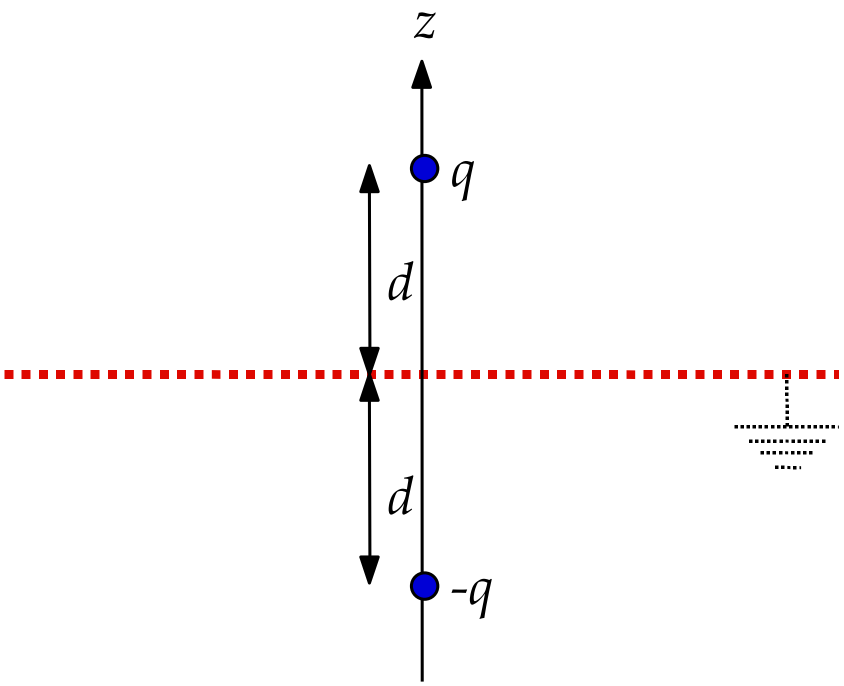
Introduction to the method of images (continued)

Consider alternatively the situation of two point charges q and $-q$, separated by $2d$.

The potential can be calculated directly and is equal to

$$V(s,\phi,z) = \frac{q}{\sqrt{s^2 + (z-d)^2}} + \frac{-q}{\sqrt{s^2 + (z+d)^2}}$$

Note that $\nabla^2 V = -4\pi\rho$ is automatically satisfied. This also gives $V = 0$ on the plane $z = 0$, just as it would need to be for the grounded plane.



Introduction to the method of images (continued)

The potential yields the electric field, as usual:

$$\boldsymbol{E}(s,\phi,z) = -\nabla V = \frac{\partial V}{\partial s}\hat{\boldsymbol{s}} + \frac{1}{s}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\varphi}} + \frac{\partial V}{\partial z}\hat{\boldsymbol{z}}$$

No ϕ dependence

$$= \frac{q\boldsymbol{s}}{\left[s^2+(z-d)^2\right]^{3/2}} - \frac{q\boldsymbol{s}}{\left[s^2+(z+d)^2\right]^{3/2}}$$

$$+ \frac{q(z-d)\hat{\boldsymbol{z}}}{\left[s^2+(z-d)^2\right]^{3/2}} - \frac{q(z+d)\hat{\boldsymbol{z}}}{\left[s^2+(z+d)^2\right]^{3/2}}$$

Note $\boldsymbol{E}(s,\phi,0) = -\dfrac{2qd\hat{z}}{\left(s^2+d^2\right)^{3/2}}$,

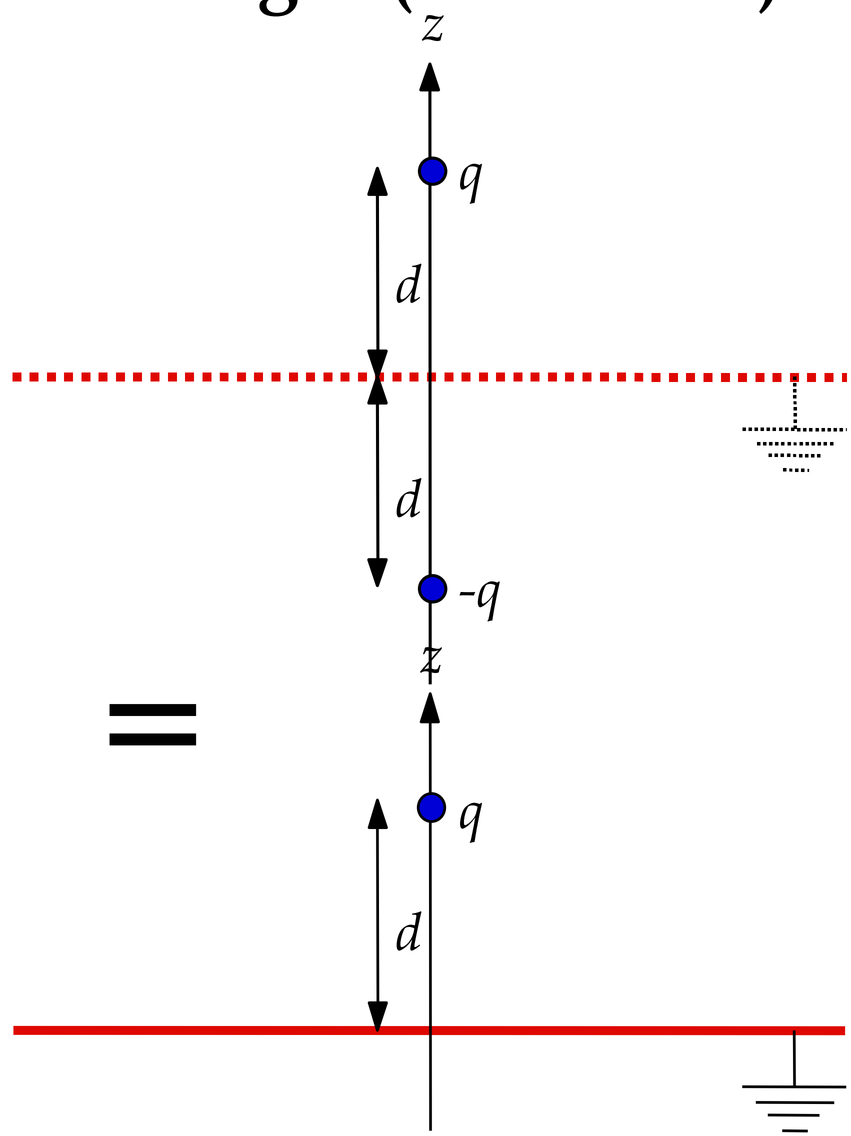
***E* perpendicular to *z* = 0 plane, at *z* = 0**, just as it would need to be with the grounded plane.

whence $\sigma(s,\phi) = \dfrac{E_{\perp,\text{ above}}(s,\phi,0)}{4\pi} = -\dfrac{qd}{2\pi\left(s^2+d^2\right)^{3/2}}.$

Introduction to the method of images (continued)

Thus, **for $z \geq 0$**, the two-charge potential satisfies the Poisson equation and the boundary conditions for the single charge – grounded plane problem: it is a solution to this problem.

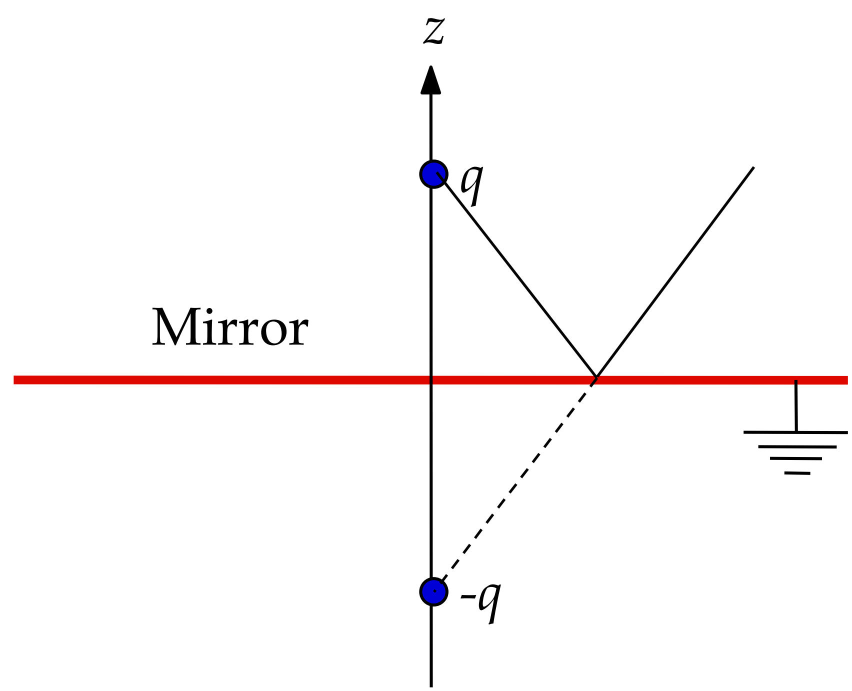
But there is no "a" solution, only "the" solution, because solutions of electrostatics problems are unique.



Introduction to the method of images (continued)

How did we know this would work? Is there a method by which we could guess these solutions in general?

Yes. The auxiliary charge is the **image** of the original charge in the "mirror" that comprises the grounded conducting plane. Other configurations of charges and grounded conductors can be treated similarly: as if they were objects, images and mirrors in geometrical optics.



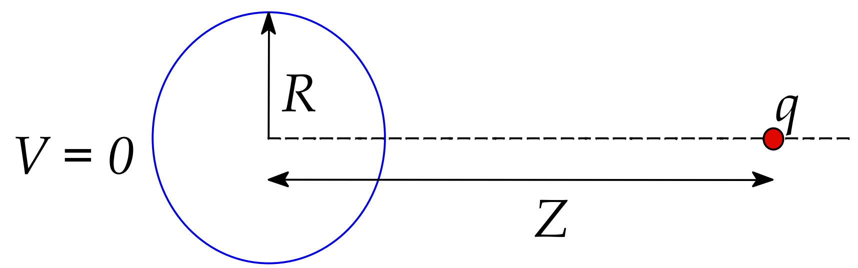
Caveats for the method of images

- The solution for the images is only the same as that for the conductor, in the region outside the conductor! In particular, the field is still zero, and the potential constant, inside the conductor. Remember that the image charge doesn't really exist.
- In particular, the potential energy of the charge+conductor arrangement is quite different from the charge-image charge combination, because the field is finite for the latter in locations where the field of the former is zero. (Got it?)
- Remind yourself of these facts by noting, in every image solution, that you can calculate the induced charge on the surface of the grounded conductor.

Potential for a point charge and a grounded sphere (Example 3.2 + Problem 3.7 in Griffiths)

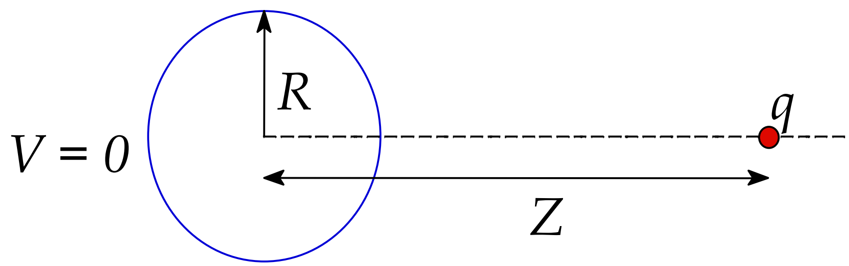
A point charge q is situated a distance Z from the center of a grounded conducting sphere of radius R.

- Find the potential everywhere.
- Find the induced surface charge on the sphere, as function of θ. Integrate this to get the total induced charge.
- Calculate the potential energy of the system.



Potential for a point charge and a grounded sphere (continued)

We need to find a position to put charge q' such that the boundary conditions on the sphere are satisfied. Start by determining where this charge should lie, using the points along the z axis:

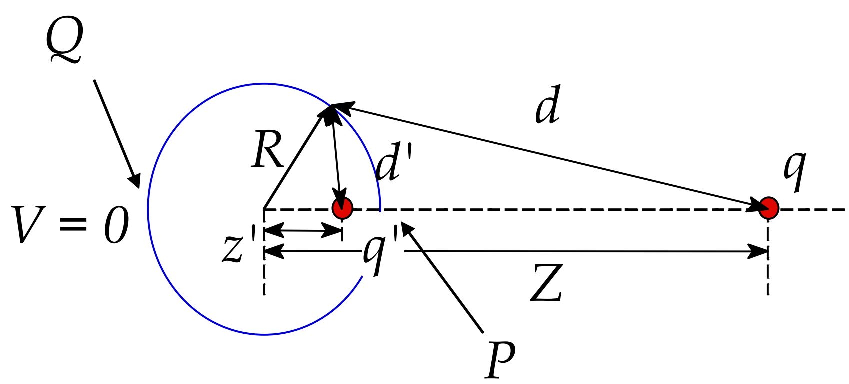


$$V_P = 0 = \frac{q}{Z-R} + \frac{q'}{R-z'}$$

$$\Rightarrow \quad q' = -\frac{q}{Z-R}(R-z')$$

$$V_Q = 0 = \frac{q}{Z+R} + \frac{q'}{R+z'}$$

$$\Rightarrow \quad q' = -\frac{q}{Z+R}(R+z')$$



Potential for a point charge and a grounded sphere (continued)

So $\frac{q}{Z-R}\left(R-z'\right)=\frac{q}{Z+R}\left(R+z'\right)$

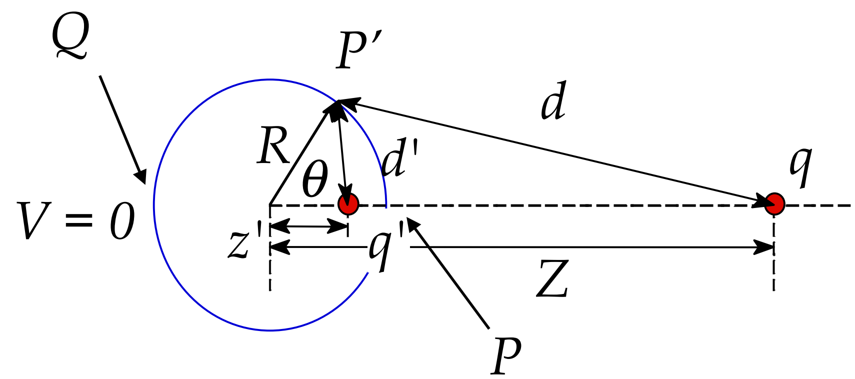
$\left(Z+R\right)\left(R-z'\right)=\left(Z-R\right)\left(R+z'\right)$

Rearrange:

$z'\left(Z-R\right)+z'\left(Z+R\right)=2Zz'$

$=R\left(Z+R\right)-R\left(Z-R\right)=2R^2$

$$\Rightarrow \quad z'=\frac{R^2}{Z}, \quad q'=-\frac{q}{Z+R}\left(R+z'\right)=-\frac{q}{Z+R}\left(R+\frac{R^2}{Z}\right)=-q\frac{R}{Z}$$



Now consider an arbitrary point on the sphere, *P′*:

$$d=\sqrt{R^2+Z^2-2RZ\cos\theta}$$

$$d'=\sqrt{R^2+z'^2-2Rz'\cos\theta}=\sqrt{R^2+\frac{R^4}{Z^2}-2R\frac{R^2}{Z}\cos\theta}$$

Potential for a point charge and a grounded sphere (continued)

The potential should come out to be zero there, and sure enough,

$$V_{P'} = \frac{q}{d} + \frac{q'}{d'} = \frac{q}{\sqrt{R^2 + Z^2 - 2RZ\cos\theta}} + \frac{-q\,\dfrac{R}{Z}}{\sqrt{R^2 + \dfrac{R^4}{Z^2} - 2R\,\dfrac{R^2}{Z}\cos\theta}}$$

$$= \frac{q}{\sqrt{R^2 + Z^2 - 2RZ\cos\theta}} + \frac{-q\,\dfrac{R}{Z}}{\sqrt{R^2 + \dfrac{R^4}{Z^2} - 2R\,\dfrac{R^2}{Z}\cos\theta}}$$

$$= \frac{q}{\sqrt{R^2 + Z^2 - 2RZ\cos\theta}} - \frac{q}{\sqrt{R^2 + Z^2 - 2RZ\cos\theta}} = 0$$

Thus the potential outside the grounded sphere is given by the superposition of the potential of the charge q and the image charge q'.

Potential for a point charge and a grounded sphere (continued)

So the potential at some point (r,θ,ϕ) outside the sphere is given by

$$V = \frac{q}{\sqrt{r^2 + Z^2 - 2rZ\cos\theta}} + \frac{-q\dfrac{R}{Z}}{\sqrt{r^2 + \dfrac{R^4}{Z^2} - 2r\dfrac{R^2}{Z}\cos\theta}}$$

$$= \boxed{\frac{q}{\sqrt{r^2 + Z^2 - 2rZ\cos\theta}} - \frac{q}{\sqrt{\left(\dfrac{rZ}{R}\right)^2 + R^2 - 2rZ\cos\theta}}}$$

Now for the induced charge density:

$$\sigma = \frac{1}{4\pi}E_r = -\frac{1}{4\pi}\frac{\partial V}{\partial r}$$

Differentiate the formula above for the potential, and evaluate it at $r = R$:

Potential for a point charge and a grounded sphere (continued)

$$\left.\frac{\partial V}{\partial r}\right|_{r=R} = q\left[\frac{-r+Z\cos\theta}{\left(r^2+Z^2-2rZ\cos\theta\right)^{3/2}} - \frac{-\dfrac{rZ^2}{R^2}+Z\cos\theta}{\left(\left(\dfrac{rZ}{R}\right)^2+R^2-2rZ\cos\theta\right)^{3/2}}\right]_{r=R}$$

$$= q\frac{-R+Z\cos\theta}{\left(R^2+Z^2-2RZ\cos\theta\right)^{3/2}} - q\frac{-\dfrac{Z^2}{R}+Z\cos\theta}{\left(Z^2+R^2-2RZ\cos\theta\right)^{3/2}}$$

$$= \frac{q}{R}\frac{Z^2-R^2}{\left(R^2+Z^2-2RZ\cos\theta\right)^{3/2}}$$

$$\sigma = -\frac{1}{4\pi}\left.\frac{\partial V}{\partial r}\right|_{r=R} = \boxed{-\frac{q}{4\pi R}\frac{Z^2-R^2}{\left(R^2+Z^2-2RZ\cos\theta\right)^{3/2}}}$$

Potential for a point charge and a grounded sphere (continued)

From this we get the total charge induced on the grounded sphere:

$$Q = \int \sigma R^2 \sin\theta \; d\theta \; d\phi = -\frac{q}{2} R\left(Z^2 - R^2\right) \int_0^\pi \frac{\sin\theta \; d\theta}{\left(R^2 + Z^2 - 2RZ\cos\theta\right)^{3/2}}$$

$$= -\frac{q}{2} R\left(Z^2 - R^2\right) \left[\frac{-\frac{1}{ZR}}{\sqrt{R^2 + Z^2 - 2RZ\cos\theta}} \right]_0^\pi$$

$$= \frac{q}{2} \frac{Z^2 - R^2}{Z} \left[\frac{1}{|R+Z|} - \frac{1}{|R-Z|} \right] = \boxed{-q\frac{R}{Z}}$$

Goes to zero as Z goes to infinity.

Potential for a point charge and a grounded sphere (continued)

Potential energy: first, the force between the charge and image,

$$\boldsymbol{F}_{qq'} = \frac{qq'}{\left(Z - z'\right)^2}\hat{\boldsymbol{z}} = \frac{q\left(-\dfrac{R}{Z}q\right)}{\left(Z - \dfrac{R^2}{Z}\right)^2}\hat{\boldsymbol{z}} = -\frac{ZRq^2}{\left(Z^2 - R^2\right)^2}\hat{\boldsymbol{z}}$$

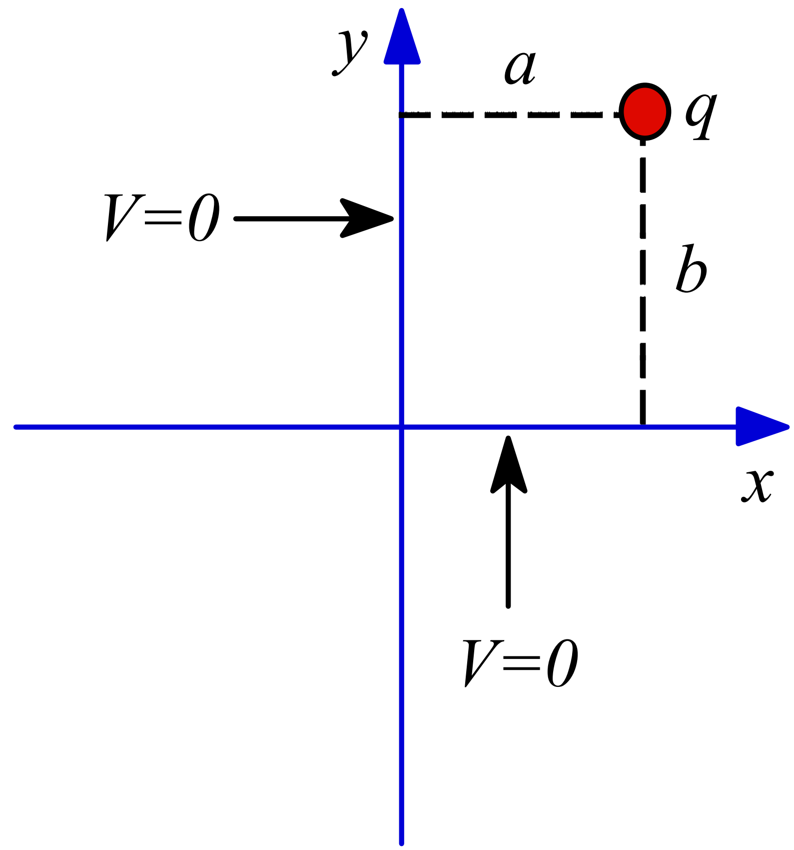
and then the work to bring q in from infinity:

$$W = \int_{\infty}^{Z} -\boldsymbol{F}_{qq'} \cdot d\boldsymbol{l} = \int_{\infty}^{Z} \frac{zRq^2}{\left(z^2 - R^2\right)^2}dz = \left.\frac{-Rq^2}{2\left(z^2 - R^2\right)}\right|_{\infty}^{Z} = \boxed{-\frac{1}{2}\frac{Rq^2}{Z^2 - R^2}}$$

Multiple images

Here's a hint for **Problem 3.10**, on this week's homework:

Two semi-infinite grounded conducting planes meet at right angles. In the region between them, there is a point charge q, situated as shown at right. Set up the image configuration, and calculate the potential in this region. What charges do you need, and where should they be located? What is the force on q? How much work did it take to bring q in from infinity?



Multiple images (continued)

In order to satisfy the boundary conditions, *three* image charges must be added to the system.

- The net force on q can be calculated by determining the vector sum of the forces on q due to the three image charges.
- The electrostatic energy of the real system is equal to 1/4 of the electrostatic energy of the image-charge system.