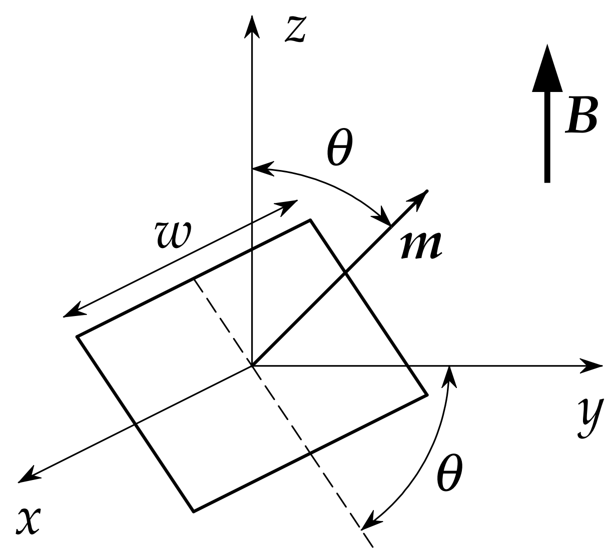
Today in Physics 217: magnetic multipoles

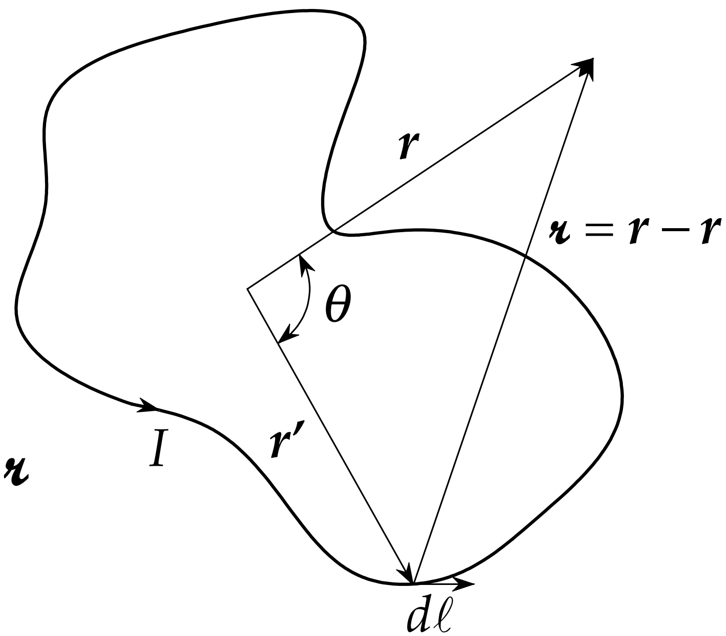
- ❑ Multipole expansion of the magnetic vector potential
- ❑ Magnetic dipoles
- ❑ Magnetic field from a magnetic dipole
- ❑ Torque on a magnetic dipole in uniform B
- ❑ Force and energy and magnetic dipoles



Multipole expansion of the magnetic vector potential

Consider an arbitrary loop that carries a current I. Its vector potential at point $\boldsymbol{r}$ is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{I}{c}\oint \frac{d\ell}{\boldsymbol{\mathscr{r}}} \quad .$$



Just as we did for V, we can expand $1/\boldsymbol{\mathscr{r}}$ in a power series and use the series as an approximation scheme:

$$\frac{1}{\boldsymbol{\mathscr{r}}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^{n} P_n(\cos\theta)$$

(see lecture notes for 21 October 2002 for derivation).

Multipole expansion of the magnetic vector potential (continued)

Put this series into the expression for $\boldsymbol{A}$:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{I}{c}\left[\frac{1}{r}\oint d\ell + \frac{1}{r^2}\oint r'\cos\theta \, d\ell \right.$$ Monopole, dipole,

$$+\frac{1}{r^3}\oint r'^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)d\ell$$ quadrupole,

$$+\frac{1}{r^4}\oint r'^3\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right)d\ell$$ octupole

$$\left. + \ldots \right]$$

Of special note in this expression:

Multipole expansion of the magnetic vector potential (continued)

- The monopole term is zero, since

$$\oint d\ell = 0 \quad .$$

  This isn't surprising, since "no magnetic monopoles" is built into the Biot-Savart law, from which we obtained $\boldsymbol{A}$.
- For points far away from the loop compared to its size, we obtain a good approximation for $\boldsymbol{A}$ by using just the first (or first two) nonvanishing terms. (For points closer by, one would need more terms for the same accuracy.)
- This is, of course, the same useful behaviour we saw in the multipole expansion of V.

Magnetic dipoles

$$\boldsymbol{A}_{\text{dipole}}(\boldsymbol{r}) = \frac{I}{cr^2}\oint r' \cos\theta \, d\ell = \frac{I}{cr^2}\oint (\hat{\boldsymbol{r}} \cdot \boldsymbol{r}') \, d\ell$$

Note that $$d\left[(\hat{\boldsymbol{r}} \cdot \boldsymbol{r}')\boldsymbol{r}'\right] = (\hat{\boldsymbol{r}} \cdot d\boldsymbol{r}')\boldsymbol{r}' + (\hat{\boldsymbol{r}} \cdot \boldsymbol{r}') d\boldsymbol{r}'$$

$$\oint d\left[(\hat{\boldsymbol{r}} \cdot \boldsymbol{r}')\boldsymbol{r}'\right] = \oint (\hat{\boldsymbol{r}} \cdot d\boldsymbol{r}')\boldsymbol{r}' + \oint (\hat{\boldsymbol{r}} \cdot \boldsymbol{r}') d\boldsymbol{r}'$$

but $\oint d\left[(\hat{\boldsymbol{r}} \cdot \boldsymbol{r}')\boldsymbol{r}'\right] = 0$, so

$$\oint (\hat{\boldsymbol{r}} \cdot d\boldsymbol{r}')\boldsymbol{r}' = -\oint (\hat{\boldsymbol{r}} \cdot \boldsymbol{r}') d\boldsymbol{r}' \quad .$$

Also,

$$\hat{\boldsymbol{r}} \times \oint \boldsymbol{r}' \times d\boldsymbol{r}' = \oint \boldsymbol{r}'(\hat{\boldsymbol{r}} \cdot d\boldsymbol{r}') - \oint (\hat{\boldsymbol{r}} \cdot \boldsymbol{r}') d\boldsymbol{r}' = -2\oint (\hat{\boldsymbol{r}} \cdot \boldsymbol{r}') d\boldsymbol{r}' \quad .$$

Magnetic dipoles (continued)

Thus $$\oint (\hat{\boldsymbol{r}}\cdot\boldsymbol{r}')\,d\boldsymbol{r}' = \oint (\hat{\boldsymbol{r}}\cdot\boldsymbol{r}')\,d\boldsymbol{\ell} = -\frac{1}{2}\hat{\boldsymbol{r}}\times\oint \boldsymbol{r}'\times d\boldsymbol{\ell}$$

so $$\boldsymbol{A}_{\text{dipole}} = -\frac{1}{2}\frac{I}{cr^2}\hat{\boldsymbol{r}}\times\oint \boldsymbol{r}'\times d\boldsymbol{\ell}$$

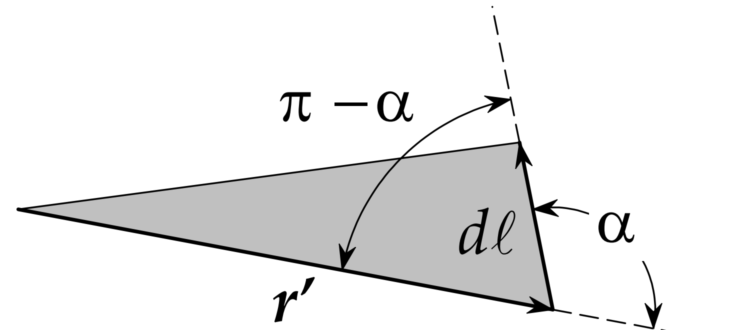
$$\equiv -\frac{\hat{\boldsymbol{r}}\times\boldsymbol{m}}{r^2} = \frac{\boldsymbol{m}\times\hat{\boldsymbol{r}}}{r^2} \quad ,$$

where $$\boldsymbol{m} \equiv \frac{I}{2c}\oint \boldsymbol{r}'\times d\boldsymbol{\ell} \quad .$$

(Compare to $V_{\text{dipole}} = \dfrac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2}$, $\boldsymbol{p} = \int \rho\,\boldsymbol{r}'\,d\tau'$.)

Magnetic dipoles (continued)

To see what this means in terms of geometry of a current loop and its dipole moment, consider the triangle formed by $\boldsymbol{r}'$ and $d\boldsymbol{\ell}$:

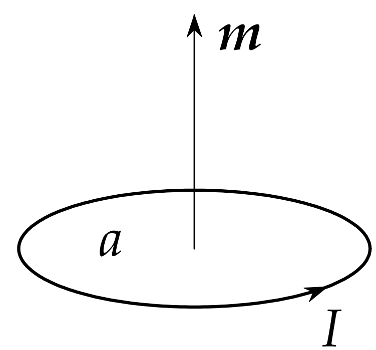


$$\left|\frac{1}{2}\boldsymbol{r}'\times d\boldsymbol{\ell}\right| = \frac{1}{2}r'd\ell\sin\alpha = \frac{1}{2}r'd\ell\sin(\pi-\alpha)$$

$$= \text{ area of triangle.}$$

So $$\frac{1}{2}\boldsymbol{r}'\times d\boldsymbol{\ell} = d\boldsymbol{a}'$$, and

$$\boldsymbol{m} = \frac{I}{c}\oint d\boldsymbol{a}' = \frac{I}{c}\boldsymbol{a}$$ for plane loops.



Magnetic field from a magnetic dipole

For a dipole with $\boldsymbol{m} = m\hat{\boldsymbol{z}}$, we have, in spherical coordinates,

$$\boldsymbol{A}_{\text{dipole}} = \frac{\boldsymbol{m}\times\hat{\boldsymbol{r}}}{r^2} = \frac{m\sin\theta}{r^2}\hat{\boldsymbol{\varphi}} \quad ,$$

so

$$\boldsymbol{B}_{\text{dipole}} = \nabla\times\boldsymbol{A}_{\text{dipole}} = \frac{\hat{\boldsymbol{r}}}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\, A_\phi - \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial r} rA_\phi$$

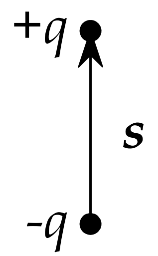
$$= \frac{m\hat{\boldsymbol{r}}}{r^3\sin\theta}\frac{\partial}{\partial\theta}\sin^2\theta - \frac{m\sin\theta\,\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial r}\frac{1}{r}$$

$$= \frac{m\hat{\boldsymbol{r}}}{r^3\sin\theta}2\sin\theta\cos\theta + \frac{m\sin\theta\,\hat{\boldsymbol{\theta}}}{r^3}$$

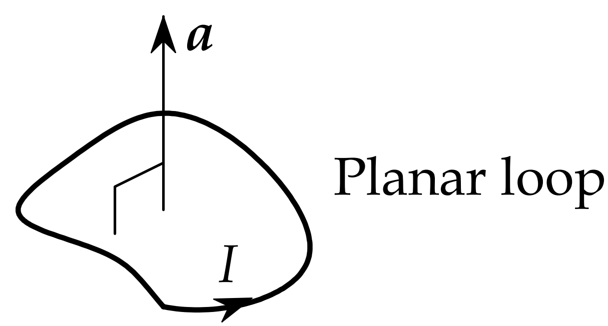
$$= \boxed{\frac{2m\cos\theta}{r^3}\hat{\boldsymbol{r}} + \frac{m\sin\theta}{r^3}\hat{\boldsymbol{\theta}}} = \frac{1}{r^3}\left[3\left(\boldsymbol{m}\cdot\hat{\boldsymbol{r}}\right)\hat{\boldsymbol{r}} - \boldsymbol{m}\right] \quad .$$

Magnetic field from a magnetic dipole (continued)

- That is to say, ***B*** from magnetic dipoles looks **exactly** like ***E*** from electric dipoles.
- We can thus work out forces, torques, energies, and even a lot about magnetostatics in magnetically-polarizable matter **in strict analogy with electrostatics.** The real-dipole paradigms:

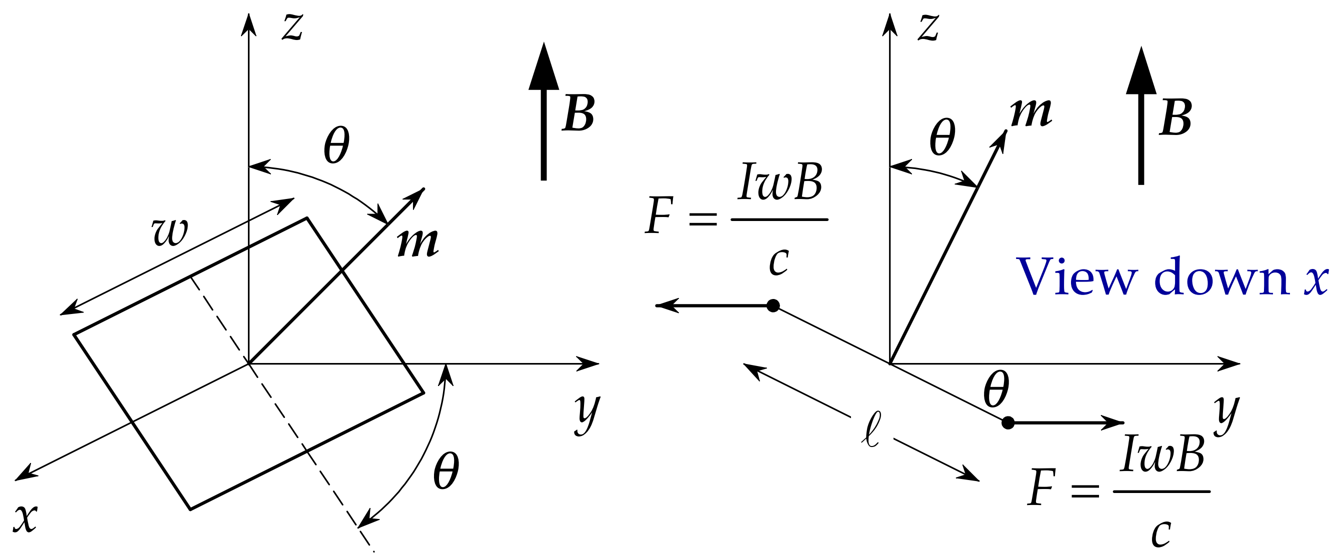


$\mathbf{p} = q\mathbf{s}$



$\mathbf{m} = I\mathbf{a}/c$ ($I\mathbf{a}$ in MKS)

Torque on a magnetic dipole in uniform B



$$N = \ell F \sin\theta = \frac{Iw\ell}{c} B \sin\theta = mB\sin\theta$$

$$\Rightarrow \quad \mathbf{N} = \mathbf{m} \times \mathbf{B} \qquad \left(\text{cf. } \mathbf{N} = \mathbf{p} \times \mathbf{E}\right).$$

Force and energy and magnetic dipoles

Thus the following should be no surprise:

$$\boldsymbol{F} = \nabla\left(\boldsymbol{m}\cdot\boldsymbol{B}\right)$$

$$= \left(\boldsymbol{m}\cdot\nabla\right)\boldsymbol{B}$$ if no currents exist at dipole's position. (cf. $\boldsymbol{F} = \left(\boldsymbol{p}\cdot\nabla\right)\boldsymbol{E}$)

$$\mathcal{W} = -\boldsymbol{m}\cdot\boldsymbol{B}$$ (cf. $\mathcal{W} = -\boldsymbol{p}\cdot\boldsymbol{E}$)

Magnetic dipoles thus tend to align in magnetic fields in the same way that electric dipoles align in electric fields.