Today in Physics 218: the Maxwell equations

- Beyond magneto-quasistatics
- Displacement current, and Maxwell's repair of Ampère's Law
- The Maxwell equations
- Symmetry of the equations: magnetic monopoles?

Rainbow over the Potala Palace, Lhasa, Tibet, by Galen Rowell.

Beyond magnetoquasistatics

In PHY 217, we came up with the basic equations for electrodynamics, namely Gauss's law, the "no magnetic monopoles" law, Faraday's law and Ampère's law:

$$\nabla\cdot\boldsymbol{E}=\frac{\rho}{\varepsilon_0}\qquad \nabla\cdot\boldsymbol{B}=0$$
$$\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}\qquad \nabla\times\boldsymbol{B}=\mu_0\boldsymbol{J}$$

in MKS units, or

$$\nabla\cdot\boldsymbol{E}=4\pi\rho\qquad \nabla\cdot\boldsymbol{B}=0$$
$$\nabla\times\boldsymbol{E}=-\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}\qquad \nabla\times\boldsymbol{B}=\frac{4\pi}{c}\boldsymbol{J}$$

in our preferred cgs units.

Beyond magnetoquasistatics (continued)

As has no doubt been mentioned to you, these equations are, *strictly* speaking, false. Why? Because we know that the divergence of a curl has to be zero, yet from these equations,

$$\nabla\cdot(\nabla\times\boldsymbol{E})=-\frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\boldsymbol{B}=0\qquad (\nabla\cdot\boldsymbol{B}\text{ always}=0),\text{ but}$$
$$\nabla\cdot(\nabla\times\boldsymbol{B})=\frac{4\pi}{c}\nabla\cdot\boldsymbol{J}=-\frac{4\pi}{c}\frac{\partial\rho}{\partial t}\qquad \left(\text{continuity: }\frac{\partial\rho}{\partial t}+\nabla\cdot\boldsymbol{J}=0\right)$$
$$=0\qquad\text{only if }\frac{\partial\rho}{\partial t}=0.$$

Thus these equations are only an approximation, good only when the rate of change of charge density is small enough. We call this approximation magnetoquasistatics.

Beyond magnetoquasistatics (continued)

The problem, of course, is Ampère's law. We derived this law from the Biot-Savart law, using along the way the magnetostatic condition $\nabla \cdot \boldsymbol{J} = 0$.

Here, I'll remind you how it went; please consult your notes from PHY 217, or view

http://www.pas.rochester.edu/~dmw/phy217/Lectures/Lect_27b.pdf

for the context of the derivation.

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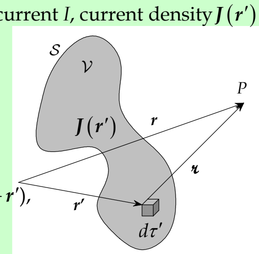
Flashback: Derivation of Ampère's Law

Any vector field is uniquely specified by its divergence and curl. What are the divergence and curl of $\boldsymbol{B}$?
Consider a volume $\mathcal{V}$ to contain current I, current density $\boldsymbol{J}(\boldsymbol{r}')$:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{1}{c}\int \frac{\boldsymbol{J}(\boldsymbol{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$



Denote gradient with respect to the components of $\boldsymbol{r}$ and $\boldsymbol{r}'$ by ∇ and ∇'. Now note that

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\nabla'\left(\frac{1}{\mathscr{r}}\right) \quad \text{(because } \mathscr{r} = \boldsymbol{r} - \boldsymbol{r}'\text{)},$$

and $$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \quad .$$

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Flashback (continued)

With these,

$$\boldsymbol{B}(\boldsymbol{r}) = -\frac{1}{c}\int_{\mathcal{V}} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \times \boldsymbol{J}(\boldsymbol{r}') d\tau' = \frac{1}{c}\int_{\mathcal{V}} \nabla\left(\frac{1}{\mathscr{r}}\right) \times \boldsymbol{J}(\boldsymbol{r}') d\tau'$$

$$= \frac{1}{c}\nabla \times \int_{\mathcal{V}} \frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}} d\tau' \quad \left(\text{remember, } \boldsymbol{J} \neq f(\boldsymbol{r})\right).$$

This is a useful form for $\boldsymbol{B}$, which we will use a lot next lecture too (the integral turns out to be the magnetic vector potential, $\boldsymbol{A}$). Take its divergence:

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = \frac{1}{c}\nabla \cdot \left(\nabla \times \int_{\mathcal{V}} \frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}} d\tau'\right) = 0 \quad .$$

The divergence of *any* curl is zero, remember.

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Flashback (continued)

Integrate this last expression over any volume:

$$\int \nabla \cdot \boldsymbol{B}(\boldsymbol{r})\, d\tau = \oint \boldsymbol{B} \cdot d\boldsymbol{a} = 0 \quad .$$

Compare these to the expressions for $\boldsymbol{E}$ in electrostatics, and we see that magnetostatics involves no counterpart of charge: there's no "magnetic charge."

Now for the curl:

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \frac{1}{c} \nabla \times \nabla \times \int_{\mathcal{V}} \frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}} d\tau' \quad .$$

Use Product Rule #10:

$$\nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \quad :$$

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Flashback (continued)

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \frac{1}{c} \nabla \left(\nabla \cdot \int_{\mathcal{V}} \frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}} d\tau' \right) - \frac{1}{c} \nabla^2 \int_{\mathcal{V}} \frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}} d\tau'$$

$$= \frac{1}{c} \nabla \left(\int_{\mathcal{V}} \nabla \cdot \frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}} d\tau' \right) - \frac{1}{c} \int_{\mathcal{V}} \boldsymbol{J}(\boldsymbol{r}') \nabla^2 \left(\frac{1}{\mathscr{r}} \right) d\tau' \quad .$$

Now use your old friend Product Rule #5,

$$\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot (\nabla f) \quad ,$$

to write

=0 ($\boldsymbol{J}$ independent of $\boldsymbol{r}$)

$$\nabla \cdot \frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}} d\tau' = \left(\frac{1}{\mathscr{r}} \right) \nabla \cdot \boldsymbol{J}(\boldsymbol{r}') + \boldsymbol{J}(\boldsymbol{r}') \cdot \nabla \left(\frac{1}{\mathscr{r}} \right) = \boldsymbol{J}(\boldsymbol{r}') \cdot \nabla \left(\frac{1}{\mathscr{r}} \right)$$

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Flashback (continued)

Also, $\nabla^2 \left(\frac{1}{\mathscr{r}} \right) = \nabla \cdot \nabla \left(\frac{1}{\mathscr{r}} \right) = \nabla \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = 4\pi \delta^3(\mathscr{r}) \quad ,$

so $\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \frac{1}{c} \nabla \int_{\mathcal{V}} \boldsymbol{J}(\boldsymbol{r}') \cdot \nabla \left(\frac{1}{\mathscr{r}} \right) d\tau' + \frac{4\pi}{c} \int_{\mathcal{V}} \boldsymbol{J}(\boldsymbol{r}') \delta^3(\boldsymbol{r} - \boldsymbol{r}') d\tau'$

$$= -\frac{1}{c} \nabla \int_{\mathcal{V}} \boldsymbol{J}(\boldsymbol{r}') \cdot \nabla' \left(\frac{1}{\mathscr{r}} \right) d\tau' + \frac{4\pi}{c} \boldsymbol{J}(\boldsymbol{r}) \quad .$$

Here's where we assumed statics:

Use Product Rule #5 again, on the first term:

$$\boldsymbol{J}(\boldsymbol{r}') \cdot \nabla' \left(\frac{1}{\mathscr{r}} \right) = \nabla' \cdot \left(\frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}} \right) - \frac{1}{\mathscr{r}} \nabla' \cdot \boldsymbol{J}(\boldsymbol{r}') = \nabla' \cdot \left(\frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}} \right)$$

=0 in magnetostatics

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Flashback (continued)

So,

$$\nabla\times B(r) = -\frac{1}{c}\nabla\left(\int_{\mathcal{V}}\nabla'\cdot\left(\frac{J(r')}{\mathscr{r}}\right)d\tau'\right)+\frac{4\pi}{c}J(r)$$

$$= -\frac{1}{c}\nabla\left(\oint_{\mathcal{S}}\frac{J(r')}{\mathscr{r}}\cdot da'\right)+\frac{4\pi}{c}J(r) \quad .$$

But by definition $J = 0$ on the surface, so the integral vanishes:

$$\boxed{\nabla\times B(r) = \frac{4\pi}{c}J(r)} \quad . \qquad \textbf{Ampère's Law}$$

Beyond magnetoquasistatics (continued)

We could go back and fix this:
- substitute $-\partial\rho/\partial t$ for $\nabla'\cdot J(r')$,
- do another integration by parts,
- arguing that two more surface integrals vanish, and
- substitute $\nabla\cdot E/4\pi$ for ρ,

and we'd naturally get a more general form of Ampère's law that is valid for any time variation in the charge density.

I encourage *you* to do this, by way of review; here, let's just take a shortcut to the answer, and demonstrate that it works:

$$\boxed{\nabla\times B = \frac{4\pi}{c}J + \frac{1}{c}\frac{\partial E}{\partial t}}$$

Beyond magnetoquasistatics (continued)

Does this work? Yes, because

$$\nabla\cdot(\nabla\times B) = \frac{4\pi}{c}\nabla\cdot J + \frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot E \qquad \text{Use Gauss's law…}$$

$$= \frac{4\pi}{c}\left(\nabla\cdot J + \frac{\partial\rho}{\partial t}\right) \qquad \text{Use continuity…}$$

$$= 0 \quad ,$$

so $\nabla\times B = \frac{4\pi}{c}J + \frac{1}{c}\frac{\partial E}{\partial t}$ $\left[\nabla\times B = \mu_0 J + \mu_0\epsilon_0\frac{\partial E}{\partial t}\text{ in MKS}\right]$.

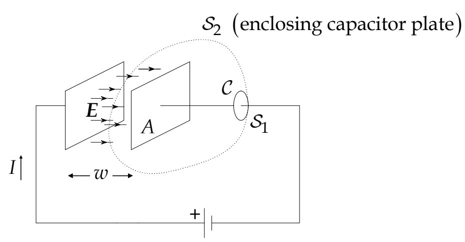
must therefore be the correct generalization of Ampère's law for time-variable charge and current densities.

Displacement current

Maxwell was, of course, the first to get this result. He didn't do it this way, though; he put in the extra term because it was the only way to get a **wave equation** by combining the four differential equations of electrodynamics, that resembled the equations for elastic waves in matter. He noted afterward that it fixed the div-curl-$\boldsymbol{B}$ problem. Maxwell thought of this extra term as related to a source he called the **displacement current density.**

The role of this term as a current density is made clearer in integral form, and applied to the simple example of a parallel-plate capacitor charging up:

Displacement current (continued)



Consider the capacitor plates to be closely spaced, even though they're not drawn that way, and consider two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$, both bounded by circle $\mathcal{C}$, with $\mathcal{S}_2$ ballooning to enclose the nearer plate.

Displacement current (continued)

Integrate the new "corrected" form of Ampère's law over either of these surfaces:

$$\int_{\mathcal{S}} (\nabla\times\boldsymbol{B})\cdot d\boldsymbol{a} = \frac{4\pi}{c}\int_{\mathcal{S}} \boldsymbol{J}\cdot d\boldsymbol{a} + \frac{1}{c}\int_{\mathcal{S}} \frac{\partial \boldsymbol{E}}{\partial t}\cdot d\boldsymbol{a}$$

or, by Stokes's theorem,

$$\oint_{\mathcal{C}} \boldsymbol{B}\cdot d\ell = \frac{4\pi}{c} I_{\text{encl}} + \frac{1}{c}\int_{\mathcal{S}} \frac{\partial \boldsymbol{E}}{\partial t}\cdot d\boldsymbol{a} \quad .$$

Considering the circle $\mathcal{C}$ to be an Ampèrean loop, we could use this to calculate $\boldsymbol{B}$. Most would use $\mathcal{S} = \mathcal{S}_1$ for the area integral, and note that the enclosed current is just the current I in the wire. But the enclosed current for $\mathcal{S} = \mathcal{S}_2$ is zero, so that term must vanish. All $\mathcal{S}_2$ intercepts is electric field, $E = V/w$:

Displacement current (continued)

$$\frac{\partial E}{\partial t}=\frac{1}{w}\frac{\partial V}{\partial t}=\frac{1}{wC}\frac{\partial q}{\partial t}=\frac{1}{w\frac{A}{4\pi w}}I=\frac{4\pi}{A}I \quad (!)$$

Since the electric field is constant between the plates and very small outside them,

$$\frac{1}{c}\int_{S_2}\frac{\partial \boldsymbol{E}}{\partial t}\cdot d\boldsymbol{a}=\frac{1}{c}\frac{4\pi}{A}IA=\frac{4\pi}{c}I \quad ,$$

just like the other surface; so $\oint_C \boldsymbol{B}\cdot d\ell=\frac{4\pi}{c}I$ no matter which surface is used.

Displacement current (continued)

If we therefore define the displacement current density as

$$\boldsymbol{J}_{\mathrm{disp}}=\frac{1}{4\pi}\frac{\partial \boldsymbol{E}}{\partial t} \quad ,$$

then there is a "displacement current" between the capacitor plates that is exactly equal to I, and there is a more general "current" that is continuous throughout the circuit.

The Maxwell equations

So here are the Maxwell equations, in vacuum, in final form:

$$\nabla\cdot\boldsymbol{E}=4\pi\rho \qquad \nabla\cdot\boldsymbol{B}=0$$
$$\nabla\times\boldsymbol{E}=-\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla\times\boldsymbol{B}=\frac{4\pi}{c}\boldsymbol{J}+\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$$

in cgs units, or

$$\nabla\cdot\boldsymbol{E}=\frac{\rho}{\varepsilon_0} \qquad \nabla\cdot\boldsymbol{B}=0$$
$$\nabla\times\boldsymbol{E}=-\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla\times\boldsymbol{B}=\mu_0\boldsymbol{J}+\mu_0\varepsilon_0\frac{\partial \boldsymbol{E}}{\partial t}$$

in MKS units.

Magnetic monopoles

The only remaining sense in which these equations may still be approximate is if magnetic charges (monopoles) exist. We will see a powerful argument for searching for magnetic monopoles in the first homework set (Griffiths problem 8.12); they would also symmetrize the Maxwell equations. Note that if there are no electric charges or currents, the Maxwell equations are symmetrical:

$$\nabla\cdot\boldsymbol{E}=0 \qquad \nabla\cdot\boldsymbol{B}=0$$

$$\nabla\times\boldsymbol{E}=-\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} \qquad \nabla\times\boldsymbol{B}=\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}$$

Magnetic monopoles (continued)

If, on the other hand, there *were* magnetic as well as electric monopoles, with magnetic charge density η and magnetic current density $\boldsymbol{K}$, then we'd have

$$\nabla\cdot\boldsymbol{E}=4\pi\rho \qquad \nabla\cdot\boldsymbol{B}=4\pi\eta$$

$$\nabla\times\boldsymbol{E}=-\frac{4\pi}{c}\boldsymbol{K}-\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} \qquad \nabla\times\boldsymbol{B}=\frac{4\pi}{c}\boldsymbol{J}+\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}$$

where, if both electric and magnetic charge were conserved,

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\boldsymbol{J}=0$$

$$\frac{\partial\eta}{\partial t}+\nabla\cdot\boldsymbol{K}=0$$