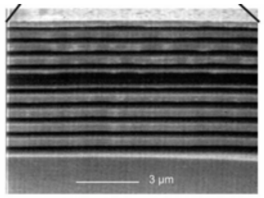
Today in Physics 218: stratified linear media II

- Characteristic matrix formulation of reflected and transmitted fields and intensity

Examples:

- Single interface
- Plane-parallel dielectric in vacuum
- Multiple quarter-wave stacks



Electron micrograph of the cross section of a twenty-layer, alternating quarter-wave stack.

Last time

Here is the relationship between $\mathbf{E}_{\|}$ and $\mathbf{H}_{\|}$ on the surfaces of a plane parallel layer of linear material, illuminated from side 1:

$$\begin{bmatrix} \tilde{E}_{\|,1} \\ \tilde{H}_{\|,1} \end{bmatrix} = \begin{bmatrix} \cos\delta_1 & -i\sin\delta_1 / Y_1 \\ -iY_1 \sin\delta_1 & \cos\delta_1 \end{bmatrix} \begin{bmatrix} \tilde{E}_{\|,2} \\ \tilde{H}_{\|,2} \end{bmatrix} \equiv M_1 \begin{bmatrix} \tilde{E}_{\|,2} \\ \tilde{H}_{\|,2} \end{bmatrix} ,$$

where

$$Y_1 = Y_{1,TE} = \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos\theta_{T1} \qquad TE: \mathbf{E} \perp \text{ incidence plane;}$$

$$= Y_{1,TM} = \sqrt{\frac{\varepsilon_1}{\mu_1}} \frac{1}{\cos\theta_{T1}} \qquad TM: \mathbf{E} \parallel \text{ incidence plane.}$$

Last time (continued)

Furthermore, the tangential components of $\mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$ for the first and p+1st surfaces of a stack of p layers are related by

$$\begin{bmatrix} \tilde{E}_{\|,1} \\ \tilde{H}_{\|,1} \end{bmatrix} = M_1 M_2 \cdots M_p \begin{bmatrix} \tilde{E}_{\|,p+1} \\ \tilde{H}_{\|,p+1} \end{bmatrix} .$$

It remains for us to show how to obtain the transmitted and reflected amplitudes from these fields.

Characteristic matrix formulation of reflected and transmitted fields and intensity

Recall that the tangential components of the fields at the first surface are related to the fields in the incident and reflected waves by

$$\tilde{E}_{\|,1} = \tilde{E}_{0I} + \tilde{E}_{0R1} \quad \text{and} \quad \tilde{H}_{\|,1} = Y_0\left(\tilde{E}_{0I} - \tilde{E}_{0R1}\right) \quad ,$$

while the tangential field components at the last surface are related to the fields of the final transmitted wave by

$$\tilde{E}_{\|,p+1} = \tilde{E}_{0T} \quad \text{and} \quad \tilde{H}_{\|,p+1} = Y_{p+1}\tilde{E}_{0T} \quad ,$$

so

$$\begin{bmatrix} \tilde{E}_{0I} + \tilde{E}_{0R1} \\ Y_0\left(\tilde{E}_{0I} - \tilde{E}_{0R1}\right) \end{bmatrix} = M \begin{bmatrix} \tilde{E}_{0T,p+1} \\ Y_{p+1}\tilde{E}_{0T,p+1} \end{bmatrix} \quad .$$

Multiply it out:

Characteristic matrix formulation of reflected and transmitted fields and intensity (continued)

$$\tilde{E}_{0I} + \tilde{E}_{0R1} = m_{11}\tilde{E}_{0T,p+1} + m_{12}Y_{p+1}\tilde{E}_{0T,p+1}$$

$$Y_0\left(\tilde{E}_{0I} - \tilde{E}_{0R1}\right) = m_{21}\tilde{E}_{0T,p+1} + m_{22}Y_{p+1}\tilde{E}_{0T,p+1} \quad .$$

In terms of the electric-field amplitude reflection and transmission coefficients, $r \equiv \tilde{E}_{0R1} / \tilde{E}_{0I}$ and $t \equiv \tilde{E}_{0T,p+1} / \tilde{E}_{0I}$, this equations are

$$1 + r = \left(m_{11} + m_{12}Y_{p+1}\right)t$$

$$Y_0(1-r) = \left(m_{21} + m_{22}Y_{p+1}\right)t \quad .$$

Multiply the first of these by Y_0 and add, to produce

$$t = \frac{2Y_0}{m_{11}Y_0 + m_{12}Y_0Y_{p+1} + m_{21} + m_{22}Y_{p+1}} \quad .$$

Characteristic matrix formulation of reflected and transmitted fields and intensity (continued)

…and substitute back to get

$$1 + r = \frac{2Y_0\left(m_{11} + m_{12}Y_{p+1}\right)}{m_{11}Y_0 + m_{12}Y_0Y_{p+1} + m_{21} + m_{22}Y_{p+1}}$$

$$r = \frac{2Y_0\left(m_{11} + m_{12}Y_{p+1}\right) - \left(m_{11}Y_0 + m_{12}Y_0Y_{p+1} + m_{21} + m_{22}Y_{p+1}\right)}{m_{11}Y_0 + m_{12}Y_0Y_{p+1} + m_{21} + m_{22}Y_{p+1}}$$

$$= \frac{m_{11}Y_0 + m_{12}Y_0Y_{p+1} - m_{21} - m_{22}Y_{p+1}}{m_{11}Y_0 + m_{12}Y_0Y_{p+1} + m_{21} + m_{22}Y_{p+1}} \quad .$$

Usually the fraction of intensity or power reflected and transmitted is also of interest. The Poynting vector is:

Characteristic matrix formulation of reflected and transmitted fields and intensity (continued)

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H} = \hat{\mathbf{k}}\frac{c}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}|E|^2 \quad ,$$

so the power per unit area flowing through any plane parallel to the surfaces is

$$\langle S_\perp \rangle = \mathbf{S}\cdot\hat{\mathbf{z}} = \frac{c}{8\pi}\sqrt{\frac{\varepsilon}{\mu}}\cos\theta|E|^2 = \frac{c}{8\pi}Y|E|^2 \quad .$$

In terms of the amplitude coefficients, we can define the intensity reflection and transmission coefficients, as

$$\rho = \frac{\langle S_{R1,\perp}\rangle}{\langle S_{I,\perp}\rangle} = |r|^2 \quad \text{and} \quad \tau = \frac{\langle S_{T,p+1,\perp}\rangle}{\langle S_{I,\perp}\rangle} = \frac{Y_{p+1}}{Y_0}|t|^2 \quad .$$

Of course $\tau + \rho = 1$, as demanded by energy conservation, and as you will show in this week's homework.

Example: a single vacuum-linear medium interface

Can our new formalism be used to re-derive the Fresnel equations? Yes, because in the expression

$$\begin{bmatrix}\tilde{E}_{\|,1}\\ \tilde{H}_{\|,1}\end{bmatrix} = \begin{bmatrix}\cos\delta_1 & -i\sin\delta_1/Y_1\\ -iY_1\sin\delta_1 & \cos\delta_1\end{bmatrix}\begin{bmatrix}\tilde{E}_{\|,2}\\ \tilde{H}_{\|,2}\end{bmatrix} \equiv M_1\begin{bmatrix}\tilde{E}_{\|,2}\\ \tilde{H}_{\|,2}\end{bmatrix} \quad ,$$

we are free to choose zero thickness (no layer at all), for which the phase delay across it is also zero, and the characteristic matrix becomes

$$M_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad .$$

Example: a single vacuum-linear medium interface (continued)

This leaves us with the two media, for which the admittances are

$$Y_{0,TE} = \cos\theta_I \qquad Y_{2,TE} = \sqrt{\frac{\varepsilon}{\mu}}\cos\theta_T$$

$$Y_{0,TM} = \frac{1}{\cos\theta_I} \qquad Y_{2,TM} = \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{\cos\theta_T}$$

so

$$t_{TE} = \frac{2Y_0}{m_{11}Y_0 + m_{12}Y_0Y_2 + m_{21} + m_{22}Y_2} = \frac{2Y_0}{Y_0 + Y_2}$$

$$= \frac{2}{1 + \dfrac{Y_2}{Y_0}} = \frac{2}{1 + \dfrac{\cos\theta_T}{\cos\theta_I}\sqrt{\dfrac{\varepsilon}{\mu}}} = \frac{2}{1 + \alpha\beta} \quad ,$$

Example: a single vacuum-linear medium interface (continued)

and
$$r_{TE} = \frac{m_{11}Y_0 + m_{12}Y_0Y_2 - m_{21} - m_{22}Y_2}{m_{11}Y_0 + m_{12}Y_0Y_2 + m_{21} + m_{22}Y_2} = \frac{Y_0 - Y_2}{Y_0 + Y_2}$$
$$= \frac{1 - \sqrt{\frac{\varepsilon}{\mu}}\frac{\cos\theta_I}{\cos\theta_T}}{1 + \sqrt{\frac{\varepsilon}{\mu}}\frac{\cos\theta_I}{\cos\theta_T}} = \frac{1-\alpha\beta}{1+\alpha\beta} \quad .$$

Similarly,
$$t_{TM} = \frac{2Y_0}{Y_0 + Y_2} = \frac{2}{\alpha+\beta} \quad \text{and} \quad r_{TM} = \frac{Y_0 - Y_2}{Y_0 + Y_2} = \frac{\alpha-\beta}{\alpha+\beta} \quad ,$$
just as before.

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Example: plane parallel dielectric in vacuum

Actually this isn't a worked example because you're doing it in this week's homework. For normal incidence on a plane-parallel linear medium with thickness d and $\mu = 1$,
$$Y_{TE} = Y_{TM} = \sqrt{\varepsilon}\cos\theta_T = n \quad ,$$
$$\delta_1 = \frac{2\pi nd}{\lambda}\cos\theta_T = \frac{2\pi nd}{\lambda} \quad ,$$
and the characteristic matrix is
$$M_1 = \begin{bmatrix} \cos\delta_1 & -i\sin\delta_1 / Y_1 \\ -iY_1\sin\delta_1 & \cos\delta_1 \end{bmatrix} = \begin{bmatrix} \cos\frac{2\pi nd}{\lambda} & -\frac{i}{n}\sin\frac{2\pi nd}{\lambda} \\ -in\sin\frac{2\pi nd}{\lambda} & \cos\frac{2\pi nd}{\lambda} \end{bmatrix}$$

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Example: plane parallel dielectric in vacuum (continued)

In vacuum, the amplitude transmission coefficient turns out to be
$$t = \frac{2}{m_{11} + m_{12} + m_{21} + m_{22}} = \frac{2}{2\cos\frac{2\pi nd}{\lambda} - i\left(n + \frac{1}{n}\right)\sin\frac{2\pi nd}{\lambda}} \quad ,$$
and the intensity transmission coefficient
$$\tau = tt^* = \frac{1}{1 + \left(\frac{n^2-1}{2n}\sin\frac{2\pi nd}{\lambda}\right)^2} \quad ,$$
just as advertised a couple of lectures back. This is plotted on the next page, for three values of the single-surface intensity reflection coefficient, $|r|^2 = \left(n-1/n+1\right)^2$.

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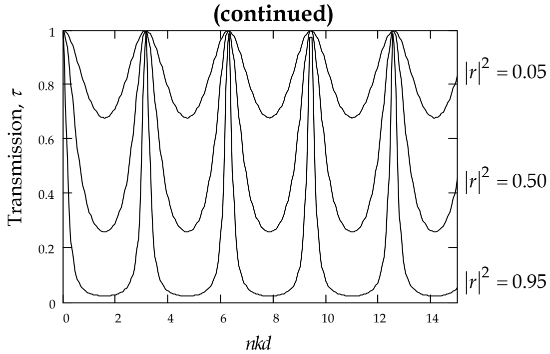
Example: plane parallel dielectric in vacuum (continued)



Example: "quarter-wave stacks"

The case of normal incidence and dielectric ($\mu = 1$) layers with thickness equal to $\lambda/4n$ at some design wavelength λ_d is encountered frequently (c.f. Crawford 5.21 on this week's homework).

$$Y = Y_{TE} = Y_{TM} = \sqrt{\varepsilon}\cos\theta_T = n \quad ,$$

$$\delta = \frac{2\pi n}{\lambda}\frac{\lambda_d}{4n}\cos\theta_T = \frac{\pi\lambda_d}{2\lambda} \quad ,$$

$$M = \begin{bmatrix} \cos\dfrac{\pi\lambda_d}{2\lambda} & -\dfrac{i}{n}\sin\dfrac{\pi\lambda_d}{2\lambda} \\ -in\sin\dfrac{\pi\lambda_d}{2\lambda} & \cos\dfrac{\pi\lambda_d}{2\lambda} \end{bmatrix} \quad .$$

Example: "quarter-wave stacks" (continued)

In particular, alternating quarter-wave layers, arranged in pairs consisting of one high-index and one low-index layer, are the main building block in the interference-filter and mirror- and lens-coating industry.

- ❑ With H as the characteristic matrix of a high-index material (n = 3-5) and L that of a low-index dielectric (n = 1.3-2), series like

  $$HLHL...HL = (HL)^m$$

  are useful in making the reflectivity or transmission change rapidly with wavelength, especially around the design wavelength.
- ❑ The more layer pairs, the sharper the change.

Example: "quarter-wave stacks" (continued)

- For instance, consider a stack that's good at transmitting long wavelengths and reflecting short wavelengths (a **low-pass filter**):

$$(\mathrm{Air})LHLHL...HL(\mathrm{Glass}) = (\mathrm{Air})L(HL)^m(\mathrm{Glass})$$

Though it looks complicated, the whole characteristic matrix is a simple product of 2x2 matrices that don't actually take very long at all to calculate. See the results on the next page:

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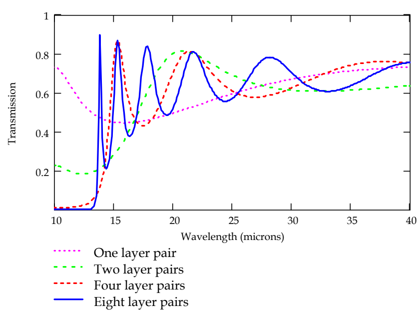
Example: "quarter-wave stacks" (continued)



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