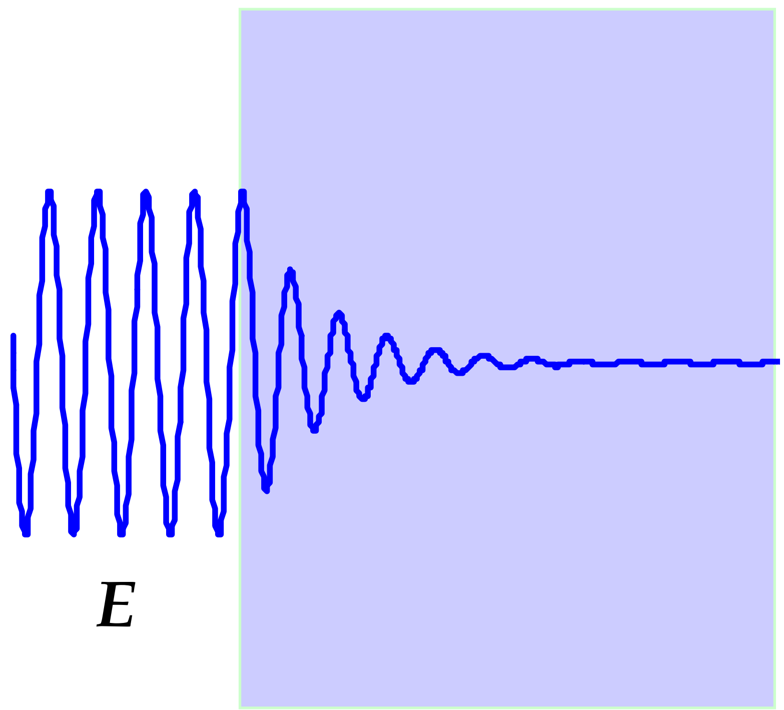
# Today in Physics 218: conductors

- ❑ Electromagnetic waves in conductors
- ❑ Attenuation of the waves, and an electronic analogy
- ❑ Penetration of waves into conductors: skin depth



# Electromagnetic waves in conductors

Consider a medium in which there is (still) no free electrical charge, but can be free currents:

$$\mathbf{J} = \sigma \mathbf{E} \quad .$$

Then the Maxwell equations become

$$\nabla \cdot \mathbf{D} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

or, getting rid of the auxiliary fields with the assumption that the medium is (still) linear,

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{4\pi\sigma\mu}{c}\mathbf{E} + \frac{\varepsilon\mu}{c}\frac{\partial \mathbf{E}}{\partial t}$$

# Electromagnetic waves in conductors (continued)

Let's make a wave equation for the electric field as we did before, by taking the curl of one of the curl equations:

$$\nabla\times(\nabla\times\boldsymbol{E}) = -\frac{1}{c}\nabla\times\frac{\partial\boldsymbol{B}}{\partial t}$$

$$\nabla(\overset{0}{\cancel{\nabla\cdot\boldsymbol{E}}}) - \nabla^2\boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\boldsymbol{B})$$

$$\nabla^2\boldsymbol{E} = \frac{\varepsilon\mu}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} + \frac{4\pi\sigma\mu}{c^2}\frac{\partial\boldsymbol{E}}{\partial t} \quad .$$

For one-dimensional propagation (plane waves, zero incidence, just like waves on a string), this is

$$\frac{\partial^2\boldsymbol{E}}{\partial z^2} = \frac{\varepsilon\mu}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} + \frac{4\pi\sigma\mu}{c^2}\frac{\partial\boldsymbol{E}}{\partial t} \quad .$$

# Electromagnetic waves in conductors (continued)

- You many have seen this equation before in classical mechanics: it is the equation of motion of a damped harmonic oscillator. Veterans of F2002's PHY 217 have seen it used in connection with *LRC* circuits.

Let's solve it for some arbitrary component of ***E*** by separation of variables. Let $E_i = Z(z)T(t)$ and divide through by ***ZT***:

$$\frac{1}{Z}\frac{d^2\mathbf{Z}}{dz^2} = \frac{1}{T}\left(\frac{\varepsilon\mu}{c^2}\frac{d^2T}{dt^2} + \frac{4\pi\sigma\mu}{c^2}\frac{dT}{dt}\right) = -k'^2 = \text{ constant} \quad .$$

For the Z part,

$$\frac{d^2\mathbf{Z}}{dz^2} = -k'^2 Z \quad \Rightarrow \quad Z = Ae^{\pm ik'z} \quad ,$$

as usual.

# Electromagnetic waves in conductors (continued)

The $T$ part is a little more interesting:

$$\frac{d^2T}{dt^2} + \frac{4\pi\sigma}{\varepsilon}\frac{dT}{dt} + \frac{c^2k'^2}{\mu\varepsilon}T = 0 \quad .$$

Try a solution of the form $T = Ce^{\alpha t}$, so that

$$\alpha^2 T + \frac{4\pi\sigma\alpha}{\varepsilon}T + \frac{c^2k'^2}{\mu\varepsilon}T = 0 \quad .$$

Unless $C = 0$ (trivial solution),

$$\alpha^2 + \frac{4\pi\sigma\alpha}{\varepsilon} + \omega^2 = 0 \quad , \qquad \omega^2 \equiv \frac{c^2k'^2}{\mu\varepsilon}$$

for which $\alpha = -\frac{2\pi\sigma}{\varepsilon} \pm \frac{1}{2}\sqrt{\left(\frac{4\pi\sigma}{\varepsilon}\right)^2 - 4\omega^2}$

## Electromagnetic waves in conductors (continued)

or, rearranged a little,

$$\alpha = -\frac{2\pi\sigma}{\varepsilon} \pm i\omega t\sqrt{1-\frac{4\pi^2\sigma^2}{\varepsilon^2\omega^2}} \quad .$$

Thus the full solution to the wave equation has the form

$$E_i = \underbrace{Ae^{\pm ik'z \pm i\omega t\sqrt{1-4\pi^2\sigma^2/\varepsilon^2\omega^2}}}_{\text{Plane waves, propagating in either direction along } z.} \ \underbrace{e^{-2\pi\sigma t/\varepsilon}}_{\text{Attenuation (damping).}} \quad .$$

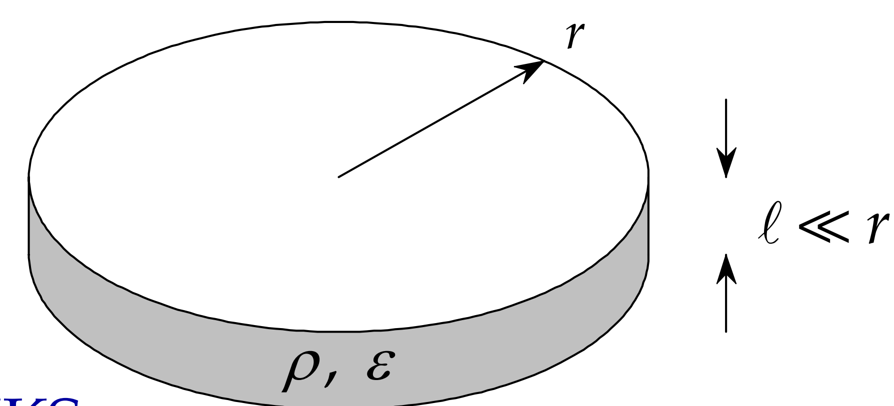
So travelling electromagnetic waves are damped out, eventually, in a conductor – attenuated exponentially, decreasing by a factor of *e* in a time constant of

$$\tau = \frac{\varepsilon}{2\pi\sigma} = \frac{\rho\varepsilon}{2\pi} \quad . \qquad \rho = \text{resistivity}$$

# Electronic analogy

What are we to make of this unlikely combination of resistivity $\rho$ and dielectric constant $\varepsilon$?

Consider a circular wafer made of very weakly conducting material with resistivity $\rho$ and dielectric constant $\varepsilon$, with radius is $r$ and thickness $\ell \ll r$, and with highly conductive, metallic electrodes covering the circular faces. Its resistance and capacitance are

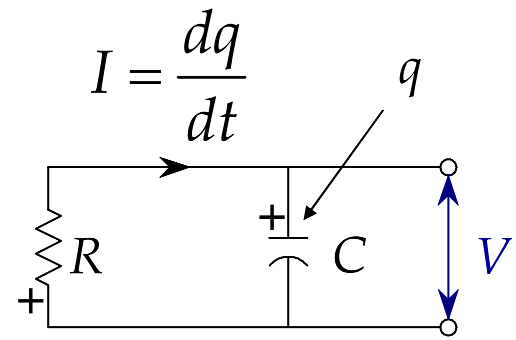



$$R = \frac{\rho\ell}{A} = \frac{\rho\ell}{\pi r^2} \quad ,$$

$$C = \frac{\varepsilon A}{4\pi\ell} = \frac{\varepsilon r^2}{4\ell} = \frac{\pi\varepsilon_0\varepsilon_r r^2}{\ell} \quad \text{in MKS.}$$

# Electronic analogy (continued)

- Suppose that the wafer is charged up with a battery with voltage $V_0$, and that the battery is disconnected at $t = 0$.
- Since the resistance and capacitance have the same voltage, they can be considered to be in parallel, so the charge on the electrodes follows




$$\frac{q}{C} + IR = 0 \quad ;$$

$$\frac{dq}{dt} + \frac{1}{RC} q = \frac{dq}{dt} + \frac{4\pi}{\varepsilon\rho} q = 0$$

Use $q = CV$:

$$\frac{dV}{dt} + \frac{4\pi}{\varepsilon\rho} V = 0 \quad .$$

# Electronic analogy (continued)

or, since the electric field in the wafer is given by $V = -E\ell$,

$$\frac{dE}{dt} + \frac{4\pi}{\varepsilon\rho} E = 0 \quad .$$

In any case it can be integrated directly:

$$\frac{dE}{dt} + \frac{4\pi}{\rho\varepsilon} E = 0 \quad \Rightarrow \quad \int \frac{dE}{E} = -\frac{4\pi}{\rho\varepsilon} \int dt$$

$$\ln \frac{E}{E_0} = -\frac{4\pi t}{\rho\varepsilon}$$

$$E(t) = E_0 e^{-4\pi t/\rho\varepsilon} = E_0 e^{-t/\tau} \quad , \quad \tau = \frac{\rho\varepsilon}{4\pi} = RC \quad .$$

Evidently, $\rho\varepsilon/4\pi$ is the "*RC* time constant" of matter!

# Penetration of electromagnetic waves in conductors: the skin depth

Evidently the wave equation for fields in conductors allows forms for plane waves *similar* to what we've been using. Suppose we want the (rightward-bound) plane wave solution to look like

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 e^{i\left(\tilde{k}z - \omega t\right)} \quad .$$

so $k$ is like the usual wavenumber, etc.

Put into the wave equation, this gives us

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} = \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{4\pi\sigma\mu}{c^2}\frac{\partial \mathbf{E}}{\partial t} \quad ,$$

$$-k^2 \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)} = -\omega^2 \frac{\varepsilon\mu}{c^2}\tilde{\mathbf{E}}_0 e^{i(kz-\omega t)} - i\omega\frac{4\pi\sigma\mu}{c^2}\tilde{\mathbf{E}}_0 e^{i(kz-\omega t)} \quad ,$$

$$k^2 = \omega^2\frac{\varepsilon\mu}{c^2} + i\omega\frac{4\pi\sigma\mu}{c^2} = \mu\varepsilon\frac{\omega^2}{c^2}\left(1 + i\frac{4\pi\sigma}{\varepsilon\omega}\right) \quad .$$

# Penetration of electromagnetic waves in conductors: the skin depth (continued)

This expression for $\tilde{k}$ differs from the corresponding expression for nonconducting media by the factor in parentheses:

$$1 + i\frac{4\pi\sigma}{\varepsilon\omega} = \sqrt{1 + \left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2}\, e^{i\theta} \quad , \text{ where } \theta = \arctan\left(4\pi\sigma/\varepsilon\omega\right).$$

Thus

$$\tilde{k} = \sqrt{\mu\varepsilon}\,\frac{\omega}{c}\left(1 + \left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2\right)^{1/4} e^{i\theta/2}$$

$$= \sqrt{\mu\varepsilon}\,\frac{\omega}{c}\left(1 + \left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2\right)^{1/4}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \equiv k + i\kappa \quad .$$

# Penetration of electromagnetic waves in conductors: the skin depth (continued)

We can simplify this with some trig work. First:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\Rightarrow \cos\theta = \cos\frac{\theta}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\sin\frac{\theta}{2} = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$\Rightarrow \cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} \quad , \quad \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} \quad .$$

But $\quad 1 = \cos^2\theta + \sin^2\theta = \cos^2\theta + \cos^2\theta\tan^2\theta$

$$= \cos^2\theta\left(1+\tan^2\theta\right), \text{ so}$$

$$\cos\theta = \sqrt{\frac{1}{1+\tan^2\theta}} = \sqrt{\frac{1}{1+\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2}} \quad .$$

# Penetration of electromagnetic waves in conductors: the skin depth (continued)

Thus,

$$k = \sqrt{\mu\varepsilon}\,\frac{\omega}{c}\left(1+\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2\right)^{1/4}\sqrt{\frac{1+\cos\theta}{2}}$$

$$= \sqrt{\mu\varepsilon}\,\frac{\omega}{c}\sqrt{\left(1+\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2\right)^{1/2}}\sqrt{\frac{1+\left(1+\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2\right)^{-1/2}}{2}}$$

$$= \sqrt{\frac{\mu\varepsilon}{2}}\,\frac{\omega}{c}\left(\left(1+\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2\right)^{1/2}+1\right)^{1/2} .$$

# Penetration of electromagnetic waves in conductors: the skin depth (continued)

Similarly,

$$\kappa = \sqrt{\frac{\mu\varepsilon}{2}}\frac{\omega}{c}\left(\left(1+\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2\right)^{1/2}-1\right)^{1/2} \quad .$$

The upshot of all this is that $\kappa$, the imaginary part of the complex wavenumber $\tilde{k}$, also represents attenuation of electromagnetic waves in conductors:

$$\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_0 e^{i\left(\tilde{k}z-\omega t\right)} = \tilde{\boldsymbol{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)} \quad .$$

The electric field amplitude decreases by a factor of $e$ with every distance $d = 1/\kappa$ the wave covers:

# Penetration of electromagnetic waves in conductors: the skin depth (continued)

$$d = \frac{1}{\kappa} = \sqrt{\frac{2}{\mu\varepsilon}}\frac{c}{\omega}\left(\left(1+\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2\right)^{1/2}-1\right)^{-1/2}$$ Skin depth

$$= \sqrt{\frac{2}{\mu_r\mu_0\varepsilon_r\varepsilon_0}}\frac{1}{\omega}\left(\left(1+\left(\frac{\sigma}{\varepsilon_r\varepsilon_0\omega}\right)^2\right)^{1/2}-1\right)^{-1/2}$$ in MKS .