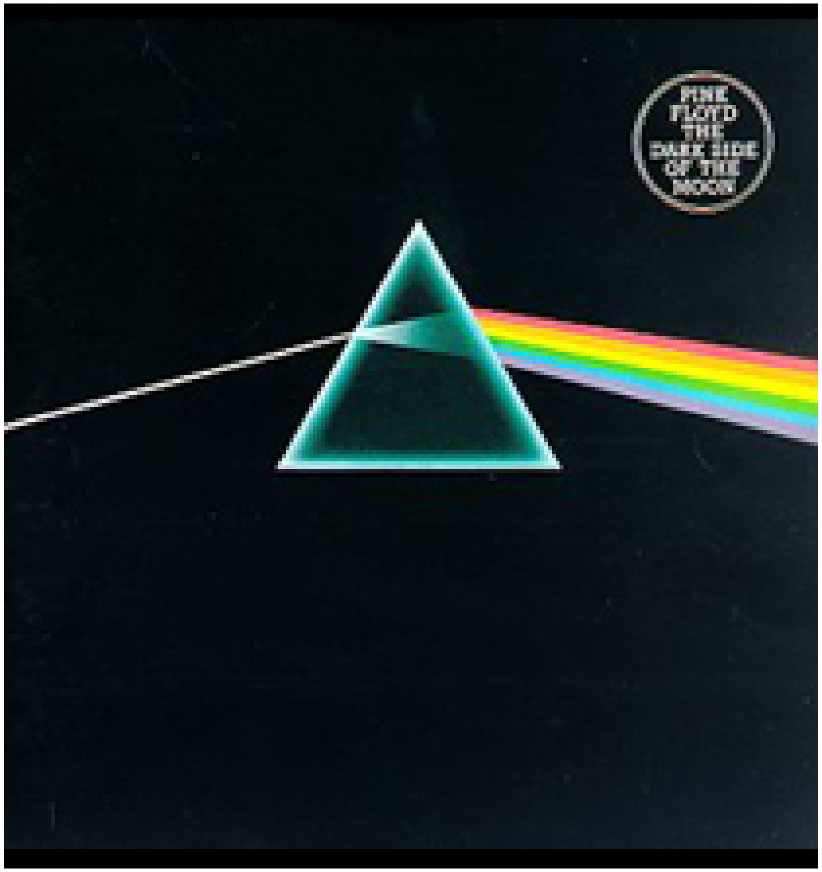
Today in Physics 218: dispersion

- ☐ Motion of bound electrons in matter, and the frequency dependence of the dielectric constant
- ☐ Dispersion relations
- ☐ Ordinary and anomalous dispersion



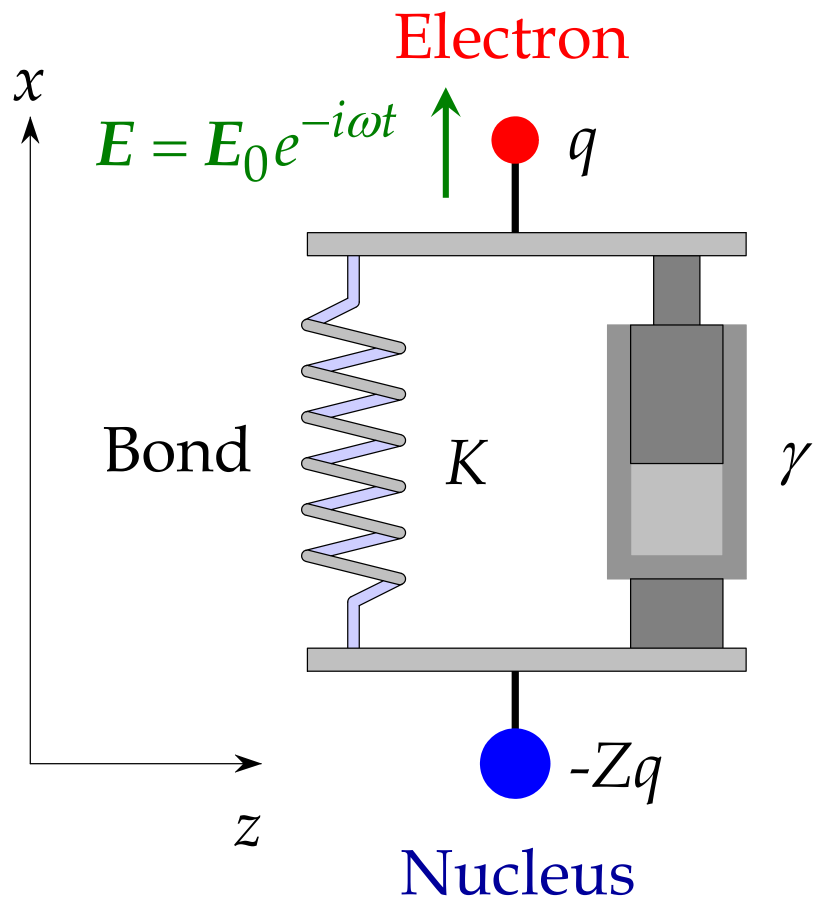
The world's largest-selling (nearly 30 million copies!) illustration of dispersion. A relatively accurate depiction, at that.

Motion of bound electrons in matter: a simple model

So far we have treating the propagation of light in matter as if the permittivity, permeability and conductivity do not depend upon the frequency of light. This is not generally true.

- Most materials that transmit light exhibit a noticeable variation of refractive index with wavelength. This is called **dispersion,** after the action of the best example of the phenomenon, the glass prism.
- To find out how, consider the effect of a plane wave of light on a simple model of a bound electron in an atom or molecule...

Motion of bound electrons in matter: a simple model (continued)



Its bond holds the electron in some equilibrium position; we can represent the bond with a restoring force (like a spring), and a "resistance" mechanism for losing energy and momentum to its surroundings (like a shock absorber). The nucleus can be considered stationary to good approximation.

Restoring force: $-Kx$

Resisting force: $-m_e \gamma \, dx/dt$

Motion of bound electrons in matter: a simple model (continued)

- ☐ The net force in the x direction is

$$F = qE - m_e \gamma \frac{dx}{dt} - Kx = m_e a = m_e \frac{d^2 x}{dt^2} \quad ,$$

so the electron's equation of motion is

$$\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = \frac{q}{m_e} E_0 e^{-i\omega t} \quad ,$$

where $\omega_0^2 = K/m_e$ .This is the equation of motion of a damped, driven harmonic oscillator, which you have solved in PHY 235 and (maybe) in PHY 217.

Motion of bound electrons in matter: a simple model (continued)

- ☐ With a trial solution $x = x_0 e^{-i\omega t}$, we have

$$\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = E(t)$$

$$-\omega^2 x_0 e^{-i\omega t} - i\omega\gamma x_0 e^{-i\omega t} + \omega_0^2 x_0 e^{-i\omega t} = \frac{q}{m_e} E_0 e^{-i\omega t} \quad \text{, or}$$

$$x_0 = \frac{\frac{q}{m_e} E_0}{\left(\omega_0^2 - \omega^2\right) - i\omega\gamma} \quad .$$

If the total dipole moment of the medium containing this atom is zero before the field displaces the electron, the dipole moment afterward is:

Motion of bound electrons in matter: a simple model (continued)

$$p = qx = \frac{\dfrac{q^2}{m_e} E_0 e^{-i\omega t}}{\left(\omega_0^2 - \omega^2\right) - i\omega\gamma} \quad .$$

Because of the complex denominator, *p* and *x* are out of phase with the electric field.

❑ Now, this is not the only electron in the medium, nor is this the only kind of bond around. Suppose there are *N* molecules per unit volume, and *M* different kinds of bonds, in all different directions, and consider for the moment just the *j*th kind.

Dispersion relations

- ☐ If there are f_j electrons per molecule that are in this situation, with damping constant γ_j and natural frequency ω_j, then the contribution of the *j*th kind of bond to the dipole moment per unit volume, P_j, is

$$P_j = Nf_j p_j = \frac{Nq^2}{m_e} \frac{f_j E_{0j} e^{-i\omega t}}{\left(\omega_j^2 - \omega^2\right) - i\gamma_j \omega} \quad ,$$

where E_{0j} is the component of $\boldsymbol{E}$ in the bond direction. The net dipole moment per unit volume from *all* of the bonds is the vector sum

$$\boldsymbol{P} = \frac{Nq^2}{m_e} \sum_{j=1}^{M} \frac{f_j}{\left(\omega_j^2 - \omega^2\right) - i\gamma_j \omega} \boldsymbol{E}(t) \quad .$$

Dispersion relations (continued)

- The f_j are commonly known as **oscillator strengths**, because they indicate the response of a each kind of bond (damped oscillator, in this simple model) to the applied electric field. To calculate them is generally a job for quantum mechanics.
- Nevertheless, we have one constraint on the quantum result. Since atoms with Z protons only come with Z electrons, that's what the f_j have to add up to:

  $$\sum_{j=1}^{M} f_j = Z \quad .$$

  This is the simplest of a family of equations relating the f_j that are known as **sum rules**.

Dispersion relations (continued)

- ❑ The vector ***P***, the total dipole moment per unit volume induced by the applied electric field, is not new to all of us. In PHY 217 we called it the **polarization**, and noted that it was related to the applied electric field by

$$\mathbf{P} = \chi_e \mathbf{E} \qquad \left(= \chi_e \varepsilon_0 \mathbf{E} \text{ in MKS}\right).$$

- ❑ In linear media, the electric susceptibility χ_e is a scalar (by definition), and is related to the dielectric constant ε by

$$\varepsilon = 1 + 4\pi\chi_e \quad \left(\varepsilon = \varepsilon_r \varepsilon_0 = \left(1 + \chi_e\right)\varepsilon_0 \text{ in MKS; } \varepsilon_{\text{cgs}} = \varepsilon_{r,\text{MKS}}\right).$$

This gives us an expression for ε in terms of the oscillator strengths, natural frequencies and damping constants:

Dispersion relations (continued)

$$\tilde{\varepsilon} = 1 + \frac{4\pi N q^2}{m_e} \sum_{j=1}^{M} \frac{f_j}{\left(\omega_j^2 - \omega^2\right) - i\gamma_j \omega} \quad .$$

Dispersion relation

- ☐ The damping term – which represents the mechanisms by which electrons can rid themselves of energy and momentum picked up from the applied field – makes the dielectric constant complex.
- ☐ The imaginary part of the dielectric constant leads to exponential attenuation of electromagnetic waves propagating in the medium: since

  $$k = \sqrt{\mu\varepsilon}\,\frac{\omega}{c} \quad ,$$

  it endows an imaginary part to k.

Dispersion in a dilute gas

- **Example:** dielectric constant of an ideal monatomic gas. For standard temperature and pressure (300 K, 1 atm $\approx 10^6$ dyne cm^{-2}),

$$N = \frac{P}{kT} \approx 10^{19} \text{ molecules cm}^{-3},$$

so that $\left(\text{recall } q = 4.8\times10^{10} \text{ esu, } m_e = 9.1\times10^{-28} \text{ gm}\right)$

$$\frac{4\pi N q^2}{m_e} \approx 8\times10^{28} \text{ sec}^{-2}.$$

For visible light, $\lambda = 0.5\ \mu\text{m}$, $\omega = 2\pi c/\lambda = 4\times10^{15}$ rad sec^{-1},

$$\frac{4\pi N q^2}{m_e\omega^2} \approx \frac{1}{200} \ll 1 \quad .$$

Dispersion in a dilute gas (continued)

- Thus, taking $\mu = 1$,

$$\tilde{k} = \sqrt{\varepsilon}\,\frac{\omega}{c} = \frac{\omega}{c}\sqrt{1 + \frac{4\pi N q^2}{m_e}\sum_{j=1}^{M}\frac{f_j}{\left(\omega_j^2 - \omega^2\right) - i\gamma_j\omega}} \quad .$$

If ω is far from any of the natural frequencies (**resonances** ω_j), then the second term under the square root is of order 1/200, as we have just seen, and we can use the binomial theorem again to approximate $\sqrt{1+x} \cong 1 + x/2$:

$$\tilde{k} = \frac{\omega}{c}\left(1 + \frac{2\pi N q^2}{m_e}\sum_{j=1}^{M}\frac{f_j}{\left(\omega_j^2 - \omega^2\right) - i\gamma_j\omega}\right) \quad , \text{ or}$$

Dispersion in a dilute gas (continued)

$$\tilde{k} = \frac{\omega}{c}\left(1 + \frac{2\pi N q^2}{m_e}\sum_{j=1}^{M}\frac{f_j}{\left(\omega_j^2 - \omega^2\right) - i\gamma_j\omega}\,\frac{\left(\omega_j^2 - \omega^2\right) + i\gamma_j\omega}{\left(\omega_j^2 - \omega^2\right) + i\gamma_j\omega}\right)$$

$$= \frac{\omega}{c}\left(1 + \frac{2\pi N q^2}{m_e}\sum_{j=1}^{M} f_j\,\frac{\left(\omega_j^2 - \omega^2\right) + i\gamma_j\omega}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2\omega^2}\right) \equiv k + i\kappa \quad ;$$

$$k = \frac{\omega}{c}\left(1 + \frac{2\pi N q^2}{m_e}\sum_{j=1}^{M}\frac{f_j\left(\omega_j^2 - \omega^2\right)}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2\omega^2}\right) \quad , \text{ and}$$

Dispersion in a dilute gas (continued)

$$\kappa = \frac{2\pi N q^2 \omega}{m_e c} \sum_{j=1}^{M} \frac{f_j \gamma_j \omega}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2 \omega^2} \quad .$$

- ☐ It is customary to define from this a complex refractive index for gases:

$$\tilde{n} \equiv \frac{c}{\omega} k \cong 1 + \frac{2\pi N q^2}{m_e} \sum_{j=1}^{M} \frac{f_j \left(\omega_j^2 - \omega^2\right)}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2 \omega^2} \quad ,$$

$$\alpha \equiv 2\kappa = \frac{4\pi N q^2 \omega}{m_e c} \sum_{j=1}^{M} \frac{f_j \gamma_j \omega}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2 \omega^2} \quad ,$$

Absorption coefficient

$$\tilde{n} = n + \frac{c}{2\omega} \alpha \quad .$$

Dispersion in a dilute gas (continued)

- ☐ As light propagates through the gas, the electric field amplitude is exponentially attenuated, as $e^{-\kappa z}$, so the intensity of light will decrease exponentially as it propagates, according to

$$I(z) = I_0 e^{-\alpha z} \quad .$$

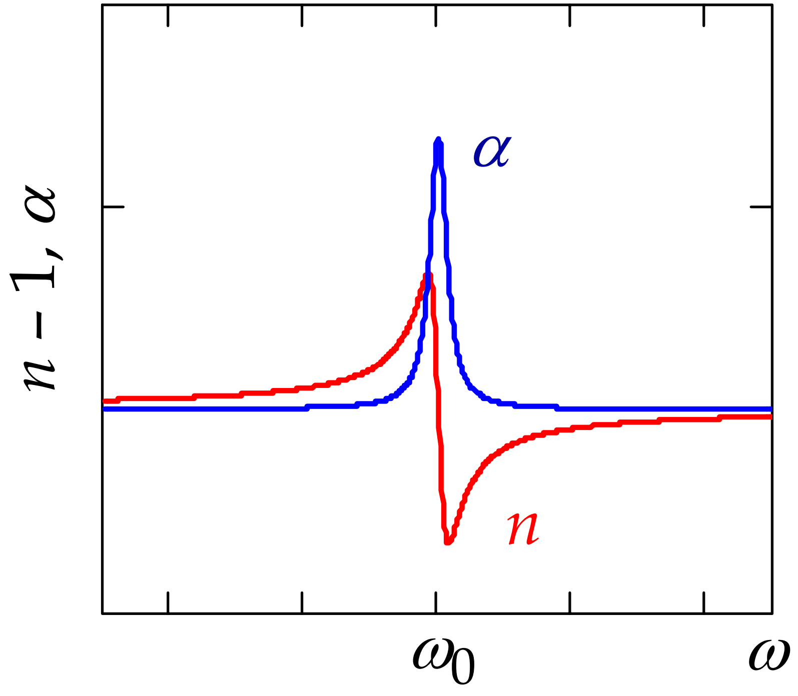
- ☐ *Caveat*: all the formulas in this example pertain to the propagation of *light* through *gases*.
  - Solids and liquids are much denser, so we could not have used the binomial approximation on them.
  - Nor do they apply so well at lower frequencies.
  - Nor do they work at all near resonances.

Ordinary and anomalous dispersion

Far from any resonances, the index of refraction rises gently as frequency increases. This is what one might call **ordinary dispersion**. Most common glasses exhibit ordinary dispersion at visible wavelengths.

- This increase, however, turns out to be due to those distant resonances.
- As frequency passes through a resonance, the index decreases sharply, then resumes its gentle increase. The sudden decrease is called **anomalous dispersion**.
- The absorption coefficient is small far from resonance, but peaks sharply on resonance. (See example on next page.)

Ordinary and anomalous dispersion (continued)



- ❑ The resonances are frequencies at which the electrons can get rid of energy and momentum efficiently – such as the frequencies corresponding to transitions between atomic energy levels.
- ❑ Especially near a resonance, the index and absorption coefficient look like they must be related by integration or differentiation. Indeed, they turn out to be related by a set of integral transforms known as the **Kramers-Kronig** relations.