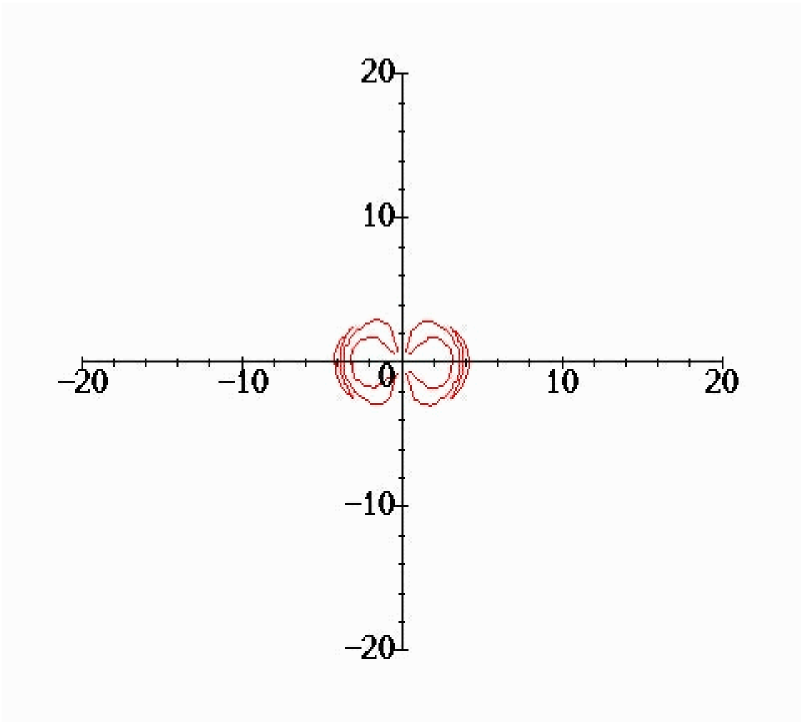
Today in Physics 218: charges, currents, and radiation

- ❑ Retarded potentials and retarded time
- ❑ Retarded potentials and the Lorentz gauge
- ❑ Retarded potentials and the inhomogeneous wave equation



Radiation by two oscillating charges. Animation by Akira Hirose, University of Saskatchewan.

Retarded potentials

- The electromagnetic waves we've been discussing have to originate somewhere. In the following we'll see that electromagnetic radiation can be generated by
  - time-variable charge and current distributions, and
  - accelerating individual charges.
- As usual when dealing with charges and currents, it is most convenient to calculate potentials first, and then to obtain fields from the potentials, rather than to calculate the fields directly.
- Also as usual, we will do our calculations mostly by construction of a solution to the relevant differential equations, demonstration that it works, and reliance upon the uniqueness of solutions.

Retarded potentials (continued)

- What are the relevant differential equations? As we first saw in lecture on 21 January, we get them from Gauss's and Ampère's laws:

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \quad \Rightarrow \quad \nabla \cdot \left(-\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \right) = 4\pi\rho$$

$$\Rightarrow \nabla^2 V + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} = -4\pi\rho \quad ,$$

$$\nabla \times \left(\nabla \times \boldsymbol{A} \right) = \frac{4\pi}{c}\boldsymbol{J} - \frac{1}{c}\frac{\partial}{\partial t}\nabla V - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2}$$

$$\nabla \left(\nabla \cdot \boldsymbol{A} \right) - \nabla^2 \boldsymbol{A} =$$

(P.R. #11)

or $$\left(\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} \right) - \nabla \left(\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial V}{\partial t} \right) = -\frac{4\pi}{c}\boldsymbol{J} \quad .$$

Retarded potentials (continued)

- With the Lorentz gauge condition, these equations become

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -4\pi\rho \quad , \quad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J} \quad ,$$

  that is, inhomogeneous wave equations.
- To construct a solution, first note that we have a lot of experience with the *static* case. For $\partial^2 V/\partial t^2 = 0 = \partial^2 \mathbf{A}/\partial t^2$, the potentials obey Poisson equations:

$$\nabla^2 V = -4\pi\rho \quad , \quad \nabla^2 \mathbf{A} = -\frac{4\pi}{c}\mathbf{J} \quad ,$$

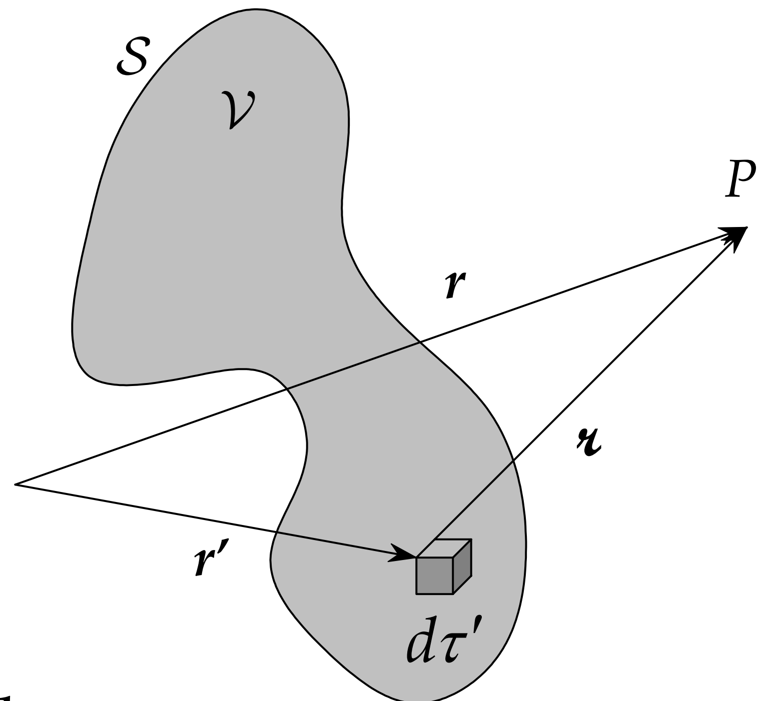
  and in PHY 217 we showed in gory detail that the solutions to these equations are:

Retarded potentials (continued)

$$V(\boldsymbol{r}) = \int_{\mathcal{V}} \frac{\rho(\boldsymbol{r}')\,d\tau'}{\boldsymbol{\mathscr{r}}} \quad ,$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{c}\int_{\mathcal{V}} \frac{\boldsymbol{J}(\boldsymbol{r}')\,d\tau'}{\boldsymbol{\mathscr{r}}} \quad ,$$

where $\mathcal{V}$ is the volume that contains the charges and currents.



- We've also seen this semester that fields and energy propagate at speed c in vacuum, when they travel in the form of electromagnetic waves.

Retarded potentials (continued)

- Here comes the guess:
  Every infinitesimal element of charge or current is a different distance $\mathscr{r}$ away from us (located at $\boldsymbol{r}$). Thus a change in the sources at time t' and position $\boldsymbol{r}'$ doesn't lead to a change in the fields at $\boldsymbol{r}$ until the **later** time $\boldsymbol{t' + \mathscr{r}/c}$.
- In other words, the fields at $\boldsymbol{r}$ depend upon the condition of the sources at $\boldsymbol{r}'$ at the **earlier** time $\boldsymbol{t - \mathscr{r}/c}$. So we'll guess that

$$V(\boldsymbol{r},t) = \int_{\mathcal{V}} \frac{\rho(\boldsymbol{r}', t - \mathscr{r}/c)\, d\tau'}{\mathscr{r}} \quad , \quad \boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{c}\int_{\mathcal{V}} \frac{\boldsymbol{J}(\boldsymbol{r}', t - \mathscr{r}/c)\, d\tau'}{\mathscr{r}} \quad .$$

  These are called the **retarded potentials**.

Retarded potentials (continued)

- $t_r = t - \mathscr{r}/c$ is called the **retarded time** for the positions $\boldsymbol{r}$ and $\boldsymbol{r}'$.
- Now we need to show that these potentials satisfy the Lorentz gauge condition, and are solutions to the inhomogeneous wave equation.
- For the former, we will need to fiddle with the divergence of $\boldsymbol{J}$ for a bit before we're ready to move on to the divergence of $\boldsymbol{A}$. Bear with me for a few slides…

Retarded potentials and the Lorentz gauge

- First, note that the product rule for derivatives means that

$$\nabla \cdot \left(\frac{\boldsymbol{J}}{\boldsymbol{\mathscr{r}}} \right) = \frac{1}{\mathscr{r}} \nabla \cdot \boldsymbol{J} + \boldsymbol{J} \cdot \nabla \left(\frac{1}{\mathscr{r}} \right) \quad \text{and}$$

$$\nabla' \cdot \left(\frac{\boldsymbol{J}}{\mathscr{r}} \right) = \frac{1}{\mathscr{r}} \nabla' \cdot \boldsymbol{J} + \boldsymbol{J} \cdot \nabla' \left(\frac{1}{\mathscr{r}} \right) \quad ,$$

where $\nabla \equiv \hat{\boldsymbol{x}} \frac{\partial}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial}{\partial y} + \hat{\boldsymbol{z}} \frac{\partial}{\partial z}$ and $\nabla' \equiv \hat{\boldsymbol{x}} \frac{\partial}{\partial x'} + \hat{\boldsymbol{y}} \frac{\partial}{\partial y'} + \hat{\boldsymbol{z}} \frac{\partial}{\partial z'}$,

as usual. Recall also that because $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$,

$$\nabla \left(\frac{1}{\mathscr{r}} \right) = -\nabla' \left(\frac{1}{\mathscr{r}} \right) \quad ,$$

as we showed and used frequently in PHY 217.

Retarded potentials and the Lorentz gauge (continued)

- Thus
$$\nabla\cdot\left(\frac{\boldsymbol{J}}{\boldsymbol{\imath}}\right)=\frac{1}{\boldsymbol{\imath}}\nabla\cdot\boldsymbol{J}-\boldsymbol{J}\cdot\nabla'\left(\frac{1}{\boldsymbol{\imath}}\right)$$
$$=\frac{1}{\boldsymbol{\imath}}\nabla\cdot\boldsymbol{J}-\nabla'\cdot\left(\frac{\boldsymbol{J}}{\boldsymbol{\imath}}\right)+\frac{1}{\boldsymbol{\imath}}\nabla'\cdot\boldsymbol{J} \quad .$$
- Now, there's an *implicit* dependence of $\boldsymbol{J}$ on $\boldsymbol{r}$ through $t_r = t-\boldsymbol{\imath}/c$ just because $\boldsymbol{\imath}=\boldsymbol{r}-\boldsymbol{r}'$. So, using the chain rule,
$$\nabla\cdot\boldsymbol{J}=\frac{\partial J_x}{\partial x}+\frac{\partial J_y}{\partial y}+\frac{\partial J_z}{\partial z}=\frac{\partial J_x}{\partial t_r}\frac{\partial t_r}{\partial x}+\frac{\partial J_y}{\partial t_r}\frac{\partial t_r}{\partial y}+\frac{\partial J_z}{\partial t_r}\frac{\partial t_r}{\partial z}$$
$$=-\frac{1}{c}\left(\frac{\partial J_x}{\partial t_r}\frac{\partial \boldsymbol{\imath}}{\partial x}+\frac{\partial J_y}{\partial t_r}\frac{\partial \boldsymbol{\imath}}{\partial y}+\frac{\partial J_z}{\partial t_r}\frac{\partial \boldsymbol{\imath}}{\partial z}\right)=-\frac{1}{c}\frac{\partial \boldsymbol{J}}{\partial t_r}\cdot\nabla\boldsymbol{\imath} \quad .$$

Retarded potentials and the Lorentz gauge (continued)

- Without this implicit dependence upon ***r***, $\nabla \cdot \boldsymbol{J}$ would be zero, as it was in the static case. Recall that we used to use $\nabla \cdot \boldsymbol{J} = 0$ in magnetostatic calculations (*viz.* the Flashback in the lecture notes for 14 January).
- But ***J*** depends *explicitly* on ***r′***, as well as implicitly through the retarded time t_r, so by the chain rule again,

$$\nabla' \cdot \boldsymbol{J} = \left(\frac{\partial J_{x'}}{\partial x'} + \frac{\partial J_{x'}}{\partial t_r} \frac{\partial t_r}{\partial x'} \right) + \left(\frac{\partial J_{y'}}{\partial y'} + \frac{\partial J_{y'}}{\partial t_r} \frac{\partial t_r}{\partial y'} \right) + \left(\frac{\partial J_{z'}}{\partial z'} + \frac{\partial J_{z'}}{\partial t_r} \frac{\partial t_r}{\partial z'} \right)$$

$$= \left[\frac{\partial J_{x'}}{\partial x'} + \frac{\partial J_{y'}}{\partial y'} + \frac{\partial J_{z'}}{\partial z'} \right] - \frac{1}{c} \left[\frac{\partial J_{x'}}{\partial t_r} \frac{\partial \boldsymbol{\mathscr{r}}}{\partial x'} + \frac{\partial J_{y'}}{\partial t_r} \frac{\partial \boldsymbol{\mathscr{r}}}{\partial y'} + \frac{\partial J_{z'}}{\partial t_r} \frac{\partial \boldsymbol{\mathscr{r}}}{\partial z'} \right]$$

$$= -\frac{\partial \rho(\boldsymbol{r}', t_r)}{\partial t} - \frac{1}{c} \frac{\partial \boldsymbol{J}}{\partial t_r} \cdot \nabla' \boldsymbol{\mathscr{r}} \quad .$$

Retarded potentials and the Lorentz gauge (continued)

Note that the continuity equation, $\nabla \cdot \boldsymbol{J} + \partial\rho/\partial t = 0$, was used in the last step.

- Combine these last three results:

$$\nabla \cdot \left(\frac{\boldsymbol{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}\nabla \cdot \boldsymbol{J} - \nabla' \cdot \left(\frac{\boldsymbol{J}}{\mathscr{r}}\right) + \frac{1}{\mathscr{r}}\nabla' \cdot \boldsymbol{J}$$

because $\nabla \mathscr{r} = -\nabla' \mathscr{r}$

$$= \frac{1}{\mathscr{r}}\left(\cancel{-\frac{1}{c}\frac{\partial \boldsymbol{J}}{\partial t_r} \cdot \nabla \mathscr{r}}\right) - \nabla' \cdot \left(\frac{\boldsymbol{J}}{\mathscr{r}}\right) + \frac{1}{\mathscr{r}}\left(-\frac{\partial \rho}{\partial t} - \cancel{\frac{1}{c}\frac{\partial \boldsymbol{J}}{\partial t_r} \cdot \nabla' \mathscr{r}}\right)$$

$$= -\frac{1}{\mathscr{r}}\frac{\partial \rho}{\partial t} - \nabla' \cdot \left(\frac{\boldsymbol{J}}{\mathscr{r}}\right) \quad .$$

- We can use this in the form of $\boldsymbol{A}$ we've guessed, and verify obedience to the Lorentz gauge condition:

Retarded potentials and the Lorentz gauge (continued)

$$\nabla\cdot\boldsymbol{A}=\nabla\cdot\frac{1}{c}\int_{\mathcal{V}}\frac{\boldsymbol{J}\left(\boldsymbol{r}',t_r\right)d\tau'}{\boldsymbol{\mathscr{r}}}=\frac{1}{c}\int_{\mathcal{V}}\nabla\cdot\left(\frac{\boldsymbol{J}}{\boldsymbol{\mathscr{r}}}\right)d\tau'$$

$$=\frac{1}{c}\int_{\mathcal{V}}\left(-\frac{1}{\boldsymbol{\mathscr{r}}}\frac{\partial\rho}{\partial t}-\nabla'\cdot\left(\frac{\boldsymbol{J}}{\boldsymbol{\mathscr{r}}}\right)\right)d\tau'$$

Use the divergence theorem:

$$=-\frac{1}{c}\frac{\partial}{\partial t}\int_{\mathcal{V}}\frac{\rho\left(\boldsymbol{r}',t_r\right)}{\boldsymbol{\mathscr{r}}}d\tau'-\frac{1}{c}\oint_{\mathcal{S}}\frac{\boldsymbol{J}\cdot d\boldsymbol{a}'}{\boldsymbol{\mathscr{r}}}=-\frac{1}{c}\frac{\partial V}{\partial t}-\frac{1}{c}\oint_{\mathcal{S}}\frac{\boldsymbol{J}\cdot d\boldsymbol{a}'}{\boldsymbol{\mathscr{r}}}\quad .$$

- The last term vanishes if we choose the surface $\mathcal{S}$ to enclose **all** of the charges and currents, because no current flows through that surface, by definition (so $\boldsymbol{J}$ = 0 there):

$$\nabla\cdot\boldsymbol{A}=-\frac{1}{c}\frac{\partial V}{\partial t}\quad .$$

Lorentz gauge

The solutions to the inhomogeneous wave equations are retarded potentials

- Now we are in a position to see whether the retarded potentials are solutions to the wave equations we derived from the Maxwell equations.
- Start by computing the Laplacian of V, and aim at showing that this is equal to $\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}+4\pi\rho$. First, we'll need to fiddle with the gradient of V a bit:

$$\nabla V(\boldsymbol{r},t)=\nabla\int_{\mathcal{V}}\frac{\rho(\boldsymbol{r}',t_r)\,d\tau'}{\mathscr{r}}=\int_{\mathcal{V}}\left[\frac{\nabla\rho}{\mathscr{r}}+\rho\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau' \quad .$$

- $\rho(r',t_r)$ depends implicitly on ***r***, through $t_r = t-\mathscr{r}/c$, so

The solutions to the inhomogeneous wave equations are retarded potentials (continued)

$$\nabla\rho = \frac{\partial\rho}{\partial t_r}\frac{\partial t_r}{\partial x}\hat{x} + \frac{\partial\rho}{\partial t_r}\frac{\partial t_r}{\partial y}\hat{y} + \frac{\partial\rho}{\partial t_r}\frac{\partial t_r}{\partial z}\hat{z} = \frac{\partial\rho}{\partial t_r}\nabla t_r$$

$$= -\frac{1}{c}\frac{\partial\rho}{\partial t_r}\nabla\mathscr{r} \quad .$$

But, as we showed in PHY 217,

$$\nabla\mathscr{r} = \hat{\mathscr{r}} \quad , \text{ and } \quad \nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \quad ,$$

so

$$\nabla V(\boldsymbol{r},t) = \int_{\mathcal{V}}\left[-\frac{1}{c}\frac{\partial\rho}{\partial t_r}\frac{\hat{\mathscr{r}}}{\mathscr{r}} - \rho\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]d\tau' \quad .$$