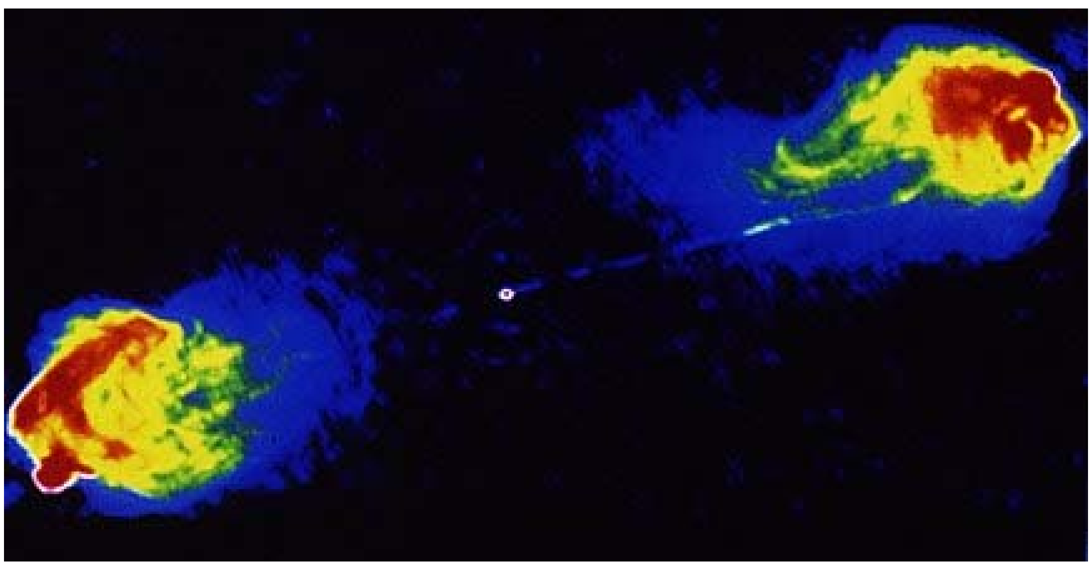
Today in Physics 218: radiation from moving charges

- ❑ Problems with moving charges
- ❑ Motion, snapshots and lengths
- ❑ The Liénard-Wiechert potentials
- ❑ Fields from moving charges



Radio galaxy Cygnus A, observed by Rick Perley et al. with the VLA. The "jets" are streams of matter, ejected at relativistic speeds, in which the electrons radiate by virtue of their acceleration in magnetic fields.

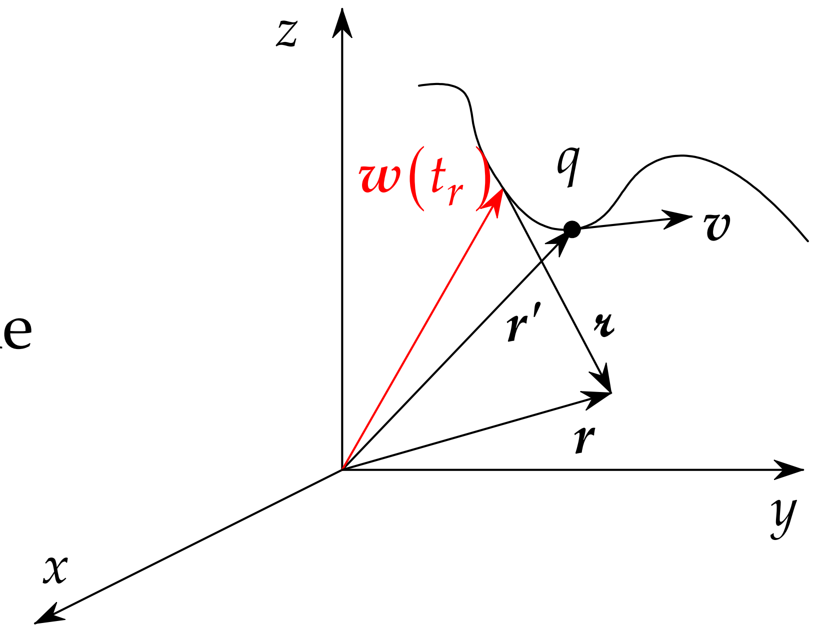
Moving charges

Last time we considered, among other things, the radiation by accelerating charges, bound to other charges by springs, and obtained the Larmor formula for the power such accelerating charges radiate in all directions:

$$P = \frac{2}{3}\frac{q^2 a^2}{c^3} \quad .$$

The use of the springs was deliberate: it keeps the charges from moving very far. Here's what else you have to account for, if the charges can move a long ways or accelerate to very high speeds…

Moving charges (continued)



Consider a point charge that moves along the path as shown.

What are the potentials at $\boldsymbol{r}$?

- Because the charge moves, the situation is different from last time. Not only are the potentials at $(\boldsymbol{r}, t)$ dependent upon the state of the charge at the earlier time t_r ,to account for the propagation delay; but we also have to consider that the charge was at a different point along its path at t_r. Thus, $t_r = t - \mathscr{r}/c$, but

$$\boldsymbol{\mathscr{r}} \neq \boldsymbol{r} - \boldsymbol{r}' \quad . \quad \text{Instead,} \quad \boxed{\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{w}(t_r)} \quad .$$

Moving charges (continued)

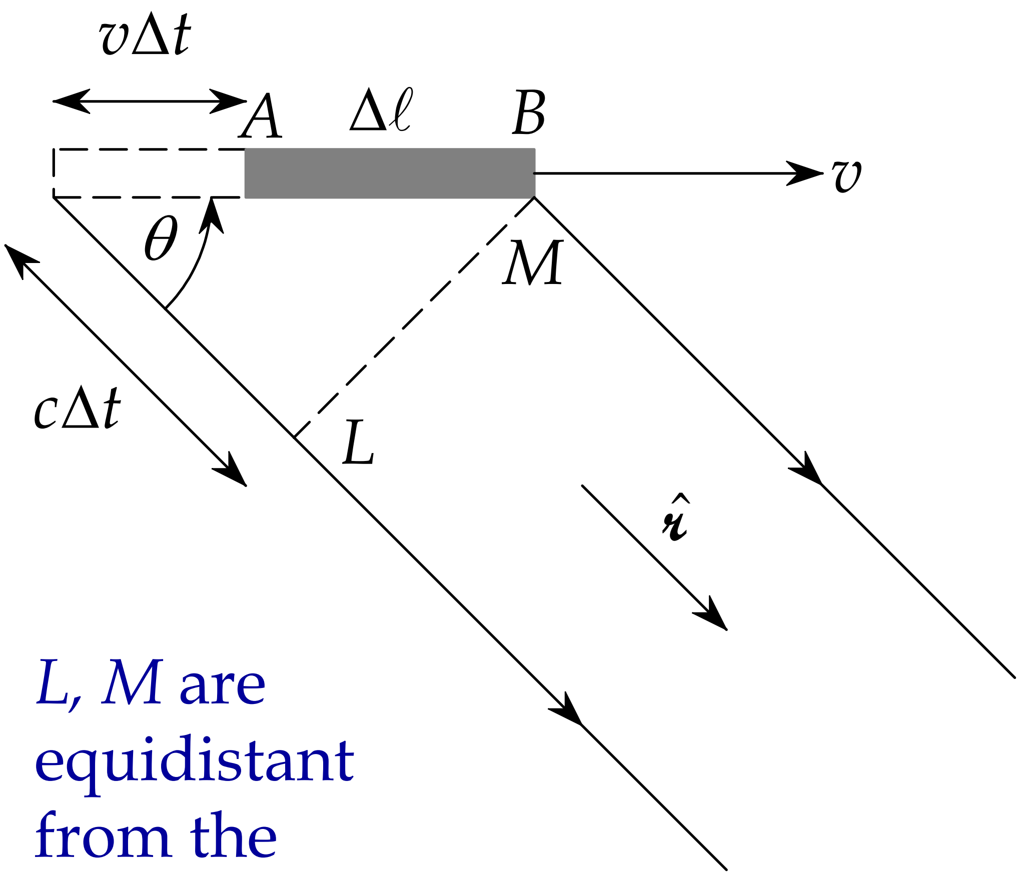
- This might seem like a simple change, but there's more to it. Suppose, for instance, we naively compute

$$V(\boldsymbol{r},t) = \int \frac{\rho(\boldsymbol{r}',t_r)}{\boldsymbol{\mathscr{r}}}\, d\tau' \quad \text{and} \quad \rho(\boldsymbol{r}',t_r) = q\delta^3\left(\boldsymbol{r}'[t_r]\right)$$

$$\Rightarrow \quad V(\boldsymbol{r},t) = \frac{q'}{\boldsymbol{\mathscr{r}}} \quad .$$

- We wouldn't be done, because it turns out in this case that $q' \neq q$.
- The reason? An object in motion looks like its size is different from its "rest" size, because light from one end of the object took longer to get there than the other end, and the object was simply in a different place:

Snapshots of moving objects



L, *M* are equidistant from the observing point.

Suppose the observation point is very far away compared to the object's size $\left(\mathscr{r} \gg \Delta\ell\right)$.

To reach the observation point at the same time as light from point *B*, light from point *A* had to leave Δt earlier, where

$$\left(v\Delta t + \Delta\ell\right)\cos\theta = c\Delta t \quad .$$

Snapshots of moving objects (continued)

Thus,
$$-v\Delta t\cos\theta + c\Delta t = \Delta\ell\cos\theta$$
$$\Rightarrow \quad \Delta t = \frac{\Delta\ell\cos\theta}{c - v\cos\theta} \quad .$$

The moving charge, which has length $\Delta\ell$ along the direction of its motion in real life, *looks* in a snapshot as it its length is

$$\Delta\ell' = \Delta\ell + v\Delta t = \Delta\ell + \frac{v\Delta\ell\cos\theta}{c - v\cos\theta} = \Delta\ell\left(\frac{c - v\cos\theta + v\cos\theta}{c - v\cos\theta}\right)$$
$$= \frac{\Delta\ell}{1 - \frac{v}{c}\cos\theta} = \frac{\Delta\ell}{1 - \frac{1}{c}\hat{\boldsymbol{\imath}}\cdot\boldsymbol{v}} \quad .$$

Snapshots of moving objects (continued)

- It looks longer if the angle between the directions of motion and observation point is acute, and shorter if it's obtuse. It looks its natural size if the angle is 90°.
- Note that this has nothing whatsoever to do with relativity; it's just geometry and the finite speed of light at work.

So we tried to do that integral too quickly. Let's consider the "snapshot" effect on the infinitesimal volume elements into which the charge is divided.

- One of its sides lines up with the direction of motion (if we like). The other two are perpendicular, and their appearance unaffected by motion. Thus it looks from $\mathbf{r}$ at an instant to have a different volume than it does at rest:

The Liénard-Wiechert potentials

$$d\tau'' = \frac{d\tau'}{1 - \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{v}}$$

Now we can do that supposedly simple integral:

$$V(\boldsymbol{r}, t) = \int \frac{\rho(\boldsymbol{r}', t_r)}{\mathscr{r}} d\tau'' = q \int \frac{\delta^3(\boldsymbol{r}'[t_r])}{\mathscr{r}} \frac{d\tau'}{1 - \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{v}}$$

$$= \boxed{\frac{q}{\mathscr{r}\left(1 - \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{v}\right)}} \quad .$$

The Liénard-Wiechert potentials (continued)

Similarly,

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{c}\int \frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{\boldsymbol{\mathscr{r}}}\,d\tau'' = \frac{1}{c}\int \frac{\rho(\boldsymbol{r}',t_r)\boldsymbol{v}}{\boldsymbol{\mathscr{r}}}\,d\tau''$$

$$= q\int \frac{\delta^3\left(\boldsymbol{r}'[t_r]\right)\boldsymbol{v}}{\boldsymbol{\mathscr{r}}}\frac{d\tau'}{1-\frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\cdot\boldsymbol{v}} = \boxed{\frac{\boldsymbol{v}}{c}\frac{q}{\boldsymbol{\mathscr{r}}\left(1-\frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\cdot\boldsymbol{v}\right)} = \frac{\boldsymbol{v}}{c}V(\boldsymbol{r},t)} \quad .$$

These results are called the Liénard-Wiechert potentials.

Fields from moving charges

As usual, we're after the fields and the power emitted by the moving charge. Evaluation of the fields from the potentials is harder than it sounds, though, because $\boldsymbol{\mathscr{r}}$ here is a retarded position, evaluated at the retarded time:

$$\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{w}(t_r),$$

$$\text{and } \boldsymbol{v} = \dot{\boldsymbol{w}}(t_r)$$

is the velocity that counts.

- Nevertheless we must proceed:

$$\boldsymbol{E} = -\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \quad , \quad \boldsymbol{B} = \nabla \times \boldsymbol{A} \quad .$$

  It will be handiest to compute derivatives of t_r first, since they appear often in the expression.

Fields from moving charges (continued)

- First, $\partial t_r / \partial t$. From the definition of retarded time, $\mathscr{r} = c\left(t - t_r\right)$, so

$$c^2\left(t - t_r\right)^2 = \mathscr{r}^2 = \mathscr{r} \cdot \mathscr{r}$$

$$2c^2\left(t - t_r\right)\left(1 - \frac{\partial t_r}{\partial t}\right) = 2\mathscr{r} \cdot \frac{\partial \mathscr{r}}{\partial t}$$

$$c\mathscr{r}\left(1 - \frac{\partial t_r}{\partial t}\right) = \mathscr{r} \cdot \frac{\partial \mathscr{r}}{\partial t} \quad .$$

Now, $\mathscr{r} = \boldsymbol{r} - \boldsymbol{w}\left(t_r\right)$, and $\boldsymbol{r}$ is a fixed point in space, so

$$\frac{\partial \mathscr{r}}{\partial t} = -\frac{\partial \boldsymbol{w}}{\partial t} = -\frac{\partial \boldsymbol{w}}{\partial t_r}\frac{\partial t_r}{\partial t} = -\boldsymbol{v}\frac{\partial t_r}{\partial t} \quad ,$$

Fields from moving charges (continued)

and so
$$c\mathscr{r}\left(1-\frac{\partial t_r}{\partial t}\right)=-\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v}\frac{\partial t_r}{\partial t}$$
$$c\mathscr{r}=\frac{\partial t_r}{\partial t}\left(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v}\right)\equiv\frac{\partial t_r}{\partial t}\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}$$
$$\Rightarrow\quad\frac{\partial t_r}{\partial t}=\frac{c\mathscr{r}}{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}}\quad,$$

where we have defined $\boldsymbol{u}=c\hat{\boldsymbol{\mathscr{r}}}-\boldsymbol{v}$.

- Next, ∇t_r :

$$\nabla t_r=-\frac{1}{c}\nabla\mathscr{r}\left(t_r\right)=-\frac{1}{c}\nabla\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}}=-\frac{1}{2c}\frac{1}{\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}}}\nabla\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}\right)$$
$$=-\frac{1}{2c\mathscr{r}}\left(2\boldsymbol{\mathscr{r}}\times\left[\nabla\times\boldsymbol{\mathscr{r}}\right]+2\left[\boldsymbol{\mathscr{r}}\cdot\nabla\right]\boldsymbol{\mathscr{r}}\right)\quad.$$

using product rule #4

Fields from moving charges (continued)

- We'll have to use the chain rule carefully here:

$$(\mathscr{r}\cdot\nabla)\mathscr{r} = (\mathscr{r}\cdot\nabla)\left(\boldsymbol{r}-\boldsymbol{w}[t_r]\right) = \left(\mathscr{r}_x\frac{\partial}{\partial x}+\mathscr{r}_y\frac{\partial}{\partial y}+\mathscr{r}_z\frac{\partial}{\partial z}\right)\left(\boldsymbol{r}-\boldsymbol{w}[t_r]\right)$$

$$= \mathscr{r}-\left(\mathscr{r}_x\frac{\partial t_r}{\partial x}\frac{d}{dt_r}+\mathscr{r}_y\frac{\partial t_r}{\partial y}\frac{d}{dt_r}+\mathscr{r}_z\frac{\partial t_r}{\partial z}\frac{d}{dt_r}\right)\boldsymbol{w}$$

$$= \mathscr{r}-\left(\mathscr{r}_x\frac{\partial t_r}{\partial x}+\mathscr{r}_y\frac{\partial t_r}{\partial y}+\mathscr{r}_z\frac{\partial t_r}{\partial z}\right)\frac{d\boldsymbol{w}}{dt_r} = \mathscr{r}-(\mathscr{r}\cdot\nabla t_r)\boldsymbol{v} \quad .$$

$$\nabla\times\mathscr{r} = \nabla\times\boldsymbol{r}+\nabla\times\boldsymbol{w}$$

$$= 0+\left(\frac{\partial w_z}{\partial y}-\frac{\partial w_y}{\partial z}\right)\hat{\boldsymbol{x}}+\left(\frac{\partial w_x}{\partial z}-\frac{\partial w_z}{\partial x}\right)\hat{\boldsymbol{y}}+\left(\frac{\partial w_y}{\partial x}-\frac{\partial w_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

Fields from moving charges (continued)

$$\nabla\times\boldsymbol{\mathscr{r}} = \left(\frac{\partial w_z}{\partial t_r}\frac{\partial t_r}{\partial y} - \frac{\partial w_y}{\partial t_r}\frac{\partial t_r}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial w_x}{\partial t_r}\frac{\partial t_r}{\partial z} - \frac{\partial w_z}{\partial t_r}\frac{\partial t_r}{\partial x}\right)\hat{\boldsymbol{y}}$$

$$+\left(\frac{\partial w_y}{\partial t_r}\frac{\partial t_r}{\partial x} - \frac{\partial w_x}{\partial t_r}\frac{\partial t_r}{\partial y}\right)\hat{\boldsymbol{z}}$$

$$= -\boldsymbol{v}\times\nabla t_r \quad ;$$

$$\boldsymbol{\mathscr{r}}\times(\nabla\times\boldsymbol{\mathscr{r}}) = \boldsymbol{\mathscr{r}}\times(-\boldsymbol{v}\times\nabla t_r) = -\boldsymbol{v}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r) + \nabla t_r(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v}) \quad .$$

Combine these last two with the formula at the start:

$$\nabla t_r = -\frac{1}{c\boldsymbol{\mathscr{r}}}\left(\boldsymbol{\mathscr{r}}\times[\nabla\times\boldsymbol{\mathscr{r}}] - [\boldsymbol{\mathscr{r}}\cdot\nabla]\boldsymbol{\mathscr{r}}\right)$$

$$= -\frac{1}{c\boldsymbol{\mathscr{r}}}\left(\cancel{-\boldsymbol{v}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)} + \nabla t_r(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v}) - \boldsymbol{\mathscr{r}} \cancel{+(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)\boldsymbol{v}}\right) \quad .$$

Fields from moving charges (continued)

or
$$\nabla t_r = -\frac{1}{c\mathscr{r}}\left(\boldsymbol{\mathscr{r}} - \nabla t_r \left(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{v}\right)\right) \quad .$$

Solving now for ∇t_r, we get
$$\nabla t_r \left(c\mathscr{r} - \boldsymbol{\mathscr{r}} \cdot \boldsymbol{v}\right) = -\boldsymbol{\mathscr{r}} \quad ;$$
$$\nabla t_r = -\frac{\boldsymbol{\mathscr{r}}}{c\mathscr{r} - \boldsymbol{\mathscr{r}} \cdot \boldsymbol{v}} \quad .$$

Next time, we'll use these results to simplify the derivatives of the fields.