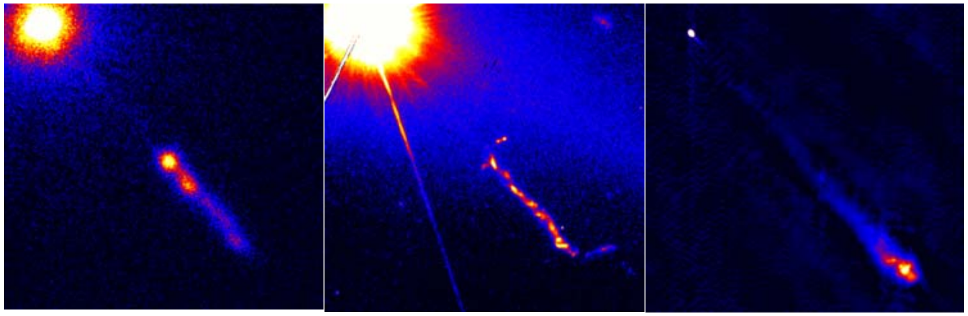
Today in Physics 218: radiation by accelerating charges

- Fields from moving charges: conclusion of derivation from last time.
- The generalized Coulomb field and the radiation field.
- Example: radiation by electric charge accelerating from rest, a rederivation of the Larmor formula.



Radiation from a jet of material ejected from the quasar 3C273, at X-ray (left, NASA Chandra X-ray Observatory), visible (center, NASA Hubble Space Telescope), and radio (right, SERC MERLIN) wavelengths.

Fields from moving charges (continued)

Last time we obtained some useful components of the calculation of the fields of moving charges from the Liénard-Wiechert potentials:

$$\frac{\partial t_r}{\partial t} = \frac{c\boldsymbol{\mathscr{r}}}{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u}} \quad , \quad \nabla t_r = -\frac{\boldsymbol{\mathscr{r}}}{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u}} \quad .$$

where $\boldsymbol{u} = c\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v}$. Now we can proceed:

$$\boldsymbol{E} = -\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \quad , \text{ where}$$

$$V = \frac{q}{\boldsymbol{\mathscr{r}}\left(1 - \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{v}\right)} = \frac{qc}{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u}} \quad \text{and} \quad \boldsymbol{A} = \boldsymbol{v}\frac{q}{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u}} \quad .$$

From last time: ∇t_r

- Next, ∇t_r :

$$\nabla t_r = -\frac{1}{c}\nabla \mathscr{r}(t_r) = -\frac{1}{c}\nabla\sqrt{\mathscr{r}\cdot\mathscr{r}} = -\frac{1}{2c}\frac{1}{\sqrt{\mathscr{r}\cdot\mathscr{r}}}\nabla(\mathscr{r}\cdot\mathscr{r})$$

$$= -\frac{1}{2c\mathscr{r}}\left(2\mathscr{r}\times[\nabla\times\mathscr{r}]+2[\mathscr{r}\cdot\nabla]\mathscr{r}\right) \quad .$$

using product rule #4

- We'll have to use the chain rule carefully here:

$$(\mathscr{r}\cdot\nabla)\mathscr{r} = (\mathscr{r}\cdot\nabla)\left(\boldsymbol{r}-\boldsymbol{w}[t_r]\right) = \left(\mathscr{r}_x\frac{\partial}{\partial x}+\mathscr{r}_y\frac{\partial}{\partial y}+\mathscr{r}_z\frac{\partial}{\partial z}\right)\left(\boldsymbol{r}-\boldsymbol{w}[t_r]\right)$$

$$= \mathscr{r} - \left(\mathscr{r}_x\frac{\partial t_r}{\partial x}\frac{d}{dt_r}+\mathscr{r}_y\frac{\partial t_r}{\partial y}\frac{d}{dt_r}+\mathscr{r}_z\frac{\partial t_r}{\partial z}\frac{d}{dt_r}\right)\boldsymbol{w}$$

$$= \mathscr{r} - \left(\mathscr{r}_x\frac{\partial t_r}{\partial x}+\mathscr{r}_y\frac{\partial t_r}{\partial y}+\mathscr{r}_z\frac{\partial t_r}{\partial z}\right)\frac{d\boldsymbol{w}}{dt_r} = \mathscr{r} - (\mathscr{r}\cdot\nabla t_r)\boldsymbol{v} \quad .$$

From last time: ∇t_r (continued)

$$
\begin{aligned}
\nabla \times \boldsymbol{\mathscr{r}} &= \nabla \times \boldsymbol{r} + \nabla \times \boldsymbol{w} \\
&= 0 + \left(\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z} \right) \hat{\boldsymbol{x}} + \left(\frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x} \right) \hat{\boldsymbol{y}} + \left(\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y} \right) \hat{\boldsymbol{z}} \\
&= \left(\frac{\partial w_z}{\partial t_r} \frac{\partial t_r}{\partial y} - \frac{\partial w_y}{\partial t_r} \frac{\partial t_r}{\partial z} \right) \hat{\boldsymbol{x}} + \left(\frac{\partial w_x}{\partial t_r} \frac{\partial t_r}{\partial z} - \frac{\partial w_z}{\partial t_r} \frac{\partial t_r}{\partial x} \right) \hat{\boldsymbol{y}} \\
&\qquad + \left(\frac{\partial w_y}{\partial t_r} \frac{\partial t_r}{\partial x} - \frac{\partial w_x}{\partial t_r} \frac{\partial t_r}{\partial y} \right) \hat{\boldsymbol{z}} \\
&= -\boldsymbol{v} \times \nabla t_r \quad ;
\end{aligned}
$$

$$
\boldsymbol{\mathscr{r}} \times \left(\nabla \times \boldsymbol{\mathscr{r}} \right) = \boldsymbol{\mathscr{r}} \times \left(-\boldsymbol{v} \times \nabla t_r \right) = -\boldsymbol{v} \left(\boldsymbol{\mathscr{r}} \cdot \nabla t_r \right) + \nabla t_r \left(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{v} \right) \quad .
$$

From last time: ∇t_r (continued)

Combine these last two with the formula at the start:

$$\nabla t_r = -\frac{1}{c\mathscr{r}}\left(\boldsymbol{\mathscr{r}}\times[\nabla\times\boldsymbol{\mathscr{r}}]-[\boldsymbol{\mathscr{r}}\cdot\nabla]\boldsymbol{\mathscr{r}}\right)$$

$$= -\frac{1}{c\mathscr{r}}\left(\cancel{-\boldsymbol{v}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)}+\nabla t_r(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v})-\boldsymbol{\mathscr{r}}+\cancel{(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)\boldsymbol{v}}\right) \quad .$$

or

$$\nabla t_r = -\frac{1}{c\mathscr{r}}\left(\boldsymbol{\mathscr{r}}-\nabla t_r(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v})\right) \quad .$$

Solving now for ∇t_r, we get

$$\nabla t_r\left(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v}\right) = -\boldsymbol{\mathscr{r}} \quad ;$$

$$\nabla t_r = -\frac{\boldsymbol{\mathscr{r}}}{c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v}} = -\frac{\boldsymbol{\mathscr{r}}}{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}} \quad .$$

Fields from moving charges (continued)

$$\nabla V = \nabla\left(\frac{qc}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}\right) = -\frac{qc}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\nabla(\boldsymbol{\mathscr{r}}\cdot\mathbf{u}) = -\frac{qc}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\nabla(c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}) \quad .$$

Now, $\nabla t_r = \nabla\left(t - \frac{\mathscr{r}}{c}\right) = -\frac{1}{c}\nabla\mathscr{r} \quad \Rightarrow \quad \nabla\mathscr{r} = -c\nabla t_r$, and

$$\nabla(\boldsymbol{\mathscr{r}}\cdot\mathbf{v}) = (\boldsymbol{\mathscr{r}}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\boldsymbol{\mathscr{r}}) \quad .$$ P.R. #4

This will take a while, but we evaluated terms like these last time:

$$(\boldsymbol{\mathscr{r}}\cdot\nabla)\mathbf{v} = \left(\mathscr{r}_x\frac{\partial}{\partial x} + \mathscr{r}_y\frac{\partial}{\partial y} + \mathscr{r}_z\frac{\partial}{\partial z}\right)\mathbf{v}$$

$$= \left(\mathscr{r}_x\frac{\partial t_r}{\partial x}\frac{d}{dt_r} + \mathscr{r}_y\frac{\partial t_r}{\partial y}\frac{d}{dt_r} + \mathscr{r}_z\frac{\partial t_r}{\partial z}\frac{d}{dt_r}\right)\mathbf{v}$$

Fields from moving charges (continued)

so
$$(\boldsymbol{\mathscr{r}}\cdot\nabla)\boldsymbol{v}=\left(\mathscr{r}_x\frac{\partial t_r}{\partial x}+\mathscr{r}_y\frac{\partial t_r}{\partial y}+\mathscr{r}_z\frac{\partial t_r}{\partial z}\right)\frac{d\boldsymbol{v}}{dt_r}=(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)\boldsymbol{a}\quad .$$

Similarly,
$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{\mathscr{r}}=(\boldsymbol{v}\cdot\nabla)\left(\boldsymbol{r}-\boldsymbol{w}[t_r]\right)=\left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right)\left(\boldsymbol{r}-\boldsymbol{w}[t_r]\right)$$
$$=\boldsymbol{v}-\left(v_x\frac{\partial t_r}{\partial x}\frac{d}{dt_r}+v_y\frac{\partial t_r}{\partial y}\frac{d}{dt_r}+v_z\frac{\partial t_r}{\partial z}\frac{d}{dt_r}\right)\boldsymbol{w}$$
$$=\boldsymbol{v}-\left(v_x\frac{\partial t_r}{\partial x}+v_y\frac{\partial t_r}{\partial y}+v_z\frac{\partial t_r}{\partial z}\right)\frac{d\boldsymbol{w}}{dt_r}=\boldsymbol{v}-(\boldsymbol{v}\cdot\nabla t_r)\boldsymbol{v}\quad .$$

Fields from moving charges (continued)

We showed last time that

$$\nabla \times \boldsymbol{\mathscr{r}} = -\boldsymbol{v} \times \nabla t_r \quad \text{, so, similarly,}$$

$$\nabla \times \boldsymbol{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\boldsymbol{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\boldsymbol{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\boldsymbol{z}}$$

$$= \left(\frac{\partial v_z}{\partial t_r} \frac{\partial t_r}{\partial y} - \frac{\partial v_y}{\partial t_r} \frac{\partial t_r}{\partial z} \right) \hat{\boldsymbol{x}} + \left(\frac{\partial v_x}{\partial t_r} \frac{\partial t_r}{\partial z} - \frac{\partial v_z}{\partial t_r} \frac{\partial t_r}{\partial x} \right) \hat{\boldsymbol{y}}$$

$$+ \left(\frac{\partial v_y}{\partial t_r} \frac{\partial t_r}{\partial x} - \frac{\partial v_x}{\partial t_r} \frac{\partial t_r}{\partial y} \right) \hat{\boldsymbol{z}}$$

$$= -\boldsymbol{a} \times \nabla t_r \quad .$$

Fields from moving charges (continued)

Thus,

$$\begin{aligned}
\nabla(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v}) &= (\boldsymbol{\mathscr{r}}\cdot\nabla)\boldsymbol{v}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{\mathscr{r}}+\boldsymbol{\mathscr{r}}\times(\nabla\times\boldsymbol{v})+\boldsymbol{v}\times(\nabla\times\boldsymbol{\mathscr{r}}) \\
&= (\boldsymbol{\mathscr{r}}\cdot\nabla t_r)\boldsymbol{a}+\boldsymbol{v}-(\boldsymbol{v}\cdot\nabla t_r)\boldsymbol{v}-\boldsymbol{\mathscr{r}}\times(\boldsymbol{a}\times\nabla t_r)-\boldsymbol{v}\times(\boldsymbol{v}\times\nabla t_r) \\
&= (\boldsymbol{\mathscr{r}}\cdot\nabla t_r)\boldsymbol{a}+\boldsymbol{v}-(\boldsymbol{v}\cdot\nabla t_r)\boldsymbol{v}-\boldsymbol{a}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)+\nabla t_r(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{a}) \\
&\qquad +\boldsymbol{v}(\boldsymbol{v}\cdot\nabla t_r)-\nabla t_r(\boldsymbol{v}\cdot\boldsymbol{v}) \\
&= \boldsymbol{v}+\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{a}-v^2\right)\nabla t_r \quad \text{, and}
\end{aligned}$$

$$\begin{aligned}
\nabla V &= -\frac{qc}{(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^2}\left[-c^2\nabla t_r-\boldsymbol{v}-\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{a}-v^2\right)\nabla t_r\right] \\
&= \frac{qc}{(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}\left[\boldsymbol{v}(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})+\left(c^2+\boldsymbol{\mathscr{r}}\cdot\boldsymbol{a}-v^2\right)(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})\nabla t_r\right] \quad .
\end{aligned}$$

Fields from moving charges (continued)

But we showed last time that $\nabla t_r = -\dfrac{\boldsymbol{\mathscr{r}}}{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u}}$, so

$$\nabla V = \frac{qc}{(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u})^3}\left[\boldsymbol{v}(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u}) - \left(c^2 + \boldsymbol{\mathscr{r}} \cdot \boldsymbol{a} - v^2\right)\boldsymbol{\mathscr{r}}\right] \quad .$$

Now for the vector-potential part:

$$\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = \frac{1}{c^2}\frac{\partial}{\partial t}(\boldsymbol{v}V) = \frac{1}{c^2}\left(V\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\frac{\partial V}{\partial t}\right) = \frac{1}{c^2}\left(V\frac{\partial \boldsymbol{v}}{\partial t_r} + \boldsymbol{v}\frac{\partial V}{\partial t_r}\right)\frac{\partial t_r}{\partial t}$$

$$= \frac{1}{c^2}\left(V\boldsymbol{a} + \boldsymbol{v}\frac{\partial}{\partial t_r}\left[\frac{qc}{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u}}\right]\right)\frac{\partial t_r}{\partial t} = \frac{1}{c^2}\left(V\boldsymbol{a} - \frac{qc\boldsymbol{v}}{(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u})^2}\frac{\partial}{\partial t_r}(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u})\right)\frac{\partial t_r}{\partial t}$$

$$= \frac{1}{c^2}\left(V\boldsymbol{a} - \frac{qc\boldsymbol{v}}{(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u})^2}\frac{\partial}{\partial t_r}(c\mathscr{r} - \boldsymbol{v} \cdot \boldsymbol{\mathscr{r}})\right)\frac{\partial t_r}{\partial t}$$

Fields from moving charges (continued)

$$\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}=\frac{1}{c^2}\left[V\boldsymbol{a}-\frac{qc\boldsymbol{v}}{(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^2}\left(c\frac{\partial \mathscr{r}}{\partial t_r}-\boldsymbol{a}\cdot\boldsymbol{\mathscr{r}}-\boldsymbol{v}\cdot\frac{\partial \boldsymbol{\mathscr{r}}}{\partial t_r}\right)\right]\frac{\partial t_r}{\partial t}$$

$$=\frac{1}{c}\left[\frac{qc}{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}}\boldsymbol{a}+\frac{qc\boldsymbol{v}}{(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^2}\left(\frac{c}{\mathscr{r}}\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v}+\boldsymbol{a}\cdot\boldsymbol{\mathscr{r}}-v^2\right)\right]\frac{\mathscr{r}}{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}}$$

$$=\frac{qc}{(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}\left[\frac{\mathscr{r}}{c}\boldsymbol{a}(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})+\frac{\mathscr{r}}{c}\boldsymbol{v}\left(\frac{c}{\mathscr{r}}\{c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\}+\boldsymbol{a}\cdot\boldsymbol{\mathscr{r}}-v^2\right)\right]$$

$$=\frac{qc}{(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}\left[\frac{\mathscr{r}}{c}\boldsymbol{a}(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})+\frac{\mathscr{r}}{c}\boldsymbol{v}\left(c^2-v^2-\frac{c}{\mathscr{r}}\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}+\boldsymbol{a}\cdot\boldsymbol{\mathscr{r}}\right)\right]\quad .$$

Thus – finally – we get:

Fields from moving charges (continued)

$$\boldsymbol{E} = -\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = -\frac{qc}{\left(\boldsymbol{\imath}\cdot\boldsymbol{u}\right)^3}\left[\boldsymbol{v}\left(\boldsymbol{\imath}\cdot\boldsymbol{u}\right) - \left(c^2 + \boldsymbol{\imath}\cdot\boldsymbol{a} - v^2\right)\boldsymbol{\imath}\right]$$

$$-\frac{qc}{\left(\boldsymbol{\imath}\cdot\boldsymbol{u}\right)^3}\left[\frac{\imath}{c}\boldsymbol{a}\left(\boldsymbol{\imath}\cdot\boldsymbol{u}\right) + \frac{\imath}{c}\boldsymbol{v}\left(c^2 - v^2 - \frac{c}{\imath}\boldsymbol{\imath}\cdot\boldsymbol{u} + \boldsymbol{a}\cdot\boldsymbol{\imath}\right)\right]$$

$$= -\frac{qc}{\left(\boldsymbol{\imath}\cdot\boldsymbol{u}\right)^3}\left[\boldsymbol{v}\left(\boldsymbol{\imath}\cdot\boldsymbol{u}\right) + \left(c^2 + \boldsymbol{\imath}\cdot\boldsymbol{a} - v^2\right)\left(-\boldsymbol{\imath} + \frac{\imath}{c}\boldsymbol{v}\right)\right.$$

$$\left. + \frac{\imath}{c}\boldsymbol{a}\left(\boldsymbol{\imath}\cdot\boldsymbol{u}\right) - \frac{\imath}{c}\boldsymbol{v}\frac{c}{\imath}\boldsymbol{\imath}\cdot\boldsymbol{u}\right]$$

$$= -\frac{qc}{\left(\boldsymbol{\imath}\cdot\boldsymbol{u}\right)^3}\left[\left(c^2 + \boldsymbol{\imath}\cdot\boldsymbol{a} - v^2\right)\left(-\frac{\imath}{c}\boldsymbol{u}\right) + \frac{\imath}{c}\boldsymbol{a}\left(\boldsymbol{\imath}\cdot\boldsymbol{u}\right)\right] \quad .$$

Fields from moving charges (continued)

$$\boldsymbol{E} = -\frac{qc}{(\boldsymbol{\imath}\cdot\boldsymbol{u})^3}\left[-\left(c^2 - v^2\right)\frac{\boldsymbol{\imath}}{c}\boldsymbol{u} + \frac{\boldsymbol{\imath}}{c}\left\{\boldsymbol{a}(\boldsymbol{\imath}\cdot\boldsymbol{u}) - \boldsymbol{u}(\boldsymbol{\imath}\cdot\boldsymbol{a})\right\}\right]$$

$$= \boldsymbol{\imath}\times(\boldsymbol{a}\times\boldsymbol{u})$$

$$= \frac{q\boldsymbol{\imath}}{(\boldsymbol{\imath}\cdot\boldsymbol{u})^3}\left[\left(c^2 - v^2\right)\boldsymbol{u} + \boldsymbol{\imath}\times(\boldsymbol{u}\times\boldsymbol{a})\right] \quad .$$

Similarly, but avoiding the tedium,

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \hat{\boldsymbol{\imath}}\times\boldsymbol{E} \quad .$$

There is a special significance to each of the two terms in ***E*****.**

The generalized Coulomb field

The first term is

$$\boldsymbol{E}_{GC} = \frac{q\mathscr{r}}{\left(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u}\right)^3}\left(c^2 - v^2\right)\boldsymbol{u} \quad .$$

This field is proportional to $1/r^2$,and its direction is the same as that of $\boldsymbol{u} = c\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v}$. Thus it is similar in some ways to the field for a static point charge. In fact, if we let $v = a = 0$, this term gives us

$$\boldsymbol{E}_{GC} = \frac{q\mathscr{r}}{\left(\boldsymbol{\mathscr{r}} \cdot \left[c\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v}\right]\right)^3}\left(c^2 - v^2\right)\left(c\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v}\right) \rightarrow \frac{q\mathscr{r}}{\left(\boldsymbol{\mathscr{r}} \cdot c\hat{\boldsymbol{\mathscr{r}}}\right)^3} c^3 \hat{\boldsymbol{\mathscr{r}}} = \frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} \quad ,$$

$$\boldsymbol{B}_{GC} = \hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{E}_{GC} = 0 \quad ,$$

just as in statics; hence the name.

The radiation field

The other term,

$$\mathbf{E}_{\text{rad}} = \frac{q\boldsymbol{\mathscr{r}}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a}) \quad ,$$

is only proportional to $1/r$. Thus, as we've seen before, in the case of dipole radiation in the far field, this term is much larger than the other one at large *r*.

- The radiation field also points perpendicular to $\hat{\boldsymbol{\mathscr{r}}}$,as befits a transverse spherical wave: $\hat{\boldsymbol{\mathscr{r}}}\cdot\left[\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right]=0.$
- Note also the presence of $\mathbf{a}$: again it is shown that an electric charge needs to accelerate in order to radiate.

Example: power radiated by accelerating charges

As just noted, the power radiated to large distances is dominated by the radiation field. Let's compute the power radiated by an electric charge q that accelerates, starting from rest at $t_r = 0$:

$$\boldsymbol{u} = c\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v} \cong c\hat{\boldsymbol{\mathscr{r}}} \quad .$$

(Actually this is a good approximation for all speeds $v << c$.) Then,

$$\boldsymbol{E}_{\text{rad}}\left(t_r = 0\right) = \frac{q\mathscr{r}}{\left(\boldsymbol{\mathscr{r}} \cdot c\hat{\boldsymbol{\mathscr{r}}}\right)^3}\boldsymbol{\mathscr{r}} \times \left(c\hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{a}\right) = \frac{q}{\mathscr{r}c^2}\left[\hat{\boldsymbol{\mathscr{r}}}\left(\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{a}\right) - \boldsymbol{a}\right] \quad ,$$

and

$$\boldsymbol{S}\left(t_r = 0\right) = \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{B} = \frac{c}{4\pi}\boldsymbol{E}_{\text{rad}} \times \left(\hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{E}_{\text{rad}}\right)$$

$$= \frac{c}{4\pi}\left[\hat{\boldsymbol{\mathscr{r}}}E_{\text{rad}}^2 - \boldsymbol{E}_{\text{rad}}\left(\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{E}_{\text{rad}}\right)\right] = \frac{cE_{\text{rad}}^2}{4\pi}\hat{\boldsymbol{\mathscr{r}}} \quad .$$

Power radiated by accelerating charges (continued)

$$\mathbf{S} = \hat{\boldsymbol{\mathscr{r}}}\frac{c}{4\pi}\mathbf{E}_{\mathrm{rad}}\cdot\mathbf{E}_{\mathrm{rad}} = \hat{\boldsymbol{\mathscr{r}}}\frac{c}{4\pi}\frac{q^2}{\mathscr{r}^2c^4}\left[\hat{\boldsymbol{\mathscr{r}}}\left(\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a}\right)-\mathbf{a}\right]^2$$

$$= \hat{\boldsymbol{\mathscr{r}}}\frac{c}{4\pi}\frac{q^2}{\mathscr{r}^2c^4}\left[a^2+\left(\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a}\right)^2-2\mathbf{a}\cdot\hat{\boldsymbol{\mathscr{r}}}\left(\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a}\right)\right]$$

$$= \hat{\boldsymbol{\mathscr{r}}}\frac{c}{4\pi}\frac{q^2}{\mathscr{r}^2c^4}\left[a^2-\left(\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a}\right)^2\right] = \hat{\boldsymbol{\mathscr{r}}}\frac{c}{4\pi}\frac{q^2}{\mathscr{r}^2c^4}\left(1-\cos^2\theta\right)$$

$$= \boxed{\frac{q^2a^2}{4\pi c^3}\frac{\sin^2\theta}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}} \quad ,$$

where θ is the angle between the acceleration and the direction to the observing point, $\mathbf{r}$ (that is, the angle of $\hat{\boldsymbol{\mathscr{r}}}$).

Power radiated by accelerating charges (continued)

The $\sin^2\theta$ factor indicates that the charge radiates no power in the forward or backward direction, and radiates most of its power perpendicular to the direction of its acceleration.

- This should remind you, again, of electric dipole radiation.

The power radiated through any sphere centered on the charge is familiar:

$$P = \oint \mathbf{S} \cdot d\boldsymbol{\sigma} = \frac{q^2 a^2}{4\pi c^3} \int \frac{\sin^2\theta}{\mathscr{r}^2} \hat{\mathscr{r}} \cdot \hat{\mathscr{r}} \mathscr{r}^2 \sin\theta d\theta d\phi$$

$$= \frac{q^2 a^2}{4\pi c^3} \int_0^{\pi} \sin^3\theta \int_0^{2\pi} d\phi = \frac{q^2 a^2}{4\pi c^3} \frac{4}{3} 2\pi = \boxed{\frac{2}{3}\frac{q^2 a^2}{c^3}} .$$

Larmor formula again