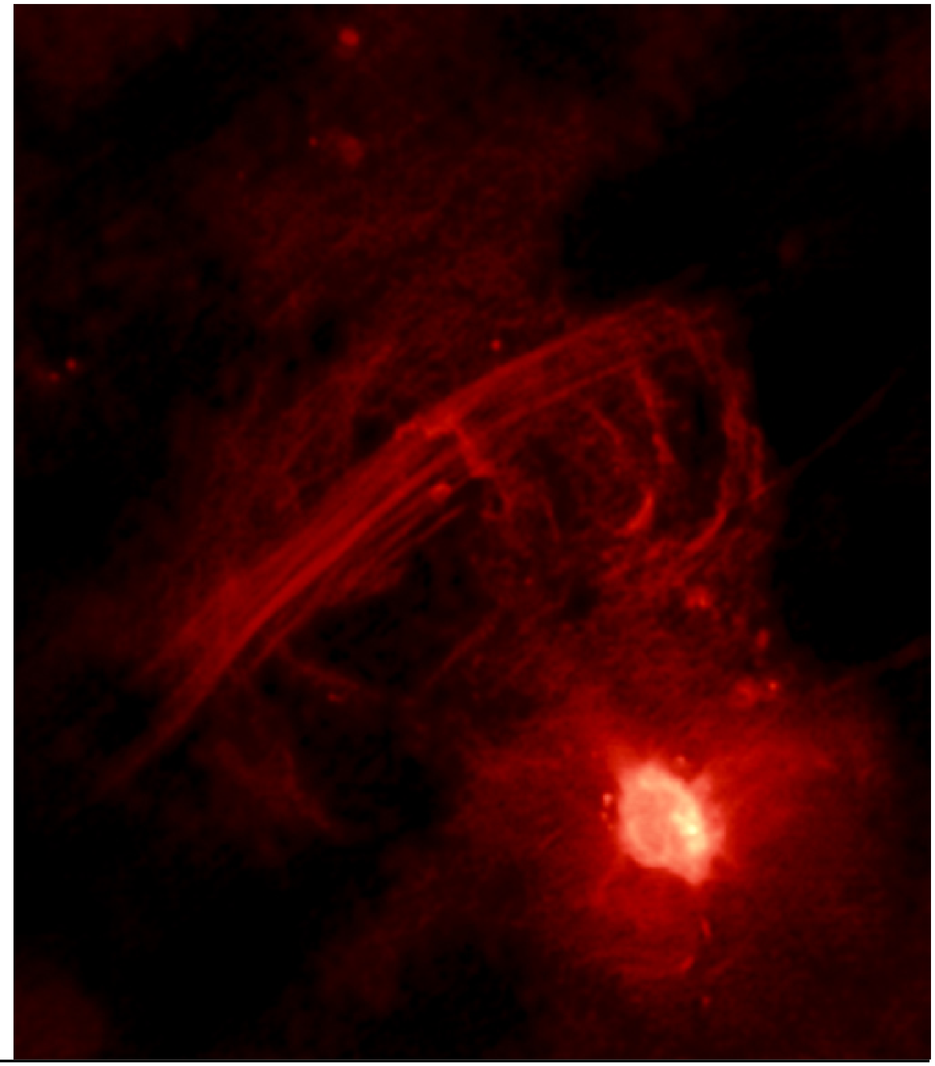
Today in Physics 218: relativistic accelerating charges

- Relativistic charges and the generalized Larmor formula
- Bremsstrahlung
- Synchrotron radiation



The radio arcs in the Milky Way's center, observed with the VLA by Farhad Yusef-Zadeh (Northwestern U.).

Relativistic charges and the generalized Larmor formula

We derived the Larmor formula,

$$P = \frac{2}{3}\frac{q^2 a^2}{c^3} \quad ,$$

under the assumption that $v \ll c$. This is not always a useful or interesting approximation.

- However, the derivation of the power is more complicated if we relax the condition $v \ll c$, so we will only sketch the derivation here.
- The extra complication arises because we, sitting at point ***r***, see the charge to be emitting $P = dW/dt$, while, from the charge's point of view, it's emitting dW/dt_r.

Relativistic charges and the generalized Larmor formula (continued)

- It's true that these are simply related:

$$P_{\text{obs.}} = \frac{dW}{dt} = \frac{dW}{dt_r}\frac{\partial t_r}{\partial t} = \frac{dW}{dt_r}\frac{\mathscr{r}c}{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}} \quad .$$

- We also know that

$$P_{\text{obs.}} = \int \frac{c}{4\pi} E_{\text{rad}}^2 \hat{\boldsymbol{\mathscr{r}}}\cdot\hat{\boldsymbol{r}}\,\mathscr{r}^2 \sin\theta\, d\theta\, d\phi$$

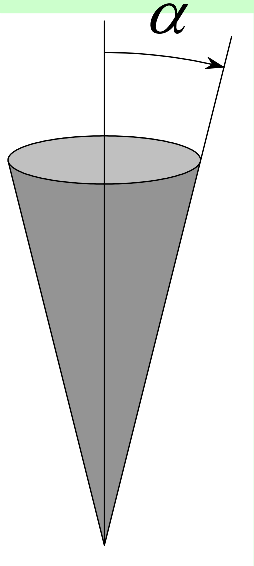
$$= \int \frac{c}{4\pi} E_{\text{rad}}^2 \mathscr{r}^2\, d\Omega \quad ,$$

and at this point it will be convenient to define the power per unit solid angle,

$$\left(\frac{dP}{d\Omega}\right)_{\text{obs.}} = \frac{c}{4\pi} E_{\text{rad}}^2 \mathscr{r}^2 \quad .$$

Reminder: solid angle

Solid angle is to angle what area is to distance, and, like angle, is bounded. The paradigm of solid angle is an infinite cone. It has units (steradians) but no dimensions.



- A cone with small opening angle α corresponds to a solid angle $\Omega = \pi\alpha^2$.
- In spherical coordinates, the infinitesimal element of solid angle is $d\Omega = \sin\theta d\theta d\phi$:

$$da = (rd\theta)(r\sin\theta d\phi) = r^2 d\Omega.$$

- The biggest a solid angle can be is that of a cone opened so far that its side collapses into a line.
  The value of this solid angle is $\Omega = \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = 4\pi.$

Relativistic charges and the generalized Larmor formula (continued)

- For a concrete example, consider the integral we did on the way to getting the Larmor formula:

$$P_{v\ll c} = \oint \boldsymbol{S}\cdot d\boldsymbol{\sigma} = \frac{q^2a^2}{4\pi c^3}\int \frac{\sin^2\theta}{\mathscr{r}^2}\hat{\mathscr{r}}\cdot\hat{\mathscr{r}}\mathscr{r}^2 \sin\theta d\theta d\phi \quad ,$$

$$\left(\frac{dP}{d\Omega}\right)_{v\ll c} = \frac{q^2a^2}{4\pi c^3}\sin^2\theta \quad .$$

- Express the emitted and observed powers in this fashion:

$$P_{\text{emitted}} = \frac{dW}{dt_r} = \frac{1}{\frac{\partial t_r}{\partial t}}\frac{dW}{dt} = \frac{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}}{\mathscr{r}c}\frac{dW}{dt} = \frac{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}}{\mathscr{r}c}P_{\text{obs}} \quad ;$$

$$\left(\frac{dP}{d\Omega}\right)_{\text{emitted}} = \frac{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}}{\mathscr{r}c}\left(\frac{dP}{d\Omega}\right)_{\text{obs}} \quad .$$

Relativistic charges and the generalized Larmor formula (continued)

- Last time we found that the field at large distance radiated from a moving charge is

$$\boldsymbol{E}_{\text{rad}} = \frac{q\boldsymbol{\imath}}{\left(\boldsymbol{\imath}\cdot\boldsymbol{u}\right)^3}\boldsymbol{\imath}\times\left(\boldsymbol{u}\times\boldsymbol{a}\right) \quad ,$$

so

$$\begin{aligned}\left(\frac{dP}{d\Omega}\right)_{\text{emitted}} &= \frac{\boldsymbol{\imath}\cdot\boldsymbol{u}}{\imath c}\frac{c}{4\pi}E_{\text{rad}}^2\imath^2 = \frac{\boldsymbol{\imath}\cdot\boldsymbol{u}}{\imath c}\frac{c}{4\pi}E_{\text{rad}}^2\imath^2 \\ &= \frac{\boldsymbol{\imath}\cdot\boldsymbol{u}}{\imath c}\frac{c\imath^2}{4\pi}\frac{q^2\imath^2}{\left(\boldsymbol{\imath}\cdot\boldsymbol{u}\right)^6}\left[\boldsymbol{\imath}\times\left(\boldsymbol{u}\times\boldsymbol{a}\right)\right]^2 \\ &= \frac{q^2}{4\pi}\frac{\left[\hat{\boldsymbol{\imath}}\times\left(\boldsymbol{u}\times\boldsymbol{a}\right)\right]^2}{\left(\hat{\boldsymbol{\imath}}\cdot\boldsymbol{u}\right)^5} \quad .\end{aligned}$$

Relativistic charges and the generalized Larmor formula (continued)

- To find the total power emitted, we "just" integrate this last result over all solid angles:

$$P_{\text{emitted}} = \frac{q^2}{4\pi}\int \frac{\left[\hat{\boldsymbol{\imath}}\times\left(\boldsymbol{u}\times\boldsymbol{a}\right)\right]^2}{\left(\hat{\boldsymbol{\imath}}\cdot\boldsymbol{u}\right)^5}\,d\Omega \quad .$$

This is very complicated and not very instructive, so, just as Griffiths does in the book, we'll skip to the answer:

$$\boxed{P_{\text{emitted}} = \frac{2}{3}\frac{q^2}{c^3}\gamma^6\left[a^2 - \left(\frac{\boldsymbol{v}}{c}\times\boldsymbol{a}\right)^2\right] \quad ,}$$

Generalized Larmor formula

where $$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \quad .$$

Relativistic charges and the generalized Larmor formula (continued)

- Note that the original Larmor formula is recovered if we let $v \ll c$ in this formula.
- Note also that the effect of moving at high speeds $(\gamma \gg 1)$ is that a charged particle emits ***much*** more power that it would alt lower speeds, for the same acceleration.
- Time to apply this in some concrete examples. There are two simple geometries that will do for illustration: velocity and acceleration collinear, or velocity and acceleration perpendicular.

Example: bremsstrahlung ("braking radiation")

Example 11.3: Suppose that $\boldsymbol{v}$ and $\boldsymbol{a}$ are instantaneously collinear at time t_r as, for example, in straight-line motion. Find the angular distribution of the radiation (i.e. $\left(dP/d\Omega\right)_{\text{emitted}}$) and the total power emitted.
Solution:

Since either $\boldsymbol{v} \parallel \boldsymbol{a}$ or $\boldsymbol{v} \parallel -\boldsymbol{a}$ would count, we'll start with something that's true for either:

$$\boldsymbol{v} \times \boldsymbol{a} = 0$$

$$\boldsymbol{u} \times \boldsymbol{a} = c\hat{\boldsymbol{\imath}} \times \boldsymbol{a} - \boldsymbol{v} \times \boldsymbol{a} = c\hat{\boldsymbol{\imath}} \times \boldsymbol{a} \quad .$$

With this, we can also write

$$\hat{\boldsymbol{\imath}} \times (\boldsymbol{u} \times \boldsymbol{a}) = c\hat{\boldsymbol{\imath}} \times (\hat{\boldsymbol{\imath}} \times \boldsymbol{a}) = c\hat{\boldsymbol{\imath}}(\hat{\boldsymbol{\imath}} \cdot \boldsymbol{a}) - c\boldsymbol{a} \quad ,$$

$$\left[\hat{\boldsymbol{\imath}} \times (\boldsymbol{u} \times \boldsymbol{a})\right]^2 = c^2\left[a^2 + (\hat{\boldsymbol{\imath}} \cdot \boldsymbol{a})^2 - 2\boldsymbol{a} \cdot \hat{\boldsymbol{\imath}}(\hat{\boldsymbol{\imath}} \cdot \boldsymbol{a})\right] = c^2\left[a^2 - (\hat{\boldsymbol{\imath}} \cdot \boldsymbol{a})^2\right] \quad .$$

Bremsstrahlung (continued)

But this is just

$$\hat{\boldsymbol{\imath}} \times (\boldsymbol{u} \times \boldsymbol{a}) = c\hat{\boldsymbol{\imath}} \times (\hat{\boldsymbol{\imath}} \times \boldsymbol{a}) = c\hat{\boldsymbol{\imath}}(\hat{\boldsymbol{\imath}} \cdot \boldsymbol{a}) - c\boldsymbol{a} \quad ,$$

$$\left[\hat{\boldsymbol{\imath}} \times (\boldsymbol{u} \times \boldsymbol{a})\right]^2 = c^2 a^2 \left[1 - \cos^2\theta\right] = c^2 a^2 \sin^2\theta \quad ,$$

where as usual θ is the angle between the acceleration and the direction from the charge to us $(\hat{\boldsymbol{\imath}})$. Meanwhile,

$$(\hat{\boldsymbol{\imath}} \cdot \boldsymbol{u})^5 = (\hat{\boldsymbol{\imath}} \cdot c\hat{\boldsymbol{\imath}} - \boldsymbol{v} \cdot \hat{\boldsymbol{\imath}})^5 = c^5 \left(1 - \frac{v}{c}\cos\theta\right)^5 \equiv c^5 (1 - \beta\cos\theta)^5 \quad .$$

Thus

$$\left(\frac{dP}{d\Omega}\right)_{\text{emitted}} = \frac{q^2}{4\pi} \frac{\left[\hat{\boldsymbol{\imath}} \times (\boldsymbol{u} \times \boldsymbol{a})\right]^2}{(\hat{\boldsymbol{\imath}} \cdot \boldsymbol{u})^5} = \boxed{\frac{q^2 a^2}{4\pi c^3} \frac{\sin^2\theta}{(1 - \beta\cos\theta)^5}} \quad .$$

Bremsstrahlung (continued)

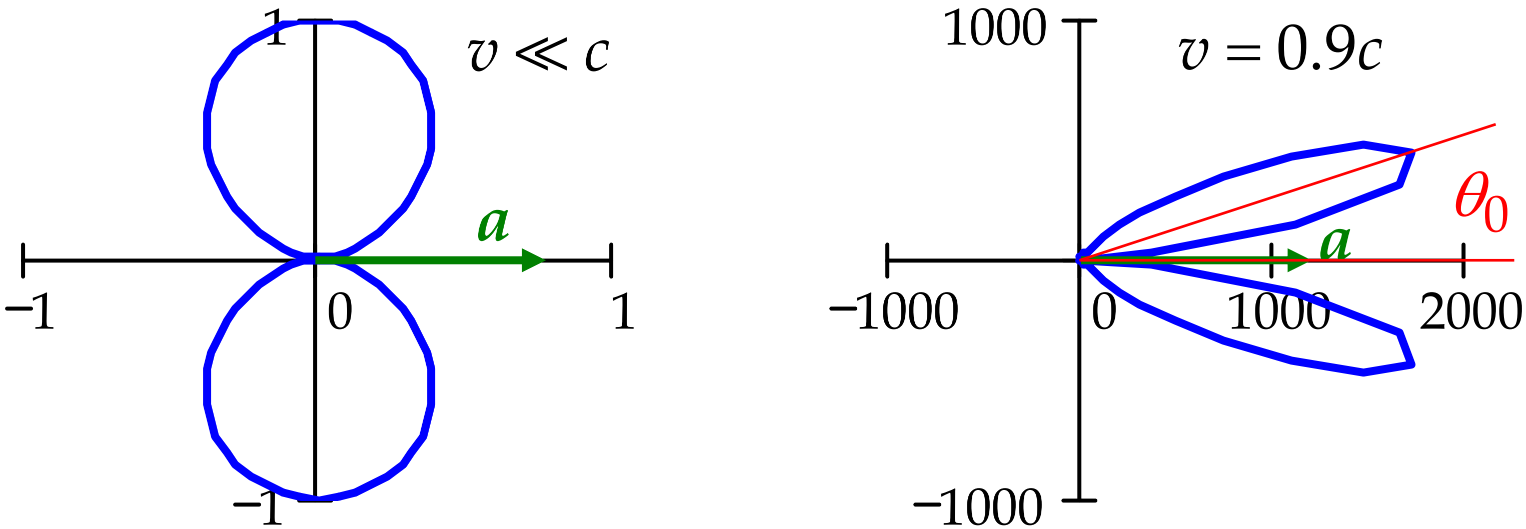
This differs from $(dP/d\Omega)_{\text{emitted}}$ at low speeds by the factor $(1-\beta\cos\theta)^5$. Thus $dP/d\Omega$ is still zero in the forward and backward directions, as it is for $v \ll c$.

- However, if $\beta \to 1$, the quantity $\beta\cos\theta$ gets very close to 1 at small angles, so

$$(1-\beta\cos\theta)^{-5} \gg 1 \quad \text{if } \beta \to 1, \theta \to 0.$$

- Thus the charge **beams** most of its energy along the wall of a narrow cone, concentrated in the forward direction (i.e. along $\mathbf{v}$) if its speed approaches that of light.

Bremsstrahlung (continued)



Angular patterns of radiation for a charge q at speeds $v \ll c$ (left) and $v = 0.9c$ (right). Note the change in scale. Here's what's plotted:

$$x = \sin^2\theta\cos\theta \qquad x = \sin^2\theta\cos\theta\big/(1-\beta\cos\theta)^5$$

$$y = \sin^2\theta\sin\theta \qquad y = \sin^2\theta\sin\theta\big/(1-\beta\cos\theta)^5$$

Bremsstrahlung (continued)

By differentiating $(dP/d\Omega)_{\text{emitted}}$ with respect to θ and setting the result equal to zero, one can find the angle at which the power per solid angle is largest, and it turns out to be

$$\theta_0 \cong \sqrt{\frac{1-\beta}{2}} \quad (\text{if } \beta \to 1).$$

At low speeds, the maximum occurs at $\theta_0 = \pi/2$. Thus

$$\frac{(dP/d\Omega)_{\max,v\to c}}{(dP/d\Omega)_{\max,v\ll c}} = \frac{\dfrac{q^2a^2}{4\pi c^3}\dfrac{\sin^2\theta_0}{(1-\beta\cos\theta_0)^5}}{\dfrac{q^2a^2}{4\pi c^3}} = \frac{1}{4}\left(\frac{8}{5}\right)^5\gamma^8 \quad .$$

Bremsstrahlung (continued)

The peak value of $\left(dP/d\Omega\right)_{\text{emitted}}$ is thus **MUCH** larger for high speeds than for low speeds; consider that

$$v = 0.9c \quad \Rightarrow \quad \gamma = 2.3 \quad \Rightarrow \quad \gamma^8 = 767 \quad ,$$

$$v = 0.99c \quad \Rightarrow \quad \gamma = 7.09 \quad \Rightarrow \quad \gamma^8 = 6.4\times 10^6 \quad .$$

The total power emitted is

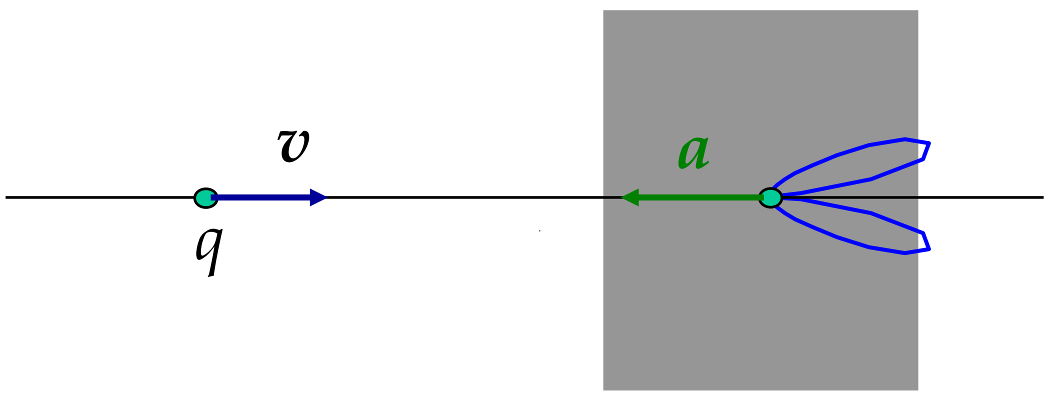
$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{q^2 a^2}{4\pi c^3} \int_0^{\pi} \frac{\sin^3\theta d\theta}{\left(1-\beta\cos\theta\right)^5} \int_0^{2\pi} d\phi$$

$$= \frac{q^2 a^2}{2c^3} \int_{-1}^{1} \frac{\left(1-u^2\right)du}{\left(1-\beta u\right)^5} = \frac{q^2 a^2}{2c^3} \frac{4}{3\left(1-\beta^2\right)^3} = \boxed{\frac{2}{3}\frac{q^2 a^2}{c^3}\gamma^6} \quad .$$

Bremsstrahlung (continued)

Since both the low-speed (Larmor formula) and high-speed versions of the emitted power depend only on the *square* of a, the same power and $dP/d\Omega$ is seen whether the charge is accelerating or decelerating.

- ❑ It is more common, terrestrially, to observe the power from decelerating particles, by first accelerating a beam of charges up to high speeds and decelerating them rapidly by sending the beam into a (lead) brick wall:



Most of the power is beamed in the forward direction.

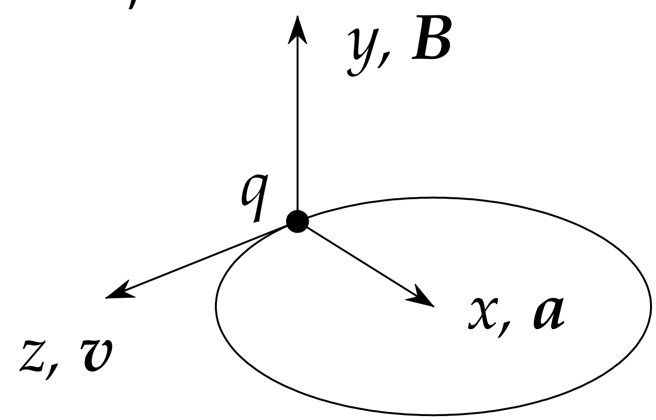
Synchrotron radiation

If $\boldsymbol{v}$ is perpendicular to $\boldsymbol{a}$ (the other "simple" geometry), as in the case of uniform circular motion, P and $dP/d\Omega$ can be calculated with just a little more effort than the previous problem. (This in fact is problem !11.16 in the book, which will not be assigned.) The answers are

$$\left(\frac{dP}{d\Omega}\right)_{\text{emitted}} = \frac{q^2 a^2}{4\pi c^3} \frac{(1-\beta\cos\theta)^2 - \left(1-\beta^2\right)\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^5} \quad ,$$

$$P = \frac{2}{3}\frac{q^2 a^2}{c^3}\gamma^4 \quad .$$

The coordinate system is described at right.



Synchrotron radiation (continued)

- The most common way to see charges in uniform circular motion in nature is of course to put some in motion in a uniform magnetic field.
- This result shows that the radiation still tends to be beamed in the forward direction.
- Charges used to be accelerated to high energies like this, in variable-***B*** machines called **synchrotrons**, and the radiation resulting from the centripetal acceleration, for which the total power is given by the expression above, has been called synchrotron radiation ever since.
- Most of the radio radiation by normal galaxies is produced in this way, by electrons spiraling around in interstellar magnetic fields.

Fields from moving charges (continued)

Last time we obtained some useful components of the calculation of the fields of moving charges from the Liénard-Wiechert potentials:

$$\frac{\partial t_r}{\partial t} = \frac{c\boldsymbol{\mathscr{r}}}{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u}} \quad , \quad \nabla t_r = -\frac{\boldsymbol{\mathscr{r}}}{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u}} \quad .$$

where $\boldsymbol{u} = c\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v}$. Now we can proceed:

$$\boldsymbol{E} = -\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \quad \text{, where}$$

$$V = \frac{q}{\boldsymbol{\mathscr{r}}\left(1 - \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{v}\right)} = \frac{qc}{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u}} \quad \text{and} \quad \boldsymbol{A} = \boldsymbol{v}\frac{q}{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u}} \quad .$$

From last time: ∇t_r

- Next, ∇t_r :

$$\nabla t_r = -\frac{1}{c}\nabla \boldsymbol{\mathscr{r}}(t_r) = -\frac{1}{c}\nabla\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}} = -\frac{1}{2c}\frac{1}{\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}}}\nabla(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}})$$

$$= -\frac{1}{2c\mathscr{r}}\left(2\boldsymbol{\mathscr{r}}\times[\nabla\times\boldsymbol{\mathscr{r}}] + 2[\boldsymbol{\mathscr{r}}\cdot\nabla]\boldsymbol{\mathscr{r}}\right) \quad .$$

using product rule #4

- We'll have to use the chain rule carefully here:

$$(\boldsymbol{\mathscr{r}}\cdot\nabla)\boldsymbol{\mathscr{r}} = (\boldsymbol{\mathscr{r}}\cdot\nabla)\left(\boldsymbol{r}-\boldsymbol{w}[t_r]\right) = \left(\mathscr{r}_x\frac{\partial}{\partial x}+\mathscr{r}_y\frac{\partial}{\partial y}+\mathscr{r}_z\frac{\partial}{\partial z}\right)\left(\boldsymbol{r}-\boldsymbol{w}[t_r]\right)$$

$$= \boldsymbol{\mathscr{r}} - \left(\mathscr{r}_x\frac{\partial t_r}{\partial x}\frac{d}{dt_r}+\mathscr{r}_y\frac{\partial t_r}{\partial y}\frac{d}{dt_r}+\mathscr{r}_z\frac{\partial t_r}{\partial z}\frac{d}{dt_r}\right)\boldsymbol{w}$$

$$= \boldsymbol{\mathscr{r}} - \left(\mathscr{r}_x\frac{\partial t_r}{\partial x}+\mathscr{r}_y\frac{\partial t_r}{\partial y}+\mathscr{r}_z\frac{\partial t_r}{\partial z}\right)\frac{d\boldsymbol{w}}{dt_r} = \boldsymbol{\mathscr{r}} - (\boldsymbol{\mathscr{r}}\cdot\nabla t_r)\boldsymbol{v} \quad .$$

From last time: ∇t_r (continued)

$$
\begin{aligned}
\nabla\times\boldsymbol{\mathscr{r}} &= \nabla\times\boldsymbol{r}+\nabla\times\boldsymbol{w}\\
&= 0+\left(\frac{\partial w_z}{\partial y}-\frac{\partial w_y}{\partial z}\right)\hat{\boldsymbol{x}}+\left(\frac{\partial w_x}{\partial z}-\frac{\partial w_z}{\partial x}\right)\hat{\boldsymbol{y}}+\left(\frac{\partial w_y}{\partial x}-\frac{\partial w_x}{\partial y}\right)\hat{\boldsymbol{z}}\\
&= \left(\frac{\partial w_z}{\partial t_r}\frac{\partial t_r}{\partial y}-\frac{\partial w_y}{\partial t_r}\frac{\partial t_r}{\partial z}\right)\hat{\boldsymbol{x}}+\left(\frac{\partial w_x}{\partial t_r}\frac{\partial t_r}{\partial z}-\frac{\partial w_z}{\partial t_r}\frac{\partial t_r}{\partial x}\right)\hat{\boldsymbol{y}}\\
&\qquad +\left(\frac{\partial w_y}{\partial t_r}\frac{\partial t_r}{\partial x}-\frac{\partial w_x}{\partial t_r}\frac{\partial t_r}{\partial y}\right)\hat{\boldsymbol{z}}\\
&= -\boldsymbol{v}\times\nabla t_r \quad ;
\end{aligned}
$$

$$
\boldsymbol{\mathscr{r}}\times\left(\nabla\times\boldsymbol{\mathscr{r}}\right)=\boldsymbol{\mathscr{r}}\times\left(-\boldsymbol{v}\times\nabla t_r\right)=-\boldsymbol{v}\left(\boldsymbol{\mathscr{r}}\cdot\nabla t_r\right)+\nabla t_r\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v}\right) \quad .
$$

From last time: ∇t_r (continued)

Combine these last two with the formula at the start:

$$\begin{aligned}\nabla t_r &= -\frac{1}{c\mathscr{r}}\left(\boldsymbol{\mathscr{r}}\times[\nabla\times\boldsymbol{\mathscr{r}}]-[\boldsymbol{\mathscr{r}}\cdot\nabla]\boldsymbol{\mathscr{r}}\right)\\ &= -\frac{1}{c\mathscr{r}}\left(\cancel{-\boldsymbol{v}\left(\boldsymbol{\mathscr{r}}\cdot\nabla t_r\right)}+\nabla t_r\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v}\right)-\boldsymbol{\mathscr{r}}+\cancel{\left(\boldsymbol{\mathscr{r}}\cdot\nabla t_r\right)\boldsymbol{v}}\right)\quad .\end{aligned}$$

or

$$\nabla t_r = -\frac{1}{c\mathscr{r}}\left(\boldsymbol{\mathscr{r}}-\nabla t_r\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v}\right)\right)\quad .$$

Solving now for ∇t_r, we get

$$\nabla t_r\left(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v}\right)=-\boldsymbol{\mathscr{r}}\quad ;$$

$$\nabla t_r = -\frac{\boldsymbol{\mathscr{r}}}{c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v}} = -\frac{\boldsymbol{\mathscr{r}}}{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}}\quad .$$

Fields from moving charges (continued)

$$\nabla V = \nabla\left(\frac{qc}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}\right) = -\frac{qc}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\nabla(\boldsymbol{\mathscr{r}}\cdot\mathbf{u}) = -\frac{qc}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\nabla(c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}) \quad .$$

Now, $\nabla t_r = \nabla\left(t - \frac{\mathscr{r}}{c}\right) = -\frac{1}{c}\nabla\mathscr{r} \quad \Rightarrow \quad \nabla\mathscr{r} = -c\nabla t_r$, and

$$\nabla(\boldsymbol{\mathscr{r}}\cdot\mathbf{v}) = (\boldsymbol{\mathscr{r}}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\boldsymbol{\mathscr{r}}) \quad .$$ P.R. #4

This will take a while, but we evaluated terms like these last time:

$$(\boldsymbol{\mathscr{r}}\cdot\nabla)\mathbf{v} = \left(\mathscr{r}_x\frac{\partial}{\partial x} + \mathscr{r}_y\frac{\partial}{\partial y} + \mathscr{r}_z\frac{\partial}{\partial z}\right)\mathbf{v}$$

$$= \left(\mathscr{r}_x\frac{\partial t_r}{\partial x}\frac{d}{dt_r} + \mathscr{r}_y\frac{\partial t_r}{\partial y}\frac{d}{dt_r} + \mathscr{r}_z\frac{\partial t_r}{\partial z}\frac{d}{dt_r}\right)\mathbf{v}$$

Fields from moving charges (continued)

so
$$(\boldsymbol{\mathscr{r}}\cdot\nabla)\boldsymbol{v}=\left(\mathscr{r}_x\frac{\partial t_r}{\partial x}+\mathscr{r}_y\frac{\partial t_r}{\partial y}+\mathscr{r}_z\frac{\partial t_r}{\partial z}\right)\frac{d\boldsymbol{v}}{dt_r}=\left(\boldsymbol{\mathscr{r}}\cdot\nabla t_r\right)\boldsymbol{a}\quad .$$

Similarly,
$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{\mathscr{r}}=(\boldsymbol{v}\cdot\nabla)\left(\boldsymbol{r}-\boldsymbol{w}\left[t_r\right]\right)=\left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right)\left(\boldsymbol{r}-\boldsymbol{w}\left[t_r\right]\right)$$
$$=\boldsymbol{v}-\left(v_x\frac{\partial t_r}{\partial x}\frac{d}{dt_r}+v_y\frac{\partial t_r}{\partial y}\frac{d}{dt_r}+v_z\frac{\partial t_r}{\partial z}\frac{d}{dt_r}\right)\boldsymbol{w}$$
$$=\boldsymbol{v}-\left(v_x\frac{\partial t_r}{\partial x}+v_y\frac{\partial t_r}{\partial y}+v_z\frac{\partial t_r}{\partial z}\right)\frac{d\boldsymbol{w}}{dt_r}=\boldsymbol{v}-\left(\boldsymbol{v}\cdot\nabla t_r\right)\boldsymbol{v}\quad .$$

Fields from moving charges (continued)

We showed last time that

$$\nabla \times \boldsymbol{\mathscr{r}} = -\boldsymbol{v} \times \nabla t_r \quad \text{, so, similarly,}$$

$$\nabla \times \boldsymbol{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\boldsymbol{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\boldsymbol{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\boldsymbol{z}}$$

$$= \left(\frac{\partial v_z}{\partial t_r} \frac{\partial t_r}{\partial y} - \frac{\partial v_y}{\partial t_r} \frac{\partial t_r}{\partial z} \right) \hat{\boldsymbol{x}} + \left(\frac{\partial v_x}{\partial t_r} \frac{\partial t_r}{\partial z} - \frac{\partial v_z}{\partial t_r} \frac{\partial t_r}{\partial x} \right) \hat{\boldsymbol{y}}$$

$$+ \left(\frac{\partial v_y}{\partial t_r} \frac{\partial t_r}{\partial x} - \frac{\partial v_x}{\partial t_r} \frac{\partial t_r}{\partial y} \right) \hat{\boldsymbol{z}}$$

$$= -\boldsymbol{a} \times \nabla t_r \quad .$$

Fields from moving charges (continued)

Thus,

$$
\begin{aligned}
\nabla\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v}\right) &= \left(\boldsymbol{\mathscr{r}}\cdot\nabla\right)\boldsymbol{v}+\left(\boldsymbol{v}\cdot\nabla\right)\boldsymbol{\mathscr{r}}+\boldsymbol{\mathscr{r}}\times\left(\nabla\times\boldsymbol{v}\right)+\boldsymbol{v}\times\left(\nabla\times\boldsymbol{\mathscr{r}}\right)\\
&= \left(\boldsymbol{\mathscr{r}}\cdot\nabla t_r\right)\boldsymbol{a}+\boldsymbol{v}-\left(\boldsymbol{v}\cdot\nabla t_r\right)\boldsymbol{v}-\boldsymbol{\mathscr{r}}\times\left(\boldsymbol{a}\times\nabla t_r\right)-\boldsymbol{v}\times\left(\boldsymbol{v}\times\nabla t_r\right)\\
&= \left(\boldsymbol{\mathscr{r}}\cdot\nabla t_r\right)\boldsymbol{a}+\boldsymbol{v}-\left(\boldsymbol{v}\cdot\nabla t_r\right)\boldsymbol{v}-\boldsymbol{a}\left(\boldsymbol{\mathscr{r}}\cdot\nabla t_r\right)+\nabla t_r\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{a}\right)\\
&\qquad\qquad +\boldsymbol{v}\left(\boldsymbol{v}\cdot\nabla t_r\right)-\nabla t_r\left(\boldsymbol{v}\cdot\boldsymbol{v}\right)\\
&= \boldsymbol{v}+\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{a}-v^2\right)\nabla t_r \quad \text{, and}
\end{aligned}
$$

$$
\begin{aligned}
\nabla V &= -\frac{qc}{\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)^2}\left[-c^2\nabla t_r-\boldsymbol{v}-\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{a}-v^2\right)\nabla t_r\right]\\
&= \frac{qc}{\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)^3}\left[\boldsymbol{v}\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)+\left(c^2+\boldsymbol{\mathscr{r}}\cdot\boldsymbol{a}-v^2\right)\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)\nabla t_r\right] \quad .
\end{aligned}
$$

Fields from moving charges (continued)

But we showed last time that $\nabla t_r = -\dfrac{\boldsymbol{\mathscr{r}}}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}$, so

$$\nabla V = \frac{qc}{\left(\boldsymbol{\mathscr{r}}\cdot\mathbf{u}\right)^3}\left[\mathbf{v}\left(\boldsymbol{\mathscr{r}}\cdot\mathbf{u}\right) - \left(c^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a} - v^2\right)\boldsymbol{\mathscr{r}}\right] \quad .$$

Now for the vector-potential part:

$$\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\mathbf{v}V\right) = \frac{1}{c^2}\left(V\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial V}{\partial t}\right) = \frac{1}{c^2}\left(V\frac{\partial \mathbf{v}}{\partial t_r} + \mathbf{v}\frac{\partial V}{\partial t_r}\right)\frac{\partial t_r}{\partial t}$$

$$= \frac{1}{c^2}\left(V\mathbf{a} + \mathbf{v}\frac{\partial}{\partial t_r}\left[\frac{qc}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}\right]\right)\frac{\partial t_r}{\partial t} = \frac{1}{c^2}\left(V\mathbf{a} - \frac{qc\mathbf{v}}{\left(\boldsymbol{\mathscr{r}}\cdot\mathbf{u}\right)^2}\frac{\partial}{\partial t_r}\left(\boldsymbol{\mathscr{r}}\cdot\mathbf{u}\right)\right)\frac{\partial t_r}{\partial t}$$

$$= \frac{1}{c^2}\left(V\mathbf{a} - \frac{qc\mathbf{v}}{\left(\boldsymbol{\mathscr{r}}\cdot\mathbf{u}\right)^2}\frac{\partial}{\partial t_r}\left(c\mathscr{r} - \mathbf{v}\cdot\boldsymbol{\mathscr{r}}\right)\right)\frac{\partial t_r}{\partial t}$$

Fields from moving charges (continued)

$$\begin{aligned}\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} &= \frac{1}{c^2}\left[V\boldsymbol{a} - \frac{qc\boldsymbol{v}}{\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)^2}\left(c\frac{\partial \mathscr{r}}{\partial t_r} - \boldsymbol{a}\cdot\boldsymbol{\mathscr{r}} - \boldsymbol{v}\cdot\frac{\partial \boldsymbol{\mathscr{r}}}{\partial t_r}\right)\right]\frac{\partial t_r}{\partial t} \\
&= \frac{1}{c}\left[\frac{qc}{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}}\boldsymbol{a} + \frac{qc\boldsymbol{v}}{\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)^2}\left(\frac{c}{\mathscr{r}}\boldsymbol{\mathscr{r}}\cdot\boldsymbol{v} + \boldsymbol{a}\cdot\boldsymbol{\mathscr{r}} - v^2\right)\right]\frac{\mathscr{r}}{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}} \\
&= \frac{qc}{\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)^3}\left[\frac{\mathscr{r}}{c}\boldsymbol{a}\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right) + \frac{\mathscr{r}}{c}\boldsymbol{v}\left(\frac{c}{\mathscr{r}}\left\{c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right\} + \boldsymbol{a}\cdot\boldsymbol{\mathscr{r}} - v^2\right)\right] \\
&= \frac{qc}{\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)^3}\left[\frac{\mathscr{r}}{c}\boldsymbol{a}\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right) + \frac{\mathscr{r}}{c}\boldsymbol{v}\left(c^2 - v^2 - \frac{c}{\mathscr{r}}\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u} + \boldsymbol{a}\cdot\boldsymbol{\mathscr{r}}\right)\right] \quad .\end{aligned}$$

Thus – finally – we get:

Fields from moving charges (continued)

$$\boldsymbol{E} = -\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = -\frac{qc}{\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)^3}\left[\boldsymbol{v}\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right) - \left(c^2 + \boldsymbol{\mathscr{r}}\cdot\boldsymbol{a} - v^2\right)\boldsymbol{\mathscr{r}}\right]$$

$$-\frac{qc}{\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)^3}\left[\frac{\mathscr{r}}{c}\boldsymbol{a}\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right) + \frac{\mathscr{r}}{c}\boldsymbol{v}\left(c^2 - v^2 - \frac{c}{\mathscr{r}}\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u} + \boldsymbol{a}\cdot\boldsymbol{\mathscr{r}}\right)\right]$$

$$= -\frac{qc}{\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)^3}\left[\boldsymbol{v}\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right) + \left(c^2 + \boldsymbol{\mathscr{r}}\cdot\boldsymbol{a} - v^2\right)\left(-\boldsymbol{\mathscr{r}} + \frac{\mathscr{r}}{c}\boldsymbol{v}\right)\right.$$

$$\left. + \frac{\mathscr{r}}{c}\boldsymbol{a}\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right) - \frac{\mathscr{r}}{c}\boldsymbol{v}\frac{c}{\mathscr{r}}\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right]$$

$$= -\frac{qc}{\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)^3}\left[\left(c^2 + \boldsymbol{\mathscr{r}}\cdot\boldsymbol{a} - v^2\right)\left(-\frac{\mathscr{r}}{c}\boldsymbol{u}\right) + \frac{\mathscr{r}}{c}\boldsymbol{a}\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)\right] \quad .$$

Fields from moving charges (continued)

$$\boldsymbol{E} = -\frac{qc}{\left(\boldsymbol{\mathfrak{r}}\cdot\boldsymbol{u}\right)^3}\left[-\left(c^2 - v^2\right)\frac{\boldsymbol{\mathfrak{r}}}{c}\boldsymbol{u} + \frac{\boldsymbol{\mathfrak{r}}}{c}\boxed{\left\{\boldsymbol{a}\left(\boldsymbol{\mathfrak{r}}\cdot\boldsymbol{u}\right) - \boldsymbol{u}\left(\boldsymbol{\mathfrak{r}}\cdot\boldsymbol{a}\right)\right\}}\right]$$

$$\boxed{= \frac{q\boldsymbol{\mathfrak{r}}}{\left(\boldsymbol{\mathfrak{r}}\cdot\boldsymbol{u}\right)^3}\left[\left(c^2 - v^2\right)\boldsymbol{u} + \boldsymbol{\mathfrak{r}}\times\left(\boldsymbol{u}\times\boldsymbol{a}\right)\right]} \quad .$$

$$\boxed{= \boldsymbol{\mathfrak{r}}\times\left(\boldsymbol{a}\times\boldsymbol{u}\right)}$$

Similarly, but avoiding the tedium,

$$\boxed{\boldsymbol{B} = \nabla\times\boldsymbol{A} = \hat{\boldsymbol{\mathfrak{r}}}\times\boldsymbol{E} \quad .}$$

There is a special significance to each of the two terms in ***E***.

The generalized Coulomb field

The first term is

$$\boxed{\boldsymbol{E}_{GC} = \frac{q\boldsymbol{\mathscr{r}}}{\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)^3}\left(c^2 - v^2\right)\boldsymbol{u} \quad .}$$

This field is proportional to $1/r^2$,and its direction is the same as that of $\boldsymbol{u} = c\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v}$. Thus it is similar in some ways to the field for a static point charge. In fact, if we let $v = a = 0$, this term gives us

$$\boldsymbol{E}_{GC} = \frac{q\boldsymbol{\mathscr{r}}}{\left(\boldsymbol{\mathscr{r}}\cdot\left[c\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v}\right]\right)^3}\left(c^2 - v^2\right)\left(c\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v}\right) \rightarrow \frac{q\boldsymbol{\mathscr{r}}}{\left(\boldsymbol{\mathscr{r}}\cdot c\hat{\boldsymbol{\mathscr{r}}}\right)^3}c^3\hat{\boldsymbol{\mathscr{r}}} = \frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} \quad ,$$

$$\boldsymbol{B}_{GC} = \hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{E}_{GC} = 0 \quad ,$$

just as in statics; hence the name.

The radiation field

The other term,

$$\boxed{\mathbf{E}_{\mathrm{rad}} = \frac{q\boldsymbol{\mathscr{r}}}{\left(\boldsymbol{\mathscr{r}}\cdot\mathbf{u}\right)^3}\boldsymbol{\mathscr{r}}\times\left(\mathbf{u}\times\mathbf{a}\right) \quad ,}$$

is only proportional to $1/r$. Thus, as we've seen before, in the case of dipole radiation in the far field, this term is much larger than the other one at large *r*.

- The radiation field also points perpendicular to $\hat{\boldsymbol{\mathscr{r}}}$, as befits a transverse spherical wave: $\hat{\boldsymbol{\mathscr{r}}}\cdot\left[\boldsymbol{\mathscr{r}}\times\left(\mathbf{u}\times\mathbf{a}\right)\right]=0.$
- Note also the presence of $\mathbf{a}$: again it is shown that an electric charge needs to accelerate in order to radiate.

Example: power radiated by accelerating charges

As just noted, the power radiated to large distances is dominated by the radiation field. Let's compute the power radiated by an electric charge q that accelerates, starting from rest at $t_r = 0$:

$$u = c\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v} \cong c\hat{\boldsymbol{\mathscr{r}}} \quad .$$

(Actually this is a good approximation for all speeds v << c.) Then,

$$\boldsymbol{E}_{\mathrm{rad}}\left(t_r = 0\right) = \frac{q\mathscr{r}}{\left(\boldsymbol{\mathscr{r}} \cdot c\hat{\boldsymbol{\mathscr{r}}}\right)^3} \boldsymbol{\mathscr{r}} \times \left(c\hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{a}\right) = \frac{q}{\mathscr{r}c^2}\left[\hat{\boldsymbol{\mathscr{r}}}\left(\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{a}\right) - \boldsymbol{a}\right] \quad ,$$

and

$$\boldsymbol{S}\left(t_r = 0\right) = \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{B} = \frac{c}{4\pi}\boldsymbol{E}_{\mathrm{rad}} \times \left(\hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{E}_{\mathrm{rad}}\right)$$

$$= \frac{c}{4\pi}\left[\hat{\boldsymbol{\mathscr{r}}}E_{\mathrm{rad}}^2 - \boldsymbol{E}_{\mathrm{rad}}\left(\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{E}_{\mathrm{rad}}\right)\right] = \frac{cE_{\mathrm{rad}}^2}{4\pi}\hat{\boldsymbol{\mathscr{r}}} \quad .$$

Power radiated by accelerating charges (continued)

$$\mathbf{S} = \hat{\boldsymbol{\imath}}\frac{c}{4\pi}\mathbf{E}_{\text{rad}}\cdot\mathbf{E}_{\text{rad}} = \hat{\boldsymbol{\imath}}\frac{c}{4\pi}\frac{q^2}{\imath^2c^4}\left[\hat{\boldsymbol{\imath}}\left(\hat{\boldsymbol{\imath}}\cdot\mathbf{a}\right)-\mathbf{a}\right]^2$$

$$= \hat{\boldsymbol{\imath}}\frac{c}{4\pi}\frac{q^2}{\imath^2c^4}\left[a^2+\left(\hat{\boldsymbol{\imath}}\cdot\mathbf{a}\right)^2-2\mathbf{a}\cdot\hat{\boldsymbol{\imath}}\left(\hat{\boldsymbol{\imath}}\cdot\mathbf{a}\right)\right]$$

$$= \hat{\boldsymbol{\imath}}\frac{c}{4\pi}\frac{q^2}{\imath^2c^4}\left[a^2-\left(\hat{\boldsymbol{\imath}}\cdot\mathbf{a}\right)^2\right] = \hat{\boldsymbol{\imath}}\frac{c}{4\pi}\frac{q^2a^2}{\imath^2c^4}\left(1-\cos^2\theta\right)$$

$$= \boxed{\frac{q^2a^2}{4\pi c^3}\frac{\sin^2\theta}{\imath^2}\hat{\boldsymbol{\imath}}} \quad ,$$

where θ is the angle between the acceleration and the direction to the observing point, $\mathbf{r}$ (that is, the angle of $\hat{\boldsymbol{\imath}}$).

Power radiated by accelerating charges (continued)

The $\sin^2\theta$ factor indicates that the charge radiates no power in the forward or backward direction, and radiates most of its power perpendicular to the direction of its acceleration.

- This should remind you, again, of electric dipole radiation.

The power radiated through any sphere centered on the charge is familiar:

$$P = \oint \boldsymbol{S}\cdot d\boldsymbol{\sigma} = \frac{q^2a^2}{4\pi c^3}\int \frac{\sin^2\theta}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\cdot\hat{\boldsymbol{\mathscr{r}}}\,\mathscr{r}^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{q^2a^2}{4\pi c^3}\int_0^{\pi}\sin^3\theta \int_0^{2\pi} d\phi = \frac{q^2a^2}{4\pi c^3}\frac{4}{3}2\pi = \boxed{\frac{2}{3}\frac{q^2a^2}{c^3}}\ .$$

Larmor formula again