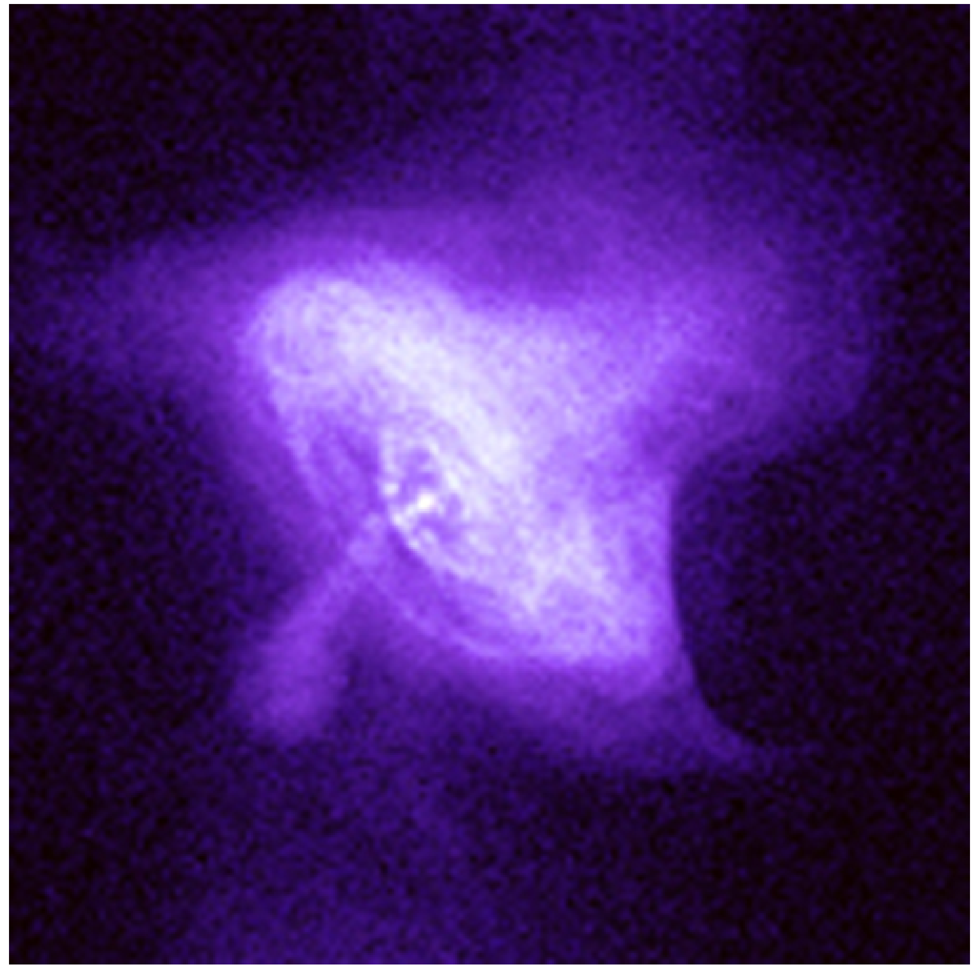
Today in Physics 218: forces in relativity

- Newton's laws in relativity
- The Minkowski force
- Relativistic transformation of forces



X-ray image of the pulsar-driven center of the Crab Nebula in Taurus, the remnant of the supernova of 1054 AD (Chandra X-ray observatory image, NASA and Center for Astrophysics).

Newton's laws in relativity

We will be interested in learning how to solve force problems in relativity, because force is ultimately how we relate to the fields. Can we still use Newton's laws? (Are those among the laws of physics that are valid in inertial reference frames?)

- **First law**: inertia. ("A body in uniform motion will remain…") Clearly this is still true in relativity; otherwise we would have had more trouble last time with momentum and energy conservation.
- **Second law**: $\mathbf{F} = m\mathbf{a}$. This is still true if one takes m to be the rest mass, and expresses it as

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{p}}{dt} \quad ,$$

Newton's laws in relativity (continued)

and uses the relativistic momentum thenceforth:

$$\boldsymbol{p} = \frac{m\boldsymbol{u}}{\sqrt{1-u^2/c^2}}$$

- The easiest way to demonstrate this is to note that mechanical work still increases mechanical energy in relativity, just as it always has:

$$W = \int \boldsymbol{F} \cdot d\ell = \int \frac{d\boldsymbol{p}}{dt} \cdot d\ell = \int \frac{d\boldsymbol{p}}{dt} \cdot \frac{d\ell}{dt} dt = \int \frac{d\boldsymbol{p}}{dt} \cdot \boldsymbol{u} dt$$

$$= \int \frac{d}{dt}\left(\frac{m\boldsymbol{u}}{\sqrt{1-u^2/c^2}} \right) \cdot \boldsymbol{u} dt$$

Newton's laws in relativity (continued)

$$W=\int\left(\frac{m\boldsymbol{u}}{\left(1-u^2/c^2\right)^{3/2}}\frac{\boldsymbol{u}}{c^2}\cdot\frac{d\boldsymbol{u}}{dt}+\frac{m}{\sqrt{1-u^2/c^2}}\frac{d\boldsymbol{u}}{dt}\right)\cdot\boldsymbol{u}\,dt$$

$$=\int\left(\frac{m\,u^2/c^2}{\left(1-u^2/c^2\right)^{3/2}}\boldsymbol{u}+\frac{m\left(1-u^2/c^2\right)}{\left(1-u^2/c^2\right)^{3/2}}\boldsymbol{u}\right)\cdot\frac{d\boldsymbol{u}}{dt}dt$$

$$=\int\left(\frac{m}{\left(1-u^2/c^2\right)^{3/2}}\boldsymbol{u}\right)\cdot\frac{d\boldsymbol{u}}{dt}dt=\int\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-u^2/c^2}}\right)dt$$

$$=\int\frac{dE}{dt}dt=E_{\text{final}}-E_{\text{initial}}\quad,\text{q.e.d.}$$

Newton's laws in relativity (continued)

- **Third law**: "for every action and equal and opposite reaction." This clearly doesn't apply in relativity:
  - Suppose two extended objects exert forces $\boldsymbol{F}(t)$ and $-\boldsymbol{F}(t)$ on each other in some reference frame, so that the third law is satisfied at all times t: $\boldsymbol{F}(t)$ and $-\boldsymbol{F}(t)$ are simultaneously applied
  - A observer in a different reference frame would see those forces applied at different times! Since the objects are not at the same spatial point, events simultaneous in the first frame will not appear so in the second frame, so unless the forces are *constant* they will not appear equal and opposite.

This won't surprise those who remember radiation reaction.

The Minkowski (four-)force

Since $\boldsymbol{F} = d\boldsymbol{p}/dt$, the values of a force seen from different inertial reference frames are not related simply by a Lorentz transformation, but instead by a transformation similar to velocity addition.

However, the vector

$$\boldsymbol{K} = \frac{d\boldsymbol{p}}{d\tau} = \frac{dt}{d\tau}\frac{d\boldsymbol{p}}{dt} = \frac{1}{\sqrt{1-u^2/c^2}}\boldsymbol{F}$$

can clearly be part of a four-vector:

$$K^0 = \frac{dp^0}{d\tau} = \frac{d}{d\tau}\frac{E}{c} \quad \Rightarrow \quad K^\mu = \frac{dp^\mu}{d\tau} \quad .$$

Minkowski force

The Minkowski force (continued)

The scalar product of K with itself is therefore Lorentz-invariant (Griffiths problem 12.39):

$$K_\mu K^\mu = -\left(K^0\right)^2 + \boldsymbol{K}\cdot\boldsymbol{K} = -\left(\frac{d}{d\tau}\frac{E}{c}\right)^2 + \frac{F^2}{1-u^2/c^2} \quad .$$

$$\frac{d}{d\tau}\frac{E}{c} = \frac{1}{c}\frac{dt}{d\tau}\frac{dE}{dt} = \frac{1}{c}\frac{1}{\sqrt{1-u^2/c^2}}\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-u^2/c^2}}\right)$$

$$= \frac{1}{c}\frac{1}{\sqrt{1-u^2/c^2}}\frac{mc^2}{\left(1-u^2/c^2\right)^{3/2}}\frac{u}{c^2}\frac{du}{dt} = \frac{m}{c}\frac{1}{\left(1-u^2/c^2\right)^2}\boldsymbol{u}\cdot\boldsymbol{a}$$

The Minkowski force (continued)

Compare this to

$$\boldsymbol{u}\cdot\boldsymbol{F} = \boldsymbol{u}\cdot\frac{d\boldsymbol{p}}{dt} = \boldsymbol{u}\cdot\frac{d}{dt}\left(\frac{m\boldsymbol{u}}{\sqrt{1-u^2/c^2}}\right) \quad .$$

We worked this out in the middle of an integral a few pages ago:

$$\boldsymbol{u}\cdot\boldsymbol{F} = \frac{m}{\left(1-u^2/c^2\right)^{3/2}}\boldsymbol{u}\cdot\frac{d\boldsymbol{u}}{dt} = \frac{m\left(\boldsymbol{u}\cdot\boldsymbol{a}\right)}{\left(1-u^2/c^2\right)^{3/2}} = c\sqrt{1-u^2/c^2}\,K^0$$

$uF\cos\theta = \quad ;$

$$K^0 = \frac{uF\cos\theta}{c\sqrt{1-u^2/c^2}} \quad .$$

The Minkowski force (continued)

Thus

$$K_\mu K^\mu = -\left(\frac{uF\cos\theta}{c\sqrt{1-u^2/c^2}}\right)^2 + \frac{F^2}{1-u^2/c^2}$$

$$= \frac{1-\left(u^2/c^2\right)\cos^2\theta}{1-u^2/c^2}F^2 \quad .$$

Utility: if ***F*** is measured at rest, an observer in a moving frame will measure

$$\frac{1-\left(u^2/c^2\right)\cos^2\bar{\theta}}{1-u^2/c^2}\bar{F}^2 = F^2 \quad .$$

The Minkowski force (continued)

Can we use the Minkowski force to cast Newton's second law? Yes, as it turns out. (Griffiths problem 12.38)

- First define the four-acceleration in terms of the four-velocity:

$$\alpha^{\mu} = d\eta^{\mu} \big/ d\tau = d^2 x^{\mu} \big/ d\tau^2 \quad .$$

- In these terms, the second law is

$$K^{\mu} = \frac{dp^{\mu}}{d\tau} = m\frac{d\eta^{\mu}}{d\tau} = m\alpha^{\mu} \quad .$$

This bears an odd relationship to the four-velocity itself, as we can see from the various Lorentz invariants we can construct:

The Minkowski force (continued)

- The inner product of the four-velocity with itself turns out to be constant:

$$\eta_\mu \eta^\mu = -\left(\eta^0\right)^2 + \boldsymbol{\eta}\cdot\boldsymbol{\eta} = -\frac{c^2}{1-u^2/c^2} + \frac{u^2}{1-u^2/c^2} = -c^2 \quad .$$

- And this turns out to mean that the four-velocity and four-acceleration are "orthogonal:"

$$\frac{d}{d\tau}\left(\eta_\mu \eta^\mu\right) = \alpha_\mu \eta^\mu + \eta_\mu \alpha^\mu = 2\eta_\mu \alpha^\mu = \frac{d}{d\tau}\left(-c^2\right) = 0 \quad ;$$

$$\eta_\mu \alpha^\mu = 0 \quad .$$

- Similarly, $K^\mu \eta_\mu = 0$.

Relativistic transformation of forces

Alas, real forces are not like the Minkowski force; we still need to derive their transformations. To wit:

- For finite intervals of momentum and time, seen from two inertial reference frames in relative motion along the x axis,

$$\begin{aligned}
\Delta\overline{p}_x &= \gamma\left(\Delta p_x - \beta\frac{E}{c}\right) \\
\Delta\overline{p}_y &= \Delta p_y \\
\Delta\overline{p}_z &= \Delta p_z \\
\Delta\overline{t} &= \gamma\left(\Delta t - \beta\frac{\Delta x}{c}\right) \quad .
\end{aligned}$$

Relativistic transformation of forces

- Thus we can, in the limit, get a component of the force:

$$\overline{F}_x = \lim_{\Delta p, \Delta t \to 0} \frac{\Delta \overline{p}_x}{\Delta \overline{t}} = \lim_{\Delta p, \Delta t \to 0} \frac{\gamma\left(\Delta p_x - \beta \dfrac{E}{c}\right)}{\gamma\left(\Delta t - \beta \dfrac{\Delta x}{c}\right)}$$

Note that

$$\Delta x = \frac{1}{2} a \Delta t^2 = \frac{1}{2m} \frac{dp_x}{dt} \Delta t^2 \quad \text{and} \quad \Delta E = \frac{1}{2m}\left(\frac{dp_x}{dt}\Delta t\right)^2$$

are both second order in $\Delta t,$ so in the limit the second term in both numerator and denominator are small compared to the first.

Relativistic transformation of forces

- Thus,

$$\bar{F}_x \cong \lim_{\Delta p,\Delta t \to 0} \frac{\gamma \Delta p_x}{\gamma \Delta t} = \frac{dp_x}{dt} = F_x \quad .$$

- By the same token

$$\bar{F}_y = \lim_{\Delta p,\Delta t \to 0} \frac{\Delta \bar{p}_y}{\Delta \bar{t}} = \lim_{\Delta p,\Delta t \to 0} \frac{\Delta p_y}{\gamma \left(\Delta t - \beta \dfrac{\Delta x}{c} \right)}$$

$$= \frac{1}{\gamma} \frac{dp_y}{dt} = \frac{F_y}{\gamma} \quad ,$$

$$\bar{F}_z = \frac{F_z}{\gamma} \quad \text{similarly.}$$

Relativistic transformation of forces

- More compactly,

$$\bar{\boldsymbol{F}}_{\|} = \boldsymbol{F}_{\|} \quad , \quad \bar{F}_{\perp} = \frac{1}{\gamma} F_{\perp} \quad ,$$

where $\|$ and $\perp$ means parallel and perpendicular to the direction of relative motion between the two different inertial frames.