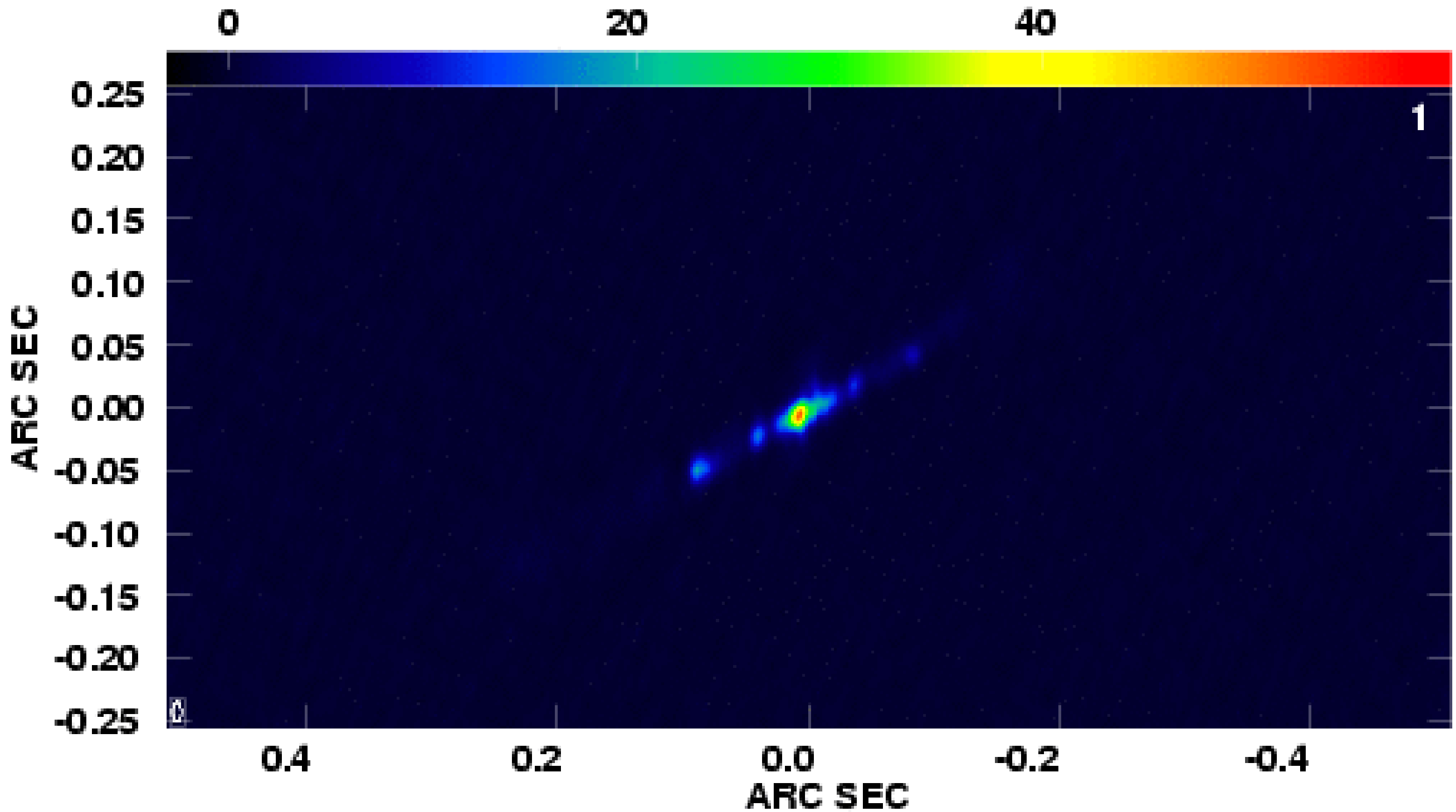
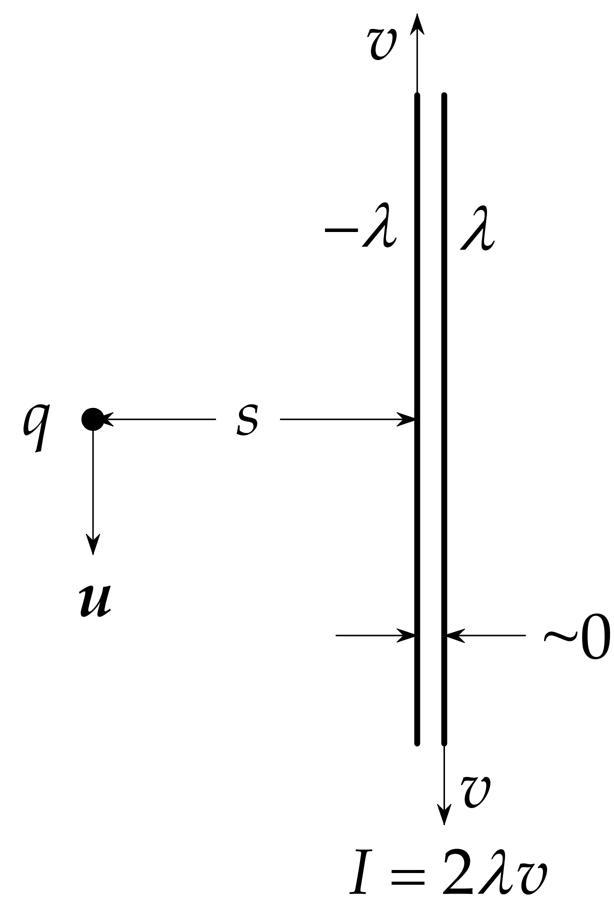
Today in Physics 218: relativistic transformation of electromagnetic forces and fields



42-day movie of VLBA observations of radio emission at 1.5 GHz from the star - black-hole binary system SS433 (Mike Rupen, NRAO). Material is ejected at a speed of about 0.26c, in the form of back-to-back precessing jets.

A single interaction viewed from two frames

- Consider a long straight current I, composed of two long line charges an infinitesimal distance apart, (charge per unit length $\pm\lambda$) travelling in opposite directions at speed v $\left(\text{so } I = 2\lambda v\right)$.
- Consider also a point charge q traveling at velocity $\boldsymbol{u}$ parallel to the current.
- What is the nature of the force on q, in our reference frame and in the charge's rest frame?



View from S.

A single interaction viewed from two frames (continued)

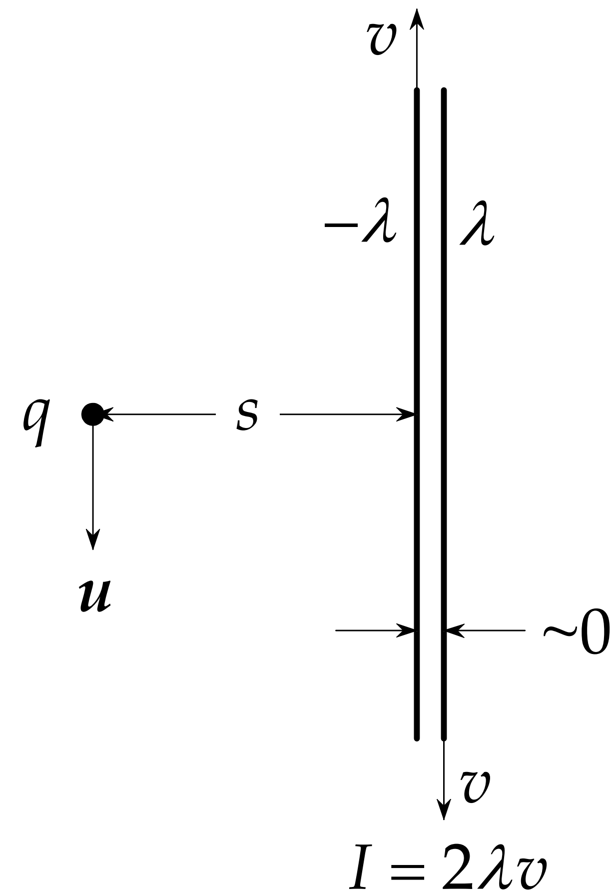
- In our frame (call it S), the wire is neutral, there's a static magnetic field, and therefore a magnetic force on the moving charge:

$$\oint \boldsymbol{B}\cdot d\ell = 2\pi s B = \frac{4\pi}{c} I$$

$$\Rightarrow \quad \boldsymbol{B} = \frac{2I}{cs}\hat{\boldsymbol{\phi}} = \frac{4\lambda v}{cs}\hat{\boldsymbol{\phi}}$$

$$\boldsymbol{F}_S = q\frac{\boldsymbol{u}}{c}\times\boldsymbol{B} = $$

$$= q\frac{u}{c}\frac{4\lambda v}{cs}\hat{\boldsymbol{z}}\times\hat{\boldsymbol{\phi}} = -\frac{4q\lambda uv}{c^2 s}\hat{\boldsymbol{s}} \quad .$$



View from S.

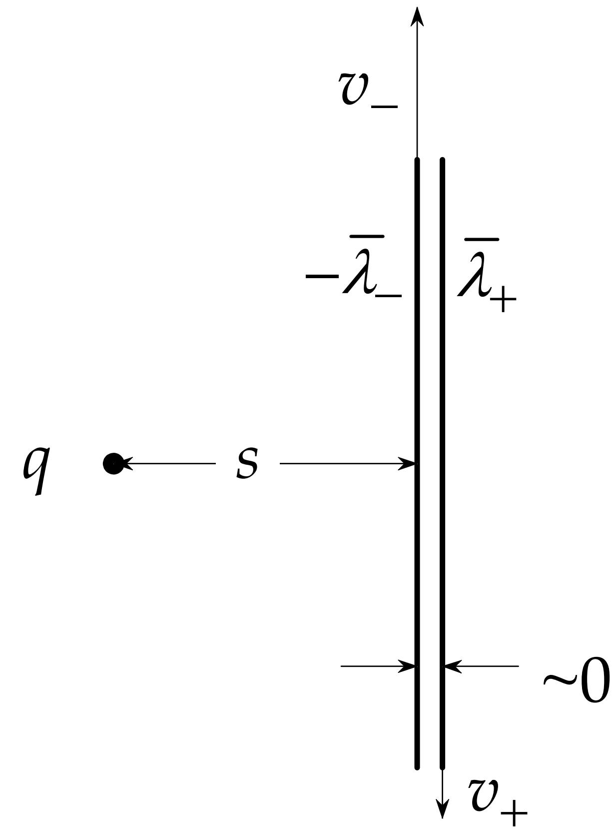
A single interaction viewed from two frames (continued)

What does this look like in the charge's rest frame (call it $\overline{S}$)?

- There's no magnetic force, because the charge is at rest.
- There is, however, an *electrostatic* force, because the two line charges are moving at speeds given by the velocity addition rule:

$$v_{\pm} = \frac{v \mp u}{1 \mp vu/c^2} \quad ,$$

they're Lorentz-contracted differently, and they no longer look equal and opposite.



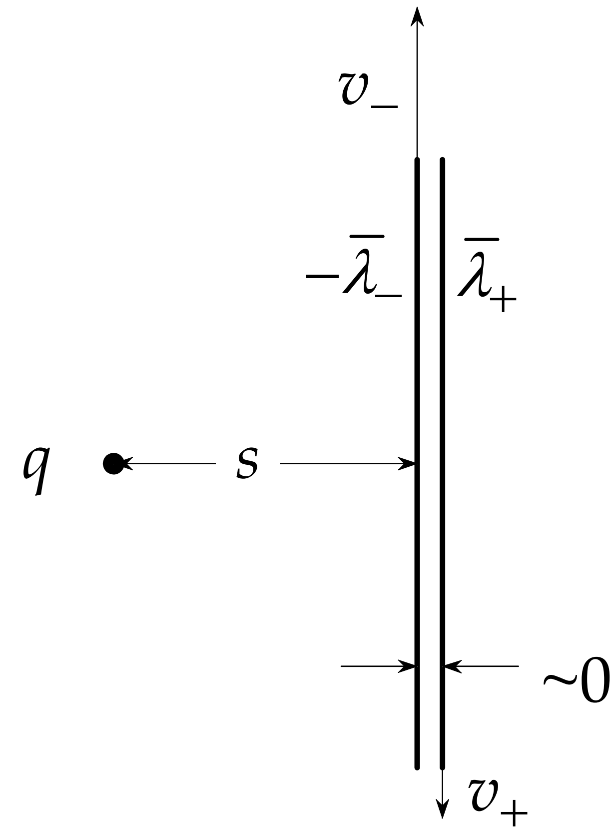
View from $\overline{S}$

A single interaction viewed from two frames (continued)

- Note that $|v_-| > |v_+|$. Thus the "-" charge Lorentz-contracts more than the "+" one, resulting in an attractive electrostatic force if the "+" charge and q have the same sign.
- In their own rest frames, the line charges have charge per unit length given by

$$\pm\lambda_0 = \pm\frac{\lambda}{\gamma} = \pm\lambda\sqrt{1-\frac{v^2}{c^2}}$$

(Lorentz-contracted in frame S, by equal amounts).



View from S'.

A single interaction viewed from two frames (continued)

- Thus in the charge's rest frame, $\bar{S}$, the charge densities are

$$\bar{\lambda}_{\pm} = \pm\gamma_{\pm}\lambda_0 \quad ,$$

where

$$\gamma_{\pm} = \frac{1}{\sqrt{1 - v_{\pm}^2/c^2}} = \frac{1}{\sqrt{1 - \frac{1}{c^2}\left(\frac{v \mp u}{1 \mp vu/c^2}\right)^2}} = \frac{1}{\sqrt{1 - c^2\left(\frac{v \mp u}{c^2 \mp vu}\right)^2}}$$

$$= \frac{c^2 \mp vu}{\sqrt{\left(c^2 \mp vu\right)^2 - c^2\left(v \mp u\right)^2}}$$

A single interaction viewed from two frames (continued)

$$
\begin{aligned}
\gamma_{\pm} &= \frac{c^2 \mp vu}{\sqrt{c^4 + u^2 v^2 \cancel{\mp 2c^2 vu} - c^2 v^2 - c^2 u^2 \cancel{\pm 2c^2 vu}}} \\
&= \frac{c^2 \mp vu}{\sqrt{\left(c^2 - v^2\right)\left(c^2 - u^2\right)}} = \frac{1 \mp vu/c^2}{\sqrt{\left(1 - v^2/c^2\right)\left(1 - u^2/c^2\right)}} \\
&= \gamma \frac{1 \mp vu/c^2}{\sqrt{1 - u^2/c^2}} \quad .
\end{aligned}
$$

- ❑ From this we can compute the total charge in the moving line charges, in the point charge’s frame:

A single interaction viewed from two frames (continued)

$$\bar{\lambda}_{\text{total}} = \bar{\lambda}_{+} + \bar{\lambda}_{-} = \lambda_0 \left(\gamma_{+} - \gamma_{-}\right)$$

$$= \lambda_0 \gamma \left(\frac{1 - uv/c^2}{\sqrt{1 - u^2/c^2}} - \frac{1 + uv/c^2}{\sqrt{1 - u^2/c^2}} \right) = \lambda \left(\frac{-2uv/c^2}{\sqrt{1 - u^2/c^2}} \right) \quad .$$

- So the static electric field that q sees is

$$\oint \bar{\boldsymbol{E}} \cdot d\boldsymbol{a} = 2\pi s \ell \bar{E} = 4\pi Q_{\text{enclosed}} = 4\pi \bar{\lambda}_{\text{total}} \ell$$

$$\Rightarrow \quad \bar{\boldsymbol{E}} = \frac{2\bar{\lambda}_{\text{total}}}{s} \hat{\boldsymbol{s}} = -\frac{4\lambda uv}{c^2 s \sqrt{1 - u^2/c^2}} \hat{\boldsymbol{s}} \quad ,$$

A single interaction viewed from two frames (continued)

- and a force given by

$$\boldsymbol{F}_{\overline{S}} = -\frac{4q\lambda uv}{c^2 s\sqrt{1-u^2/c^2}}\hat{\boldsymbol{s}} \quad .$$

BUT… last time we showed that forces transform from one inertial frame to another, in relative motion at speed u, as

$$\overline{\boldsymbol{F}}_{\|} = \boldsymbol{F}_{\|} \quad , \quad \overline{F}_{\perp} = \frac{1}{\gamma_u}F_{\perp} = F_{\perp}\sqrt{1-u^2/c^2} \quad ,$$

so we can transform the force seen in frame $\overline{S}$, to see how it would appear in frame S:

A single interaction viewed from two frames (continued)

$$\boldsymbol{F}_S = \frac{1}{\gamma_u}\boldsymbol{F}_{\overline{S}} = -\frac{4q\lambda uv}{c^2 s\sqrt{1-u^2/c^2}}\sqrt{1-u^2/c^2}\,\hat{\boldsymbol{s}}$$

$$= -\frac{4q\lambda uv}{c^2 s}\hat{\boldsymbol{s}} \quad ,$$

exactly the same as the magnetostatic force we computed within frame S several slides ago.

The Lorentz transformation is built into the Maxwell equations

So we see in this example two remarkable facts:

- ❑ The same force can appear as an electrostatic force in one inertial reference frame, and as a magnetostatic force in another one.
- ❑ The Lorentz transformation of forces seems already to be built into the properties of ***E*** and ***B***!
  - Because in this example – in which we started with a purely magnetic force – computation of the force in the new frame from scratch, and transformation of the old force to the new frame, give the same answer.

The Lorentz transformation is built into the Maxwell equations (continued)

- By built-in, we mean that the Maxwell equations *look* the same in all inertial reference frames. Though two observers in frames S and $\overline{S}$ may measure the fields to be $(\boldsymbol{E},\boldsymbol{B})$ and $(\overline{\boldsymbol{E}},\overline{\boldsymbol{B}})$, respectively, they agree that their fields satisfy the Maxwell equations:

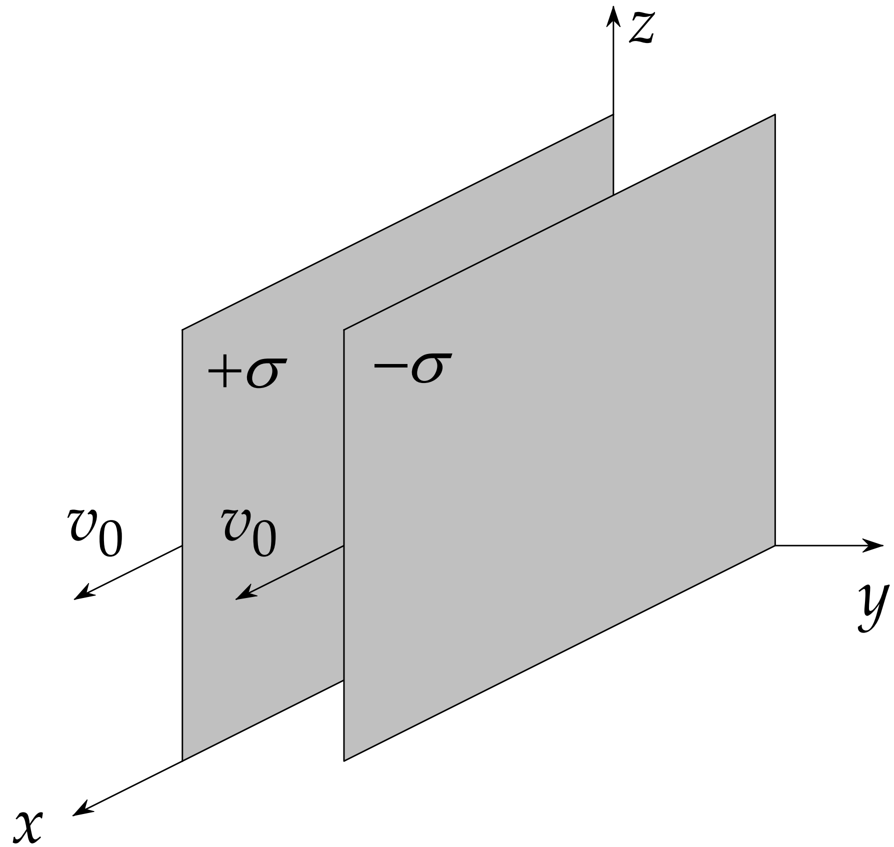
$$\begin{aligned}
\nabla \cdot \boldsymbol{E} &= 4\pi\rho \\
\nabla \cdot \boldsymbol{B} &= 0 \\
\nabla \times \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \\
\nabla \times \boldsymbol{B} &= \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
\overline{\nabla} \cdot \overline{\boldsymbol{E}} &= 4\pi\overline{\rho} \\
\overline{\nabla} \cdot \overline{\boldsymbol{B}} &= 0 \\
\overline{\nabla} \times \overline{\boldsymbol{E}} &= -\frac{1}{c}\frac{\partial \overline{\boldsymbol{B}}}{\partial \overline{t}} \\
\overline{\nabla} \times \overline{\boldsymbol{B}} &= \frac{4\pi}{c}\overline{\boldsymbol{J}} + \frac{1}{c}\frac{\partial \overline{\boldsymbol{E}}}{\partial \overline{t}}
\end{aligned}\ .$$

Transformation of *E* and *B* in special relativity

Consider two infinite charged planes, with charge per unit area $\pm\sigma$, and which move at equal speeds v_0 in the $+x$ direction from our point of view. Let's work out the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields in two different reference frames.

- ❑ In our frame (S), the sheets are equivalent to a surface current (current per unit transverse length):

$$\boldsymbol{K} = \pm\sigma v_0 \hat{\boldsymbol{x}} \quad .$$



Planes infinite; only fragments shown.

Transformation of *E* and *B* in special relativity (continued)

- Thus we see both electric and magnetic fields between the sheets, given by

$$\boldsymbol{E} = 4\pi\sigma\hat{\boldsymbol{y}} = E_y\hat{\boldsymbol{y}}$$

$$\boldsymbol{B} = \frac{4\pi}{c}\mathbf{K}\times\hat{\boldsymbol{n}} = \frac{4\pi\sigma v_0}{c}\hat{\boldsymbol{z}} = B_z\hat{\boldsymbol{z}} \quad ,$$

  as one can easily work out using Gauss's law, Ampère's law, and superposition.
- Now consider an observer in frame $\bar{S}$, moving at speed v along x. According to him/her, the sheet speed is

$$\bar{v}_0 = \frac{v_0 - v}{1 - v_0 v/c^2} = c\,\frac{\beta_0 - \beta}{1 - \beta_0\beta} \quad .$$

Transformation of *E* and *B* in special relativity (continued)

- In the sheets's own rest frame, they have charge per unit area $\pm\sigma/\gamma_0 = \pm\sigma\sqrt{1 - v_0^2/c^2}$, so in frame $\overline{S}$ their charge density looks like

$$\pm\overline{\sigma} = \pm\sigma\frac{\overline{\gamma}_0}{\gamma_0} = \pm\sigma\frac{\sqrt{1 - v_0^2/c^2}}{\sqrt{1 - \overline{v}_0^2/c^2}} = \pm\sigma\frac{\sqrt{1 - \beta_0^2}}{\sqrt{1 - \left(\dfrac{\beta_0 - \beta}{1 - \beta_0\beta}\right)^2}}$$

$$= \pm\sigma\sqrt{\frac{\left(1 - \beta_0^2\right)\left(1 - \beta_0\beta\right)^2}{\left(1 - \beta_0\beta\right)^2 - \left(\beta_0 - \beta\right)^2}}$$

Transformation of *E* and *B* in special relativity (continued)

$$\pm\bar{\sigma} = \pm\sigma\sqrt{\frac{\left(1-\beta_0^2\right)\left(1-\beta_0\beta\right)^2}{1+\beta_0^2\beta^2 - \cancel{2\beta_0\beta} - \beta_0^2 - \beta^2 + \cancel{2\beta_0\beta}}}$$

$$= \pm\sigma\sqrt{\frac{\cancel{\left(1-\beta_0^2\right)}\left(1-\beta_0\beta\right)^2}{\cancel{\left(1-\beta_0^2\right)}\left(1-\beta^2\right)}} = \pm\sigma\gamma\left(1-\beta_0\beta\right) \quad ,$$

where $\gamma = 1\Big/\sqrt{1-\beta^2}$ (the Lorentz factor between us and the observer in frame $\bar{S}$). Thus,

$$\bar{\boldsymbol{K}} = \pm\bar{\sigma}\bar{v}_0\hat{\boldsymbol{x}} = \pm\sigma\gamma\left(1-\beta_0\beta\right)c\,\frac{\beta_0-\beta}{1-\beta_0\beta}\,\hat{\boldsymbol{x}} = \pm\sigma\gamma\left(v_0-v\right)\hat{\boldsymbol{x}} \quad .$$

Transformation of *E* and *B* in special relativity (continued)

- Thus the fields in $\bar{S}$ look like

$$\bar{E}_y = 4\pi\bar{\sigma} = \gamma\left[4\pi\sigma - \left(\frac{4\pi\sigma v_0}{c}\right)\beta\right] = \gamma\left(E_y - \beta B_z\right) \quad ,$$

$$\bar{B}_z = \frac{4\pi\bar{K}_x}{c} = \gamma\left[\frac{4\pi\sigma v_0}{c} - 4\pi\sigma\beta\right] = \gamma\left(B_z - \beta E_z\right) \quad ,$$

in terms of the fields in *S*. This looks like a reasonable start to the general transformation.