Today in Physics 218: review I

You learned a lot this semester, in principle. Here's a laundry-list-like reminder of the first half of it:

- Generally useful things
- Electrodynamics
- Electromagnetic plane wave propagation in a variety of media (linear, conducting, dispersive, guides)

"The Scream," by Edvard Munch (1893).

Generally useful math facts

- Vector and vector-calculus product relation from the inside covers of the book
- Properties of the delta function
- Orthonormality of sines and cosines
- $Ae^{iau} + Be^{ibu} = Ce^{icu}$
  $\Rightarrow A + B = C, \;\; a = b = c$

$$\int_0^{2\pi} \cos mx \cos nx dx = \int_0^{2\pi} \sin mx \sin nx dx = \pi\delta_{mn}$$

$$\int_0^{2\pi} \cos mx \sin nx dx = 0 \quad \text{, so}$$

$$\left\langle \cos^2 \omega t \right\rangle = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \cos^2 \omega t dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos^2 x dx = \frac{1}{2} = \left\langle \sin^2 \omega t \right\rangle$$

Electrodynamics (as opposed to statics or quasistatics)

- Beyond magneto-quasistatics
- Displacement current, and Maxwell's repair of Ampère's Law
- The Maxwell equations
- Symmetry of the equations: magnetic monopoles?

cgs units:

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \qquad \nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$$

MKS units:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

Electrodynamics (continued)

- The Maxwell equations in matter

$$\nabla \cdot \boldsymbol{D} = 4\pi\rho_f \qquad \nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{J}_f + \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t}$$

- Boundary conditions for electrodynamics

$B_\perp$ and $E_\parallel$ are continuous;
$D_\perp$ is discontinuous by $4\pi\sigma_f$;
$H_\parallel$ is discontinuous by $(4\pi/c)\boldsymbol{K}_f \times \hat{\boldsymbol{n}}$.
In linear media $(\boldsymbol{D} = \varepsilon\boldsymbol{E}, \boldsymbol{B} = \mu\boldsymbol{H})$:

$$\varepsilon_{\text{above}}E_{\perp,\text{above}} - \varepsilon_{\text{below}}E_{\perp,\text{below}} = 4\pi\sigma_f$$

$$B_{\perp,\text{above}} - B_{\perp,\text{below}} = 0$$

$$E_{\parallel,\text{above}} - E_{\parallel,\text{below}} = 0$$

$$\frac{1}{\mu_{\text{above}}}B_{\parallel,\text{above}} - \frac{1}{\mu_{\text{below}}}B_{\parallel,\text{below}} = \frac{4\pi}{c}\left|\boldsymbol{K}_f \times \hat{\boldsymbol{n}}\right|$$

Electrodynamics (continued)

- Potentials and fields
- Gauge transformations, especially the Lorentz gauge
- Energy conservation in electrodynamics: Poynting's theorem

$$\boldsymbol{E} = -\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial V}{\partial t} = 0$$

$$\frac{dW_{\text{mech.}}}{dt} = -\frac{c}{4\pi}\oint_{\mathcal{S}}(\boldsymbol{E}\times\boldsymbol{B})\cdot d\boldsymbol{a} - \frac{1}{8\pi}\frac{d}{dt}\int_{\mathcal{V}}\left(B^2 + E^2\right)d\tau$$

$$\frac{\partial}{\partial t}\left(u_{\text{mech.}} + u_{EB}\right) + \nabla \cdot \boldsymbol{S} = 0$$

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B}, \quad u_{EB} = \frac{1}{8\pi}\left(E^2 + B^2\right)$$

Electrodynamics (continued)

- Momentum conservation in electrodynamics and the Maxwell stress tensor

$$T_{ij} = \frac{1}{4\pi}\left(E_iE_j + B_iB_j - \frac{1}{2}\left(E^2 + B^2\right)\delta_{ij}\right)$$

$$\frac{d\boldsymbol{p}_{\text{mech.}}}{dt} = \oint_{\mathcal{S}}\overleftrightarrow{\boldsymbol{T}}\cdot d\boldsymbol{a} - \frac{d}{dt}\int_{\mathcal{V}}\boldsymbol{g}_{EB}\,d\tau$$

$$\frac{\partial}{\partial t}\left(\boldsymbol{g}_{\text{mech.}} + \boldsymbol{g}_{EB}\right) - \nabla \cdot \overleftrightarrow{\boldsymbol{T}} = 0$$

$$\boldsymbol{g}_{EB} \equiv \frac{1}{4\pi c}\boldsymbol{E}\times\boldsymbol{B}$$

$$\mathcal{L}_{EB} = \boldsymbol{r}\times\boldsymbol{g}_{EB} = \frac{1}{4\pi c}\boldsymbol{r}\times\left(\boldsymbol{E}\times\boldsymbol{B}\right)$$

Waves

- Electromagnetic waves $\nabla^2 \boldsymbol{E} = \frac{\mu\varepsilon}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}, \quad \nabla^2 \boldsymbol{B} = \frac{\mu\varepsilon}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2}$
- Waves on a string $\frac{\partial^2 f}{\partial x^2} = \left(\frac{\mu}{T}\right)\frac{\partial^2 f}{\partial t^2}$
- The simple solutions to the wave equation $f = g(x \pm vt) = g(z)$.
- Sinusoidal waves $\tilde{f}(x,t) = \tilde{A}e^{i(kx-\omega t)}, \quad \tilde{A} = Ae^{i\delta}$
- Polarization

Waves (continued)

- Reflection and transmission of waves on a string
- Impedance

$$\left.\begin{aligned} f_I &= \tilde{A}_I e^{i(k_1 z-\omega t)} \\ f_R &= \tilde{A}_R e^{i(-k_1 z-\omega t)} \end{aligned}\right\} \quad z \le 0 : f = f_I + f_R$$

$$\left. f_T = \tilde{A}_T e^{i(k_2 z-\omega t)} \right\} \quad z \ge 0 : f = f_T$$

$$f\left(0^-,t\right) = f\left(0^+,t\right), \quad \frac{\partial f}{\partial z}\left(0^-,t\right) = \frac{\partial f}{\partial z}\left(0^+,t\right)$$

$$\tilde{A}_R = \frac{v_2 - v_1}{v_2 + v_1}\tilde{A}_I = \frac{Z_1 - Z_2}{Z_1 + Z_2}\tilde{A}_I$$

$$\tilde{A}_T = \frac{2v_2}{v_1 + v_2}\tilde{A}_I = \frac{2Z_1}{Z_1 + Z_2}\tilde{A}_I ; \quad Z = T/v = \sqrt{T\mu}$$

Plane electromagnetic waves in linear media

- Plane electro-magnetic waves
- Energy and momentum in plane electro-magnetic waves
- Radiation pressure
- Waves in linear media

$$\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \quad , \quad \tilde{\boldsymbol{B}} = \hat{\boldsymbol{k}} \times \tilde{\boldsymbol{E}}$$

$$u = \frac{E^2}{4\pi} = \frac{B^2}{4\pi} \quad , \quad \boldsymbol{S} = \frac{cE^2}{4\pi}\hat{\boldsymbol{k}} = cu\hat{\boldsymbol{k}}$$

$$\boldsymbol{g} = \frac{E^2}{4\pi c}\hat{\boldsymbol{k}} = \frac{\boldsymbol{S}}{c^2} = \frac{u}{c}\hat{\boldsymbol{k}}$$

$$\boldsymbol{B} = \sqrt{\mu\varepsilon}\,\hat{\boldsymbol{z}} \times \boldsymbol{E}$$

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{H} = \frac{c}{4\pi\mu}\boldsymbol{E} \times \boldsymbol{B}$$

$$u = \frac{1}{8\pi}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}\cdot\boldsymbol{H}) = \frac{1}{8\pi}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right)$$

$$\boldsymbol{g} = \frac{\varepsilon\mu\boldsymbol{S}}{c^2} = \frac{\varepsilon\mu}{4\pi c}\boldsymbol{E} \times \boldsymbol{H} = \frac{\varepsilon}{4\pi c}\boldsymbol{E} \times \boldsymbol{B}$$

Plane electromagnetic waves in linear media (continued)

- The impedance of linear media $$Z = \frac{4\pi}{c}\sqrt{\frac{\mu}{\varepsilon}}$$
- Spacecloth
- Boundary conditions for reflection and transmission of electromagnetic plane waves at interfaces

$$\varepsilon_1 E_{\perp,1} - \varepsilon_2 E_{\perp,2} = 0 \qquad B_{\perp,1} - B_{\perp,2} = 0$$
$$E_{\parallel,1} - E_{\parallel,2} = 0 \qquad \frac{1}{\mu_1} B_{\parallel,1} - \frac{1}{\mu_2} B_{\parallel,2} = 0$$

or

$$\varepsilon_1\left(\tilde{E}_{0Iz} + \tilde{E}_{0Rz}\right) = \varepsilon_2 \tilde{E}_{0Tz} \qquad \tilde{B}_{0Iz} + \tilde{B}_{0Rz} = \tilde{B}_{0Tz}$$
$$\tilde{E}_{0Ix} + \tilde{E}_{0Rx} = \tilde{E}_{0Tx} \qquad \frac{1}{\mu_1}\left(\tilde{B}_{0Ix} + \tilde{B}_{0Rx}\right) = \frac{1}{\mu_2}\tilde{B}_{0Tx}$$
$$\tilde{E}_{0Iy} + \tilde{E}_{0Ry} = \tilde{E}_{0Ty} \qquad \frac{1}{\mu_1}\left(\tilde{B}_{0Iy} + \tilde{B}_{0Ry}\right) = \frac{1}{\mu_2}\tilde{B}_{0Ty}$$

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Plane electromagnetic waves in linear media (continued)

- Snell's Law

$$\theta_I = \theta_R$$
$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{k_I}{k_T} = \frac{v_2}{v_1} = \frac{n_1}{n_2}$$

- The Fresnel equations

$$\mathbf{E} \perp \mathbf{k}_I, \mathbf{k}_R, \mathbf{k}_T: \quad \tilde{E}_{0T} = \frac{2\tilde{E}_{0I}}{1+\alpha\beta} \quad , \quad \tilde{E}_{0R} = \frac{1-\alpha\beta}{1+\alpha\beta}\tilde{E}_{0I} \quad .$$
$$\mathbf{E} \parallel \mathbf{k}_I, \mathbf{k}_R, \mathbf{k}_T: \quad \tilde{E}_{0T} = \frac{2\tilde{E}_{0I}}{\alpha+\beta} \quad , \quad \tilde{E}_{0R} = \frac{\alpha-\beta}{\alpha+\beta}\tilde{E}_{0I} \quad .$$
$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} \cong \frac{1}{\cos\theta_I}\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_I\right)^2}$$
$$\beta = \sqrt{\mu_1\varepsilon_2/\mu_2\varepsilon_1} = Z_1/Z_2 \cong \sqrt{\varepsilon_2/\varepsilon_1} = n_2/n_1$$

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Plane electromagnetic waves in linear media (continued)

- Total internal reflection $$\theta_{IC} > \arcsin\left(\frac{n_2}{n_1}\right)1$$
- Polarization on reflection $$\tan\theta_{IB} = \beta = \frac{n_2}{n_1} \quad .$$
- Interference in layers of linear media $$\lambda_m = \frac{2dn\cos\theta_t}{m} \quad (m = 0,1,2,\ldots)$$
- Transmission and reflection in stratified linear media, viewed as a boundary-value problem

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Plane electromagnetic waves in linear media (continued)

- Matrix formulation of the fields at the interfaces in stratified linear media

$$Y_{1,TE} = \sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta_{T1} = \frac{4\pi}{c}\frac{1}{Z_1}\cos\theta_{T1}$$

$$Y_{1,TM} = \sqrt{\frac{\varepsilon_1}{\mu_1}}\frac{1}{\cos\theta_{T1}}$$

$$\begin{bmatrix}\tilde{E}_{\|,1}\\ \tilde{H}_{\|,1}\end{bmatrix} = \begin{bmatrix}\cos\delta_1 & -i\sin\delta_1/Y_1\\ -iY_1\sin\delta_1 & \cos\delta_1\end{bmatrix}\begin{bmatrix}\tilde{E}_{\|,2}\\ \tilde{H}_{\|,2}\end{bmatrix} \equiv M_1\begin{bmatrix}\tilde{E}_{\|,2}\\ \tilde{H}_{\|,2}\end{bmatrix}$$

$$\begin{bmatrix}\tilde{E}_{\|,1}\\ \tilde{H}_{\|,1}\end{bmatrix} = M_1 M_2\cdots M_p\begin{bmatrix}\tilde{E}_{\|,p+1}\\ \tilde{H}_{\|,p+1}\end{bmatrix}$$

$$M = M_1 M_2\cdots M_p = \begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix} .$$

Plane electromagnetic waves in linear media (continued)

- Characteristic matrix formulation of reflected and transmitted fields and intensity
- Examples:
  - Single interface
  - Plane-parallel dielectric in vacuum
  - Multiple quarter-wave stacks

$$r = \frac{m_{11}Y_0 + m_{12}Y_0Y_{p+1} - m_{21} - m_{22}Y_{p+1}}{m_{11}Y_0 + m_{12}Y_0Y_{p+1} + m_{21} + m_{22}Y_{p+1}}$$

$$t = \frac{2Y_0}{m_{11}Y_0 + m_{12}Y_0Y_{p+1} + m_{21} + m_{22}Y_{p+1}}$$

$$\rho = \frac{\langle S_{R1,\perp}\rangle}{\langle S_{I,\perp}\rangle} = |r|^2$$

$$\tau = \frac{\langle S_{T,p+1,\perp}\rangle}{\langle S_{I,\perp}\rangle} = \frac{Y_{p+1}}{Y_0}|t|^2$$

$$\tau + \rho = 1$$

Plane electromagnetic waves in conductors

- Electromagnetic waves in conductors

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} = \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{4\pi\sigma\mu}{c^2}\frac{\partial \mathbf{E}}{\partial t} .$$

- Attenuation of the waves, and an electronic analogy

$$\tau = \frac{\varepsilon}{2\pi\sigma} = \frac{\rho\varepsilon}{2\pi} .$$

- Penetration of waves into conductors: skin depth

$$d = \frac{1}{\kappa} = \sqrt{\frac{2}{\mu\varepsilon}}\frac{c}{\omega}\left(\left(1 + \left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2\right)^{1/2} - 1\right)^{-1/2}$$

Plane electromagnetic waves in conductors (continued)

- Good and bad conductors

$$\sigma \gg \frac{\varepsilon\omega}{4\pi} \text{ good,} \quad \sigma \ll \frac{\varepsilon\omega}{4\pi} \text{ bad.}$$

- Relative phase of **E** and **B** of waves in conductors

$$\tilde{k} = \frac{\sqrt{2\pi\omega\mu\sigma}}{c}(1+i),\ k = \kappa = \frac{\sqrt{2\pi\omega\mu\sigma}}{c} \quad \text{good,}$$

$$\tilde{k} = k + i\kappa,\ k \cong \sqrt{\mu\varepsilon}\frac{\omega}{c} \gg \kappa,\ \kappa \cong \frac{2\pi\sigma}{c}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\sigma}{2}Z \quad \text{bad.}$$

$$\tilde{B}_0 = \frac{(k+i\kappa)}{\omega}c\tilde{E}_0 = \frac{c\left|\tilde{k}\right|e^{i\phi}}{\omega}\tilde{E}_0$$

$$\langle \mathbf{S} \rangle = \hat{z}\frac{c^2}{8\pi\mu}\frac{k}{\omega}E_0^2 e^{-2\kappa z}$$

Plane electromagnetic waves in conductors (continued)

- Reflection from conducting surfaces

$$\tilde{E}_{0T} = \frac{2}{1+\tilde{\beta}}\tilde{E}_{0I},\ \tilde{E}_{0R} = \frac{1-\tilde{\beta}}{1+\tilde{\beta}}\tilde{E}_{0I}$$

$$\tilde{\beta} = \sqrt{\frac{\mu_1}{\varepsilon_1}}\frac{c\tilde{k}_2}{\mu_2\omega} \xrightarrow[\text{good}]{} \gamma(1+i),\ \gamma = \sqrt{\frac{\mu_1}{\varepsilon_1\mu_2}}\sqrt{\frac{2\pi\sigma}{\omega}}$$

$$\frac{I_R}{I_I} = \frac{1-(2\gamma)+2\gamma^2}{1+(2\gamma)+2\gamma^2}$$

- The characteristic matrix of a conducting layer

$$Y_1 = \frac{1}{\mu_1}\frac{c\tilde{k}_1}{\omega} \quad \text{and} \quad \delta_1 = \tilde{k}_1 d \quad ,$$

$$\tilde{k}_1 = \begin{cases} \dfrac{\sqrt{2\pi\omega\mu_1\sigma_1}}{c}(1+i) & \text{good} \\ \sqrt{\mu_1\varepsilon_1}\dfrac{\omega}{c} + i\dfrac{2\pi\sigma_1}{c}\sqrt{\dfrac{\mu_1}{\varepsilon_1}} & \text{bad} \end{cases}$$

Plane electromagnetic waves in dispersive media

- Motion of bound electrons in matter, and the frequency dependence of the dielectric constant
- Dispersion relations
- Ordinary and anomalous dispersion

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{q}{m_e}E_0 e^{-i\omega t}$$

$$\tilde{\varepsilon} = 1 + \frac{4\pi Nq^2}{m_e}\sum_{j=1}^{M}\frac{f_j}{\left(\omega_j^2 - \omega^2\right) - i\gamma_j\omega}$$

Dilute gas: $I(z) = I_0 e^{-\alpha z}$,

$$n \equiv \frac{c}{\omega}k \cong 1 + \frac{2\pi Nq^2}{m_e}\sum_{j=1}^{M}\frac{f_j\left(\omega_j^2 - \omega^2\right)}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2\omega^2} \quad ,$$

$$\alpha \equiv 2\kappa = \frac{4\pi Nq^2\omega}{m_e c}\sum_{j=1}^{M}\frac{f_j\gamma_j\omega}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2\omega^2} \quad ,$$

$$\tilde{n} = n + i\frac{c}{2\omega}\alpha \quad .$$

Plane electromagnetic waves in dispersive media (continued)

- Semiclassical theory of conductivity
- Conductivity and dispersion in metals and in very dilute conductors
- Light propagation in very dilute conductors: group velocity, plasma frequency

$$\sigma = \frac{Nf_0 q^2}{m_e}\frac{1}{\gamma_0 - i\omega}$$

$$\sigma \cong \frac{Nf_0 q^2}{m_e \gamma_0} \quad \text{metals,} \quad \sigma \cong i\frac{Nf_0 q^2}{m_e \omega} \quad \text{gases.}$$

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) \quad , \quad \omega_p \equiv \sqrt{\frac{4\pi N f_0 q^2}{m_e}}$$

$$v = \frac{\omega}{k} = \frac{c}{n} = \frac{c}{\sqrt{1-\left(\omega_p/\omega\right)^2}} > c$$

$$v_g = \frac{d\omega}{dk} = c\sqrt{1-\omega_p^2/\omega^2} < c$$

- $\frac{\omega}{k} = \frac{d\omega}{dk} = \frac{c}{n} < c$, always, in nondispersive media.

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Guided waves

- Metallic waveguides
- Light propagation in hollow conductive waveguides
- $\tilde{E}_{0z} = 0 \Rightarrow$ TE waves
  $\tilde{B}_{0z} = 0 \Rightarrow$ TM waves

$$\tilde{E}_{0x} = \frac{i}{\frac{\omega^2}{c^2} - k^2}\left(k\frac{\partial \tilde{E}_{0z}}{\partial x} + \frac{\omega}{c}\frac{\partial \tilde{B}_{0z}}{\partial y}\right)$$

$$\tilde{E}_{0y} = \frac{i}{\frac{\omega^2}{c^2} - k^2}\left(k\frac{\partial \tilde{E}_{0z}}{\partial y} - \frac{\omega}{c}\frac{\partial \tilde{B}_{0z}}{\partial x}\right)$$

$$\tilde{B}_{0x} = \frac{i}{\frac{\omega^2}{c^2} - k^2}\left(k\frac{\partial \tilde{B}_{0z}}{\partial x} - \frac{\omega}{c}\frac{\partial \tilde{E}_{0z}}{\partial y}\right)$$

$$\tilde{B}_{0y} = \frac{i}{\frac{\omega^2}{c^2} - k^2}\left(k\frac{\partial \tilde{B}_{0z}}{\partial y} + \frac{\omega}{c}\frac{\partial \tilde{E}_{0z}}{\partial x}\right) \quad .$$

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Guided waves (continued)

- The TE modes of rectangular metal waveguides

$$\tilde{B}_{0z} = B_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} ,$$

$m, n = 0, 1, 2, ...$ (but not both 0)

$$\tilde{E}_{0x} = \frac{i}{\frac{\omega^2}{c^2} - k^2}\frac{\omega}{c}\frac{\partial \tilde{B}_{0z}}{\partial y} , \quad \tilde{E}_{0y} = \frac{-i}{\frac{\omega^2}{c^2} - k^2}\frac{\omega}{c}\frac{\partial \tilde{B}_{0z}}{\partial x} ,$$

$$\tilde{B}_{0x} = \frac{-ik}{\frac{\omega^2}{c^2} - k^2}\frac{\partial \tilde{B}_{0z}}{\partial x} , \quad \tilde{B}_{0y} = \frac{ik}{\frac{\omega^2}{c^2} - k^2}\frac{\partial \tilde{B}_{0z}}{\partial y} .$$

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Guided waves (continued)

- Waveguide modes, e.g. TE:

$$\langle \mathbf{S} \rangle = \frac{B_0^2}{8\pi}\left[\frac{i\omega}{\omega^2/c^2 - k^2}\left(\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b}\hat{\mathbf{x}} + \frac{n\pi}{b}\cos^2\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{n\pi y}{b}\hat{\mathbf{y}} \right) + \frac{k\omega}{\left(\omega^2/c^2 - k^2\right)^2}\left(\left[\frac{n\pi}{b}\right]^2 \cos^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b} + \left[\frac{m\pi}{b}\right]^2 \sin^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b} \right)\hat{\mathbf{z}} \right] .$$

Guided waves (continued)

- Dispersion and cut-off in waveguides
- Massive photons?
- The real reason there are no TEM modes in hollow conducting waveguides
- TEM modes in coaxial waveguides

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}}$$

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \omega_{mn}^2/\omega^2}} > c$$

$$v_g = \frac{d\omega}{dk} = c\sqrt{1 - \omega_{mn}^2/\omega^2} < c$$