Today in Physics 218: review II

Here's a laundry-list-like reminder of the contents of the *second* half of the course:

- ❑ Retarded potentials and radiation by time-variable charge distributions
- ❑ Pathlength differences and diffraction
- ❑ Electrodynamics and the special theory of relativity

Left and right panels from "The Garden of Earthly Delights," Hieronymus Bosch, c. 1504.

Generally useful math facts

- ❑ Divergence and delta function

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}', \int_{\mathcal{V}} \delta^3(\mathbf{r})\,d\tau = 1 \Rightarrow$$

$$\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}} = \frac{1}{\mathscr{r}^2}, \quad \nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla \mathscr{r} = \hat{\boldsymbol{\mathscr{r}}}, \quad \nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}, \quad \nabla\left(\frac{1}{\mathscr{r}}\right) = -\nabla'\left(\frac{1}{\mathscr{r}}\right)$$

- ❑ Trig identities

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta :$$

- ❑ Solid angle

$$d\Omega = \sin\theta\, d\theta\, d\phi, \quad \Omega \cong \pi\alpha^2, \alpha \ll 1$$

$$\Omega = \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi$$

Generally useful math facts (continued)

- ❑ First-order approximations

$$\sin x = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots \cong x$$

$$\cos x = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!} = 1 - \frac{x^2}{2} + \frac{x^4}{120} - \ldots \cong 1$$

$$\tan x = \sum_{i=0}^{\infty} \frac{2^{2i+2}\left(2^{2i+2}-1\right)B_i x^{2i+1}}{(2i+2)!} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots \cong x$$

$$\arctan x = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{2i+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \cong x$$

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2} + \ldots \cong 1 + x$$

$$\ln(1+x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{i+1}}{i+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \cong x$$

$$(1+x)^n = \sum_{i=0}^{\infty} \frac{n!}{i!(n-i)!} x^i = 1 + nx + \frac{n(n-1)}{2} x + \ldots \cong 1 + nx$$

Generally useful math facts (continued)

- Fourier transforms, 2-D

$$f(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(s,t)e^{-i(xs+yt)}dsdt$$

$$F(s,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{i(xs+yt)}dxdy$$

- Rayleigh's theorem

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|f(x,y)\right|^2 dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|F(s,t)\right|^2 dsdt$$

- Bessel functions

$$J_m(u)=\frac{i^{-m}}{2\pi}\int_0^{2\pi}e^{i(mv+u\cos v)}dv$$

$$\frac{d}{du}\left[u^m J_m(u)\right]=u^m J_{m-1}(u)$$

$$u^m J_m(u)=\int_0^u v^m J_{m-1}(v)dv$$

Retarded potentials and radiation

- Retarded potentials and retarded time
- Retarded potentials and the Lorentz gauge
- Retarded potentials as solutions to the inhomogeneous wave equation

$$t_r=t-\mathscr{r}/c$$

$$V(\boldsymbol{r},t)=\int_{\mathcal{V}}\frac{\rho(\boldsymbol{r}',t-\mathscr{r}/c)d\tau'}{\mathscr{r}}$$

$$\boldsymbol{A}(\boldsymbol{r},t)=\frac{1}{c}\int_{\mathcal{V}}\frac{\boldsymbol{J}(\boldsymbol{r}',t-\mathscr{r}/c)d\tau'}{\mathscr{r}}$$

Retarded potentials and radiation (continued)

- Retarded potentials for an oscillating electric dipole

$$V=2\frac{p_0\cos\theta}{2r^2}\cos\omega\left(t-\frac{r}{c}\right)-2\frac{p_0\omega\cos\theta}{2rc}\sin\omega\left(t-\frac{r}{c}\right)=V_{\text{near}}+V_{\text{rad}}$$

$$\boldsymbol{A}_{\text{rad}}=\frac{p_0\omega\sin\theta\sin\omega(t-r/c)}{rc}\hat{\boldsymbol{\theta}}-\frac{p_0\omega\cos\theta\sin\omega(t-r/c)}{rc}\hat{\boldsymbol{r}}$$

- The far field

Far field: $r\gg\lambda\gg d$

Retarded potentials and radiation (continued)

- Radiated fields and intensity for an oscillating electric dipole

$$\mathbf{E}_{\text{rad}} = -\hat{\boldsymbol{\theta}}\frac{p_0\omega^2 \sin\theta}{rc^2}\cos\omega\left(t-\frac{r}{c}\right)$$

$$\mathbf{B}_{\text{rad}} = -\hat{\boldsymbol{\phi}}\frac{p_0\omega^2}{c^2}\frac{\sin\theta}{r}\cos\omega\left(t-\frac{r}{c}\right)$$

$$\langle \mathbf{S}\rangle = \frac{c}{8\pi}\left(\frac{p_0\omega^2}{c^2}\right)^2\left(\frac{\sin\theta}{r}\right)^2\hat{\mathbf{r}} = I\hat{\mathbf{r}}$$

$$\langle P\rangle = \frac{p_0^2\omega^4}{3c^3}$$

- Total scattering cross section of a dielectric sphere

$$P_{\text{scattered}} = \sigma_{sc} I_I \;,\; \sigma_{sc} = 2\left(\frac{4\pi}{3}\right)^3\frac{a^6\chi_e^2\omega^4}{c^4}$$

Retarded potentials and radiation (continued)

- The color and polarization of the sky; reddening in sunsets and interstellar clouds
- Demonstration of the wavelength and polarization dependence of Rayleigh scattering
- Magnetic dipole radiation

Electric dipole with $p_0 \leftrightarrow m_0$, $\mathbf{E} \leftrightarrow \mathbf{B}$, $\mathbf{B} \leftrightarrow -\mathbf{E}$ = magnetic dipole:

$$\mathbf{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{m_0\omega^2}{c^2}\frac{\sin\theta}{r}\cos\omega\left(t-\frac{r}{c}\right)\hat{\boldsymbol{\phi}}$$

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A} = -\frac{m_0\omega^2}{c^2}\frac{\sin\theta}{r}\cos\omega\left(t-\frac{r}{c}\right)\hat{\boldsymbol{\theta}}$$

$$\langle \mathbf{S}\rangle = \frac{c}{8\pi}\left(\frac{m_0\omega^2}{c^2}\right)^2\left(\frac{\sin\theta}{r}\right)^2\hat{\mathbf{r}} = I\hat{\mathbf{r}}$$

$$\frac{\langle S_{\text{mag}}\rangle}{\langle S_{\text{elec}}\rangle} = \left(\frac{\omega b}{c}\right)^2 \ll 1$$

Retarded potentials and radiation (continued)

- Multipole expansion for the potentials in radiating systems
- Radiation field in the dipole approximation
- Radiation by accelerating charges: the Larmor formula

$$V(\mathbf{r},t) \cong \frac{Q}{r} + \hat{\mathbf{r}}\cdot\frac{\mathbf{p}(t-r/c)}{r^2} + \hat{\mathbf{r}}\cdot\frac{\dot{\mathbf{p}}(t-r/c)}{rc}$$

$$\mathbf{A}(\mathbf{r},t) \cong \frac{1}{rc}\dot{\mathbf{p}}$$

$$\mathbf{E}_{\text{rad}}(\mathbf{r},t) = \frac{\ddot{p}}{c^2}\frac{\sin\theta}{r}\hat{\boldsymbol{\theta}}$$

$$\mathbf{B}_{\text{rad}} = \frac{\ddot{p}}{c^2}\frac{\sin\theta}{r}\hat{\boldsymbol{\phi}}$$

$$\mathbf{S} = \frac{1}{4\pi}\frac{\ddot{p}^2}{c^3}\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}} \;,\; P = \frac{2}{3}\frac{\ddot{p}^2}{c^3}$$

Retarded potentials and radiation (continued)

- ❑ Problems with moving charges
- ❑ Motion, snapshots and lengths
- ❑ The Liénard-Wiechert potentials
- ❑ Fields from moving charges

$\boldsymbol{\mathscr{r}} \neq \boldsymbol{r} - \boldsymbol{r}'$: instead,

$\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{w}(t_r)$.

$$V(\boldsymbol{r},t) = \frac{q}{\mathscr{r}\left(1 - \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\cdot\boldsymbol{v}\right)}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\boldsymbol{v}}{c}\frac{q}{\mathscr{r}\left(1 - \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\cdot\boldsymbol{v}\right)} = \frac{\boldsymbol{v}}{c}V(\boldsymbol{r},t)$$

Retarded potentials and radiation (continued)

- ❑ Fields from moving charges.
- ❑ The generalized Coulomb field and the radiation field.
- ❑ Example: radiation by electric charge accelerating from rest, a rederivation of the Larmor formula.

$$\boldsymbol{E} = \frac{q\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}\left[\left(c^2 - v^2\right)\boldsymbol{u} + \boldsymbol{\mathscr{r}}\times(\boldsymbol{u}\times\boldsymbol{a})\right]$$

$$= \boldsymbol{E}_{\mathrm{GC}} + \boldsymbol{E}_{\mathrm{rad}}$$

$$\boldsymbol{B} = \hat{\boldsymbol{\mathscr{r}}}\times\boldsymbol{E}$$

$$\boldsymbol{S} = \frac{q^2 a^2}{4\pi c^3}\frac{\sin^2\theta}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$

$$P = \frac{2}{3}\frac{q^2 a^2}{c^3}$$

$$\left(\frac{dP}{d\Omega}\right)_{v\ll c} = \frac{q^2 a^2}{4\pi c^3}\sin^2\theta$$

Retarded potentials and radiation (continued)

- ❑ Relativistic charges and the generalized Larmor formula

$$\left(\frac{dP}{d\Omega}\right)_{\mathrm{emitted}} = \frac{q^2}{4\pi}\frac{\left[\hat{\boldsymbol{\mathscr{r}}}\times(\boldsymbol{u}\times\boldsymbol{a})\right]^2}{(\hat{\boldsymbol{\mathscr{r}}}\cdot\boldsymbol{u})^5}$$

$$P_{\mathrm{emitted}} = \frac{2}{3}\frac{q^2}{c^3}\gamma^6\left[a^2 - \left(\frac{\boldsymbol{v}}{c}\times\boldsymbol{a}\right)^2\right]$$

- ❑ Bremsstrahlung

$$\left(\frac{dP}{d\Omega}\right)_{\mathrm{emitted,B}} = \frac{q^2 a^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

$$P_{\mathrm{B}} = \frac{2}{3}\frac{q^2 a^2}{c^3}\gamma^6$$

$$\theta_0 \cong \sqrt{\frac{1-\beta}{2}} \quad (\text{if } \beta \to 1)$$

$$\frac{(dP/d\Omega)_{\max, v\to c}}{(dP/d\Omega)_{\max, v\ll c}} = \frac{1}{4}\left(\frac{8}{5}\right)^5\gamma^8$$

Retarded potentials and radiation (continued)

- Synchrotron radiation
- Radiation reaction
- The Abraham-Lorentz formula; radiation reaction force
- Radiation reaction: a fundamental inconsistency of electrodynamics.
- Runaway solutions and acausal "preaccelerations."

$$\left(\frac{dP}{d\Omega}\right)=\frac{q^2a^2}{4\pi c^3}\frac{1}{(1-\beta\cos\theta)^3}-\frac{q^2a^2}{4\pi c^3}\frac{\left(1-\beta^2\right)\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^5}$$

$$P=\frac{2}{3}\frac{q^2a^2}{c^3}\gamma^4$$

$$F_{\mathrm{rad}}=-\frac{2}{3}\frac{q^2}{c^3}\dot{a}$$

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Diffraction

- Fields as sources of radiation: Huygens's principle.
- The Kirchhoff integral: "the far field is the Fourier transform of the near field."

$$dE_F=\frac{\mathcal{E}_A(x',y')da'}{\mathscr{r}}e^{i(k\mathscr{r}-\omega t)}$$

$$E_F\left(k_x,k_y,t\right)=\frac{e^{ikr}}{\lambda r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E_N(x',y',t)e^{-i\left(k_xx'+k_yy'\right)}dx'dy' \quad .$$

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Diffraction (continued)

- Circular-aperture diffraction and the Airy pattern
- Circular obstacles, and Poisson's spot.

$$I_F(0)=\frac{cE_{N0}^2A^2}{8\pi\lambda^2r^2}$$

$$I_F(ka\theta)=I_F(0)\left[\frac{2J_1(ka\theta)}{ka\theta}\right]^2$$

$$\theta_1=1.22\frac{\lambda}{D}$$

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Diffraction (continued)

- The facts about rainbows, and the short explanation of all the facts
- Brief survey of the history of the study of rainbows
- The geometrical optics of raindrops
- Dispersion and the color of rainbows
- Brewster's angle and the polarization of rainbows

$$\sin\theta = \frac{y}{r}\ ,\ \cos\theta = -\sqrt{1-\frac{y^2}{r^2}}$$

$$\sin\theta' = \frac{1}{n}\sin\theta = \frac{y}{nr}$$

$$y_0 = \frac{r}{3}\sqrt{12-3n^2}$$

$$\Delta\theta = 2\theta - 4\theta' + \pi$$

$$= 2\arcsin\left(\frac{y}{r}\right) - 4\arcsin\left(\frac{y}{nr}\right) + \pi$$

Diffraction (continued)

- Supernumerary arcs
- Caustics and diffraction
- Airy's theory of the rainbow and the supernumerary arcs

$$I = \frac{cE_0^2}{8\pi\lambda^2 R^2}\left(\frac{3\lambda r^2}{4h}\right)^{2/3}\left(\int_{\text{drop}} \cos\frac{\pi}{2}\left(\zeta w - w^3\right)dw\right)^2 \quad .$$

Electrodynamics and special relativity

- Relativity and the four basic areas of physics
- Brief review of the basics of the special theory of relativity

$$L_{\parallel} = \sqrt{1-\left(\frac{v}{c}\right)^2} = \frac{L_{\parallel 0}}{\gamma}\ ,\ L_{\perp} = L_{\perp 0}$$

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \gamma\Delta t_0$$

$$x' = \gamma\left(x - vt\right)\quad ,$$

$$y' = y \quad , \quad z' = z \quad ,$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

Electrodynamics and special relativity (continued)

- The Lorentz transformation and four-vectors
- Scalar products of four-vectors, and Lorentz invariants

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$a^\mu = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}, \quad a_\mu = \begin{pmatrix} -a^0 & a^1 & a^2 & a^3 \end{pmatrix}$$

$$\bar{a}_\mu \bar{b}^\mu = a_\mu b^\mu$$

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Electrodynamics and special relativity (continued)

- The Einstein summation convention
- The Minkowski invariant interval
- Proper time and four-velocity
- Four-momentum and the relativistic energy

$$I = \Delta x_\mu \Delta x^\mu = -c^2 \Delta t^2 + d^2$$

$$d\tau = dt\sqrt{1 - \frac{u^2}{c^2}}, \quad \eta^\mu = \frac{dx^\mu}{d\tau}$$

$$\boldsymbol{\eta} = \frac{d\mathbf{x}}{d\tau} = \frac{\mathbf{u}}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma_u \mathbf{u}$$

$$\eta^0 = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma_u c$$

$$p^0 = m\eta^0 = \frac{E}{c}, \quad \mathbf{p} = m\boldsymbol{\eta}$$

$$\bar{p}_\mu \bar{p}^\mu = p_\mu p^\mu = E^2 - p^2 c^2 = \left(mc^2\right)^2$$

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Electrodynamics and special relativity (continued)

- Newton's laws in relativity

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{p}}{dt}, \quad \mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}}$$

- The Minkowski force

$$\mathbf{K} = \frac{d\mathbf{p}}{d\tau} = \frac{dt}{d\tau}\frac{d\mathbf{p}}{dt} = \frac{1}{\sqrt{1 - u^2/c^2}}\mathbf{F}$$

$$K^0 = \frac{dp^0}{d\tau} = \frac{d}{d\tau}\frac{E}{c} \quad \Rightarrow \quad K^\mu = \frac{dp^\mu}{d\tau}$$

- Relativistic transformation of forces

$$\bar{\mathbf{F}}_\parallel = \mathbf{F}_\parallel, \quad \bar{F}_\perp = \frac{1}{\gamma}F_\perp$$

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Electrodynamics and special relativity (continued)

- Relativistic transformations of ***E*** and ***B***.

$$\bar{E}_x = E_x \quad , \quad \bar{E}_y = \gamma\left(E_y - \beta B_z\right) \quad , \quad \bar{E}_z = \gamma\left(E_z + \beta B_y\right) \quad ,$$
$$\bar{B}_x = B_x \quad , \quad \bar{B}_y = \gamma\left(B_y + \beta E_z\right) \quad , \quad \bar{B}_z = \gamma\left(B_z - \beta E_y\right) \quad .$$

Or:

$$\bar{\mathbf{E}}_\parallel = \mathbf{E}_\parallel \quad , \quad \bar{\mathbf{E}}_\perp = \gamma\left(\mathbf{E}_\perp + \boldsymbol{\beta}\times\mathbf{B}_\perp\right) \quad ,$$
$$\bar{\mathbf{B}}_\parallel = \mathbf{B}_\parallel \quad , \quad \bar{\mathbf{B}}_\perp = \gamma\left(\mathbf{B}_\perp - \boldsymbol{\beta}\times\mathbf{E}_\perp\right) \quad .$$

Electrodynamics and special relativity (continued)

- The electromagnetic field four-tensor.

$$F^{\mu\nu} = \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$
$$= \partial^\mu A^\nu - \partial^\nu A^\mu \quad .$$

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix} \quad .$$

Electrodynamics and special relativity (continued)

- Charge and current densities, the Maxwell equations, and the Lorentz force, in tensor form
- The four-potential and gauge transformations
- The relativistic analogue of the inhomogeneous wave equation for potentials.

$$J^\mu = \left(c\rho, \mathbf{J}\right) \;, \; \partial_\mu J^\mu = 0$$
$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} J^\mu \;, \; \partial_\nu G^{\mu\nu} = 0$$
$$K^\mu = \frac{q\eta_\nu}{c} F^{\mu\nu}$$
$$A^\mu = \left(V, \mathbf{A}\right) \;, \; \partial_\nu A^\nu = 0$$
$$\Box^2 A^\mu \equiv \partial_\nu \partial^\nu A^\mu = -\frac{4\pi}{c} J^\mu$$